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A NOTE ON SINGULAR QUARTIC MOMENT PROBLEM

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ABSTRACT. Let $\gamma \equiv \gamma^{(2n)}$ denote a sequence of complex numbers $\gamma_{00}, \gamma_{01}, \gamma_{10}, \cdots, \gamma_{0,2n}, \cdots, \gamma_{2n,0}$ with $\gamma_{00} > 0, \gamma_{ji} = \overline{\gamma_{ij}}$, and let K denote a closed subset of the complex plane $\mathbb{C}$. The truncated K complex moment problem entails finding a positive Borel measure μ such that $\gamma_{ij} = \int \bar{z}^i z^j d\mu$ $(0 \le i+j \le 2n)$ and supp $\mu \subseteq K$. If $n = 2$, then it is called the quartic moment problem. In this paper, we give partial solutions for the singular quartic moment problem with rank $M(2) = 5$ and $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2, \bar{Z}^2 \rangle$.

1. Introduction and preliminaries

Given a closed subset $K \subseteq \mathbb{C}$ and a doubly indexed finite sequence of complex numbers

(1.1)
$$\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \cdots, \gamma_{0,2n}, \gamma_{1,2n-1}, \cdots, \gamma_{2n-1,1}, \gamma_{2n,0},$$
$$\text{where} \quad \gamma_{00} > 0 \quad \text{and} \quad \gamma_{ji} = \overline{\gamma_{ij}},$$

the *truncated K complex moment problem* entails finding a positive Borel measure μ such that

$$(1.2) \qquad \gamma_{ij} = \int \bar{z}^i z^j d\mu \quad (0 \le i+j \le 2n) \quad \text{and} \quad \text{supp } \mu \subseteq K.$$

Any sequence γ as in (1.1) is a *truncated moment sequence* and any measure μ as in (1.2) is a *representing measure* for γ.

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For $n \geq 1$, let $m \equiv m(n) = (n+1)(n+2)/2$. For $A \in M_m(\mathbb{C})$ (the $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering: $1, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \cdots, Z^n, \cdots, \bar{Z}^n$; rows and columns indexed by $1, Z, Z^2, \cdots, Z^n$ are said to be *analytic*. For the truncated moment sequence (1.1), we define $M(n)(\gamma) \in M_m(\mathbb{C})$ as follows: for $0 \leq i+j \leq n, 0 \leq l+k \leq n$, the entry in row $\bar{Z}^l Z^k$ and column $\bar{Z}^i Z^j$ is

$$M(n)_{(l,k)(i,j)} = \gamma_{i+k,j+l}. \tag{1.3}$$

For example, if $n = 1$, the *quadratic moment problem* for $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}$ corresponds to

$$M(1) = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} \end{bmatrix},$$

and if $n = 2$, the *quartic moment problem* for $\gamma : \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{02}, \gamma_{11}, \gamma_{20}, \gamma_{03}, \gamma_{12}, \gamma_{21}, \gamma_{30}, \gamma_{04}, \gamma_{13}, \gamma_{22}, \gamma_{31}, \gamma_{40}$ corresponds to

$$M(2) = \begin{bmatrix} \gamma_{00} & \gamma_{01} & \gamma_{10} & \gamma_{02} & \gamma_{11} & \gamma_{20} \\ \gamma_{10} & \gamma_{11} & \gamma_{20} & \gamma_{12} & \gamma_{21} & \gamma_{30} \\ \gamma_{01} & \gamma_{02} & \gamma_{11} & \gamma_{03} & \gamma_{12} & \gamma_{21} \\ \gamma_{20} & \gamma_{21} & \gamma_{30} & \gamma_{22} & \gamma_{31} & \gamma_{40} \\ \gamma_{11} & \gamma_{12} & \gamma_{21} & \gamma_{13} & \gamma_{22} & \gamma_{31} \\ \gamma_{02} & \gamma_{03} & \gamma_{12} & \gamma_{04} & \gamma_{13} & \gamma_{22} \end{bmatrix}. \tag{1.4}$$

The quadratic moment problem was solved completely. In fact, it was shown that γ has a representing measure if and only if $M(1) \geq 0$ [6, Theorem 6.1].

Let $\mathcal{P}_n \subseteq \mathbb{C}[z, \bar{z}]$ denote the complex polynomials in $z, \bar{z}$ of total degree $\leq n$. For $p \in \mathcal{P}_n, p(z, \bar{z}) = \sum_{0 \leq i+j \leq n} a_{ij} \bar{z}^i z^j$, let $\bar{p}(z, \bar{z}) = \sum_{0 \leq i+j \leq n} \overline{a}_{ij} z^i \bar{z}^j$ and let $\hat{p} \equiv (a_{00}, a_{01}, a_{10}, \cdots, a_{0n}, \cdots, a_{n0}) \in \mathbb{C}^{m(n)}$. The basic connection between $M(n)(\gamma)$ and any representing measure μ is provided by the identity

$$\int f\bar{g}d\mu = (M(n)\hat{f}, \hat{g}) \qquad (f, g \in \mathcal{P}_n); \tag{1.5}$$

in particular $(M(n)\hat{f}, \hat{f}) = \int |f|^2 d\mu \geq 0$, so $M(n) \geq 0$.

THEOREM 1.1 ([6, Corollary 5.14]). *If $M(n) \geq 0$ and $M(n)$ is flat, i.e.,* rank $M(n)$ = rank $M(n-1)$, *then γ has a unique representing measure, which is* rank $M(n)$*-atomic.*

THEOREM 1.2 ([6, Theorem 5.13]). *γ has a* rank $M(n)$*-atomic representing measure if and only if $M(n) \geq 0$ and $M(n)$ admits a flat extension $M(n+1)$, i.e., $M(n)$ can be extended to a moment matrix $M(n+1)$ satisfying* rank $M(n+1)$ = rank $M(n)$.

Recall from [6, Proposition 3.1] that if μ is a representing measure for γ, then

$$\begin{aligned} &\text{For } \ p \in \mathcal{P}_n, p(Z,\bar{Z}) = 0 \\ &\Longleftrightarrow \text{supp } \mu \subseteq \mathcal{Z}(p) := \{z \in \mathbb{C} : p(z,\bar{z}) = 0\}. \end{aligned} \tag{1.6}$$

It follows from [6, Corollary 3.5] that

$$\begin{aligned} &\text{If } \ \mu \ \text{ is a representing measure for } \ \gamma, \\ &\text{then card supp } \mu \geq \text{rank } M(n). \end{aligned} \tag{1.7}$$

Further, let $r :=$ rank $M(n)$ and let $\mathcal{C}_{M(n)}$ denote the column space of $M(n)$, so that in $\mathcal{C}_{M(r)}$ there is a dependence relation of the form $Z^r = c_0 1 + c_1 Z + \cdots + c_{r-1} Z^{r-1}$. The polynomial $z^r - (c_0 + \cdots + c_{r-1} z^{r-1})$ has r distinct roots, $z_0, \cdots, z_{r-1}$, which provide the support for the unique representing measure for $\gamma^{(2n+2)}$ corresponding to the flat extension $M(n+1)$. The densities of this measure, $\rho_0, \cdots, \rho_{r-1}$, are determined by the Vandermonde equation

$$V(z_0, \cdots, z_{r-1})(\rho_0, \cdots, \rho_{r-1})^t = (\gamma_{00}, \cdots, \gamma_{0,r-1})^t;$$

where $(*, \cdots, *)^t$ is the transpose of $(*, \cdots, *)$. Thus, the representing measure is $\mu := \sum_{i=0}^{r-1} \rho_i \delta_{z_i}$.

For the quartic moment problem, R. Curto and L. Fialkow obtained the following results.

THEOREM 1.3 ([8, Theorem 1.10]). *Suppose $M(2)(\gamma)$ is positive and recursively generated. Then γ has a* rank $M(2)$*-atomic representing measure in each of the following cases:*

(i) *$\{1, Z, \bar{Z}, Z^2\}$ is linearly dependent in $\mathcal{C}_{M(2)}$;*

(ii) $\{1, Z, \bar{Z}, Z^2\}$ *is a basis for* $\mathcal{C}_{M(2)}, \bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, *and the moments* γ_{ij} *are all real, with the possible exception of* γ_{04};

(iii) $\{1, Z, \bar{Z}, Z^2\}$ *is a basis for* $\mathcal{C}_{M(2)}, \bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, *and the reduced C-block test* $c_{11} = c_{22}$ *passes;*

(iv) $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ *is a basis for* $\mathcal{C}_{M(2)}, \bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, *and the reduced C-block test* $c_{11} = c_{22}$ *passes for some choice of* γ_{05}.

In our recent work [11], we have obtained the following further results on the quartic moment problem.

THEOREM 1.4 ([11]). *Suppose* $M(2)(\gamma)$ *is positive.*

(1) *Let* $\operatorname{rank} M(2) = 4$.
   (i) *If* $\{1, Z, \bar{Z}, Z^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$, *then there exists* $M(2)$ *admitting no representing measure.*
   (ii) *If* $\{1, Z, \bar{Z}, Z^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$ *and* $\bar{Z}Z \in \langle 1, Z, \bar{Z}\rangle$, *then* $M(2)$ *admits a unique representing measure.*

(2) *Let* $\operatorname{rank} M(2) = 5$. *If* $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ *is a basis for* $\mathcal{C}_{M(2)}$ *and* $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2\rangle$, *then* $M(2)$ *admits a representing measure.*

Thus the remaining problems for the quartic moment problem are as follows:

PROBLEM 1.5. *Assume that* $M(2)$ *is positive and* $\operatorname{rank} M(2) = 5$. *If* $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2, \bar{Z}^2\rangle$, *does* $M(2)$ *admit a representing measure?*

PROBLEM 1.6. *Does arbitrary nonsingular positive quartic moment matrix* $M(2)$ *admit a representing measure?*

For Problem 1.6, we also obtained some affirmative partial solutions [11]. In this paper, we consider Problem 5 and give partial solutions for the singular quartic moment problem with rank $M(2) = 5$ and $\bar{Z}Z \in \langle 1, Z, \bar{Z}, Z^2, \bar{Z}^2\rangle$.

We conclude this section with an introduction to the extension problem for positive moment matrices. For $k, l \in \mathbb{Z}_+$, let $A \in M_k(\mathbb{C}), A = A^*, B \in M_{k,l}(\mathbb{C}), C \in M_l(\mathbb{C})$; we refer to any matrix of the form

$$\tilde{A} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{1.8}$$

as an *extension* of A.

Proposition 1.7 ([6, Proposition 2.2]). *For $A \geq 0$, the following are equivalent:*

1) $\tilde{A} \geq 0$;
2) *There exists $W \in M_{k,l}(\mathbb{C})$ such that $AW = B$ and $C \geq W^*AW$.*

If $A \geq 0$ and $AW = B$, i.e., Ran $B \subseteq$ Ran A, there is the unique flat extension of the form (1.8), which we denote by $[A; B]$. For $M(n) \geq 0$, we want to construct a positive flat extension of the form $M(n+1) = [M(n); B(n)]$.

Proposition 1.8 ([6, Proposition 2.3]). *If Ran $B(n) \subseteq$ Ran $M(n)$, then $M := [M(n); B(n)]$ satisfies $M_{(p,q)(r,s)} = M_{(s,r)(q,p)}$ for all choices of $p, q, r, s \geq 0$ such that $p + q = r + s = n + 1$.*

2. Main results

Throughout this section, we assume that $M(2)$ is positive and rank $M(2) = 5$.

Lemma 2.1. *If $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$, then $\bar{Z}Z = a1 + bZ + \bar{b}\bar{Z} + dZ^2 + \bar{d}\bar{Z}^2$, where $a \in \mathbb{R}$ and $b, d \in \mathbb{C}$.*

Proof. Suppose $\bar{Z}Z = a1 + bZ + c\bar{Z} + dZ^2 + e\bar{Z}^2$. Since $p(Z, \bar{Z}) = 0$ implies $\bar{p}(Z, \bar{Z}) = 0$ [6, Lemma 3.10], it follows $\bar{Z}Z = \bar{a}1 + \bar{b}\bar{Z} + \bar{c}Z + \bar{d}\bar{Z}^2 + \bar{e}Z^2$. Thus we have

$$(a - \bar{a})1 + (b - \bar{c})Z + (c - \bar{b})\bar{Z} + (d - \bar{e})Z^2 + (e - \bar{d})\bar{Z}^2 = 0.$$

But since $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is linearly independent, it follows that $a = \bar{a}, b = \bar{c}, d = \bar{e}$. □

Let $C = (c_{ij})_{1 \leq i,j \leq n+1}$ in $[M(n); B(n)]$. By Proposition 1.8, we have the following proposition.

Proposition 2.2. *If Ran $B(2) \subseteq$ Ran $M(2)$, then $[M(2); B(2)]$ is a flat extension of $M(2)$ if and only if $c_{11} = c_{22}$ and $c_{21} = c_{32}$.*

From now we consider the quartic moment problem with the moment sequence $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = 0, \gamma_{12} = 0, \gamma_{04} =$

$h, \gamma_{13} = w, \gamma_{22} = x$. Hence the corresponding moment matrix is given by

$$(2.1) \qquad M(2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & \bar{w} & \bar{h} \\ 1 & 0 & 0 & w & x & \bar{w} \\ 0 & 0 & 0 & h & w & x \end{bmatrix}.$$

We first analyze the positivity of $M(2)$ in (2.1).

LEMMA 2.3. *Let $M(2)$ be a moment matrix as in* (2.1). *Then $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$ if and only if*

$$(2.2) \qquad x > |h| \quad \text{and} \quad (x^2 - |h|^2)(1-x) + 2x|w|^2 - 2\mathrm{Re}(w^2\bar{h}) = 0.$$

Proof. Suppose $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$. If we let

$$A := M(2)_{\{1,2,3,4,6\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & x & \bar{h} \\ 0 & 0 & 0 & h & x \end{bmatrix},$$

then

$$\begin{aligned} &\det\ A = x^2 - |h|^2 > 0, \\ &\det\ M(2) = 2\mathrm{Re}(w^2\bar{h}) - 2x|w|^2 - (x^2 - |h|^2)(1-x) = 0. \end{aligned}$$

Thus we have (2.2). Conversely, suppose (2.2) is satisfied. Then it is clear that $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is linearly independent. $\square$

For the moment matrix (2.1), we want to construct a positive flat extension of the form $[M(2); B(2)]$. Thus by Proposition 2.2, we should detect the conditions $c_{11} = c_{22}$ and $c_{21} = c_{32}$.

THEOREM 2.4. *Let $M(2)$ be a moment matrix as in (2.1). If*
(i) $x > |h|$ *and* $(x^2 - |h|^2)(1-x) + 2x|w|^2 - 2\text{Re}(w^2\bar{h}) = 0$,
(ii) *there exist* y, z, v *in* $\mathbb{C}$ *such that*

$$\begin{cases} (x^2 - |h|^2)z = (xw - h\bar{w})y + (x\bar{w} - w\bar{h})v, \\ (x^2 - |h|^2)y = (xw - h\bar{w})\bar{y} + (x\bar{w} - w\bar{h})z, \\ (x^2 - |h|^2)^2 - 2\text{Re}(\bar{h}yz) + x|z|^2 = x|v|^2 - 2\text{Re}(hy\bar{v}), \\ (\bar{w}h - xw)(x^2 - |h|^2) + xy^2 - hy\bar{z} + x\bar{z}v - \bar{h}yv = 2x\bar{y}z - h\bar{y}^2 - \bar{h}z^2 \end{cases}$$

then $M(2)$ has a flat extension $M(3)$.

Proof. First note that condition (i) implies $M(2) \geq 0$ and $\{1, Z, \bar{Z}, Z^2, \bar{Z}^2\}$ is a basis for $\mathcal{C}_{M(2)}$ by Lemma 2.3. If we construct $B(2)$ satisfying $\text{Ran}\, B(2) \subseteq \text{Ran}\, M(2)$, $c_{11} = c_{22}$ and $c_{21} = c_{32}$, then Proposition 2.2 guarantees the existence of a flat extension. To define Z^3 so that $Z^3 \in \text{Ran}M(2)$, let $\gamma_{23} = y, \gamma_{14} = z, \gamma_{05} = v$. Since $[M(2)]_{\{1,2,3,4,6\}} > 0$, there exist unique scalars k_1, k_2, k_3, k_4, k_6 such that

$$[M(2)]_{\{1,2,3,4,6\}}(k_1, k_2, k_3, k_4, k_6)^t = [Z^3]_{\{1,2,3,4,6\}} = (0, w, h, \gamma_{23}, \gamma_{05})^t.$$

In fact,

$$(k_1, k_2, k_3, k_4, k_6) = \left(0, w, h, \frac{xy - v\bar{h}}{x^2 - |h|^2}, \frac{xv - hy}{x^2 - |h|^2}\right).$$

To ensure that $Z^3 \in \text{Ran}\, M(2)$, we are forced to define

$$Z^3 := M(2)\left(0, w, h, \frac{xy - v\bar{h}}{x^2 - |h|^2}, 0, \frac{xv - hy}{x^2 - |h|^2}\right)^t.$$

This relation defines γ_{14}, i.e.,

$$\gamma_{14} = \frac{wxy - wv\bar{h} + xv\bar{w} - hy\bar{w}}{x^2 - |h|^2}.$$

This is the first equation of (ii). The same argument for $\bar{Z}Z^2$ gives

$$(x^2 - |h|^2)y = (xw - h\bar{w})\bar{y} + (x\bar{w} - w\bar{h})z.$$

Since $\{Z^3, \bar{Z}Z^2\}$ and $\{\bar{Z}Z^2, \bar{Z}^2Z\}$ form normal bands and since $\bar{Z}^2Z$ and $\bar{Z}^3$ are symmetric with respect to Z^3 and $\bar{Z}Z^2$, respectively, we can conclude that the B block of a proposed $M(3)$ is symmetric and satisfies normality. Thus we have constructed moment matrix extension block B satisfying $\operatorname{Ran} B \subseteq \operatorname{Ran} M(2)$. Now the existence of a flat extension depends on the reduced C-test, that is, $c_{11} = c_{22}$ and $c_{21} = c_{32}$. But since

$$c_{11} = (0, \bar{w}, \bar{h}, \bar{y}, \bar{v}) \left(0, w, h, \frac{xy - v\bar{h}}{x^2 - |h|^2}, \frac{xv - yh}{x^2 - |h|^2}\right)^t,$$

$$c_{22} = (0, x, \bar{w}, y, \bar{z}) \left(0, x, w, \frac{x\bar{y} - z\bar{h}}{x^2 - |h|^2}, \frac{xz - h\bar{y}}{x^2 - |h|^2}\right)^t,$$

$$c_{21} = (0, x, \bar{w}, y, \bar{z}) \left(0, w, h, \frac{xy - v\bar{h}}{x^2 - |h|^2}, \frac{xv - yh}{x^2 - |h|^2}\right)^t,$$

$$c_{32} = (0, w, x, z, \bar{y}) \left(0, x, w, \frac{x\bar{y} - z\bar{h}}{x^2 - |h|^2}, \frac{xz - h\bar{y}}{x^2 - |h|^2}\right)^t,$$

the third and the last equations of condition (ii) implies $c_{11} = c_{22}$ and $c_{21} = c_{32}$. This completes the proof. □

Unfortunately, we have not been able to decide if the condition (ii) of Theorem 2.4 holds. However for specific forms we have the following:

COROLLARY 2.5. *Let $M(2)$ be a moment matrix as in (2.1). If $x = 1, |h| < 1$ and $w = 0$, then $M(2)$ has a flat extension.*

Proof. In Theorem 2.4, take $y = z = 0$ and v with $|v| = 1 - |h|^2$. □

3. Examples

In this section, we give two examples by Corollary 2.5 in order to find the representing measure of them.

EXAMPLE 3.1. If we consider the truncated moment sequence $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = 0, \gamma_{12} = 0, \gamma_{04} = 0, \gamma_{13} = 0, \gamma_{22} = 1,$

then the corresponding moment matrix is

$$M(2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is clear that the conditions of Corollary 2.5 are satisfied. So, $M(2)$ has a flat extension. Thus, by Theorem 1.2 the truncated moment sequence admits a 5-atomic representing measure. In the following, we shall find a representing measure. To do so, if we choose $\gamma_{23} = 0, \gamma_{14} = 0, \gamma_{05} = i$, then $\bar{Z}Z^2 = Z$. Hence, $\gamma_{24} = 0, \gamma_{33} = 1, \gamma_{34} = 0, \gamma_{44} = 1$. So

$$[M(2)]_{\{1,2,4,7,11\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $[Z^5]_{\{1,2,4,7,11\}} = (i, 0, 0, 0, 0)^t$. Thus we have $z^5 - i = 0$. By direct simple calculation, we can obtain the following five atoms

$$z_0 = e^{\frac{\pi}{10}i},\ z_1 = e^{\frac{\pi}{2}i},\ z_2 = e^{\frac{9\pi}{10}i},\ z_3 = e^{\frac{13\pi}{10}i},\ z_4 = e^{\frac{17\pi}{10}i},$$

and the densities $\rho_0 = \rho_1 = \rho_2 = \rho_3 = \rho_4 = \frac{1}{5}$. Thus, one of the representing measures can be given by the following form

$$\mu = \frac{1}{5}\left(\delta_{e^{\frac{\pi}{10}i}} + \delta_{\frac{\pi}{2}i} + \delta_{\frac{9\pi}{10}i} + \delta_{\frac{13\pi}{10}i} + \delta_{\frac{17\pi}{10}i}\right).$$

Note that the five atoms are in the unit circle $|z| = 1$ and these form regular pentagon.

EXAMPLE 3.2. If we consider the truncated moment sequences $\gamma_{00} = 1, \gamma_{01} = 0, \gamma_{02} = 0, \gamma_{11} = 1, \gamma_{03} = 0, \gamma_{12} = 0, \gamma_{04} = \frac{i}{2}, \gamma_{13} = 0, \gamma_{22} = 1$, then the corresponding moment matrix is given by

$$M(2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{i}{2} \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{i}{2} & 0 & 1 \end{bmatrix}.$$

It is clear that $M(2) \geq 0$ and rank $M(2) = 5$ and the conditions of Corollary 2.5 are satisfied. Thus also by Theorem 1.2 the truncated moment sequence admits a 5-atomic representing measure. Our goal is finding a representing measure. To do so, if we choose $\gamma_{23} = 0, \gamma_{14} = 0, \gamma_{05} = \frac{3}{4}i$ (s.t. $|\gamma_{05}| = \frac{3}{4}$), then $\bar{Z}Z^2 = Z$. Hence, $\gamma_{24} = 0, \gamma_{33} = 1, \gamma_{34} = 0, \gamma_{44} = 1$. So

$$[M(2)]_{\{1,2,4,7,11\}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{i}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{i}{2} & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $[Z^5]_{\{1,2,4,7,11\}} = (\frac{3}{4}i, \frac{i}{2}, 0, 0, 0)^t$. Thus we have

$$z^5 + \frac{1}{2}z^4 - \frac{i}{2}z - i = 0.$$

By calculation through Mathematica software, we can obtain the following atoms and densities

$$\begin{aligned} z_0 &\approx -0.974219 + 0.225605i, & \rho_0 &\approx 0.14882, \\ z_1 &\approx -0.734671 - 0.678423i, & \rho_1 &\approx 0.183305, \\ z_2 &\approx -0.197957 + 0.980211i, & \rho_2 &\approx 0.212184, \\ z_3 &\approx 0.471733 - 0.881741i, & \rho_3 &\approx 0.225448, \\ z_4 &\approx 0.935113 + 0.354349i, & \rho_4 &\approx 0.230243. \end{aligned}$$

The five atoms are all in the unit circle $|z| = 1$. Note that $\rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4 \approx 1$. Hence the representing measure is $\mu = \rho_0\delta_{z_0} + \rho_1\delta_{z_1} + \rho_2\delta_{z_2} + \rho_3\delta_{z_3} + \rho_4\delta_{z_4}$.

REMARK 3.3. In the above two examples, the atoms of representing measure are all in the unit circle $|z| = 1$. In fact, Theorem 1.1 of [9] said that if let $p \in \mathbb{C}[z, \bar{z}], p \not\equiv 0, \deg\ p = 2k$ or $\deg\ p = 2k - 1$, then there exists a rank $M(n)$-atomic (minimal) representing measure for $\gamma^{(2n)}$ supported in $K_p = \{z \in \mathbb{C} : p(z, \bar{z}) \geq 0\}$ if and only if $M(n) \geq 0$ and there is some flat extension $M(n + 1)$ for which the localizing matrix $M_p(n + k) \geq 0$. In this case, the measure is a rank $M(n)$-atomic representing measure supported in K_p, with precisely rank $M(n)$ − rank $M_p(n+k)$ atoms in $\mathcal{Z}(p) \equiv \{z \in \mathbb{C} : p(z, \bar{z}) = 0\}$. For the moment matrix (2.1), if we take $p(z, \bar{z}) := 1 - z\bar{z}$, then the localizing matrix is

$$M_p(3) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - x & -\bar{w} \\ 0 & -w & 1 - x \end{bmatrix}.$$

Hence, for the moment matrix (2.1), we know that there is 5-atomic representing measure and supp $\mu \subseteq \{z \in \mathbb{C} : |z| \leq 1\}$ if and only if $M(2) \geq 0$ and there is a flat extension $M(3)$ for which $x \leq 1$ and $1 \geq x + |w|$. So, if we take $x = 1, w = 0$, then the localizing matrix $M_p(3)$ is a zero matrix, where $p(z, \bar{z}) = 1 - z\bar{z}$. Hence, if we further assume that $M(2) \geq 0$ and there is a flat extension $M(3)$, then there is 5-atomic representing measure and supp $\mu \subseteq \{z \in \mathbb{C} : |z| = 1\}$.

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