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ITERATIVE APPROXIMATIONS OF FIXED POINTS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to study the iterative approximations of fixed points for asymptotically pseudo-contractive mappings and asymptotically nonexpansive mappings in Banach spaces. The results presented in this paper extend and improve the main results in Geobel-Kirk [4], Liu [5] and Schu [7].

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the topological dual space of E, $\langle \cdot, \cdot \rangle$ denotes the dual pair between E and E^* and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \ \|x\| = \|f\|\}$$

for all $x \in E$.

Definition 1.1. Let D be a nonempty subset of E and $T : D \to D$ be a mapping.

(1) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

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for all $x, y \in D$ and $n = 1, 2, \cdots$.

(2) T is said to be *uniformly L-Lipschitzian* with the constant $L \geq 1$ if

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in D$ and $n = 1, 2, \cdots$.

(3) T is said to be *asymptotically pseudo-contractive* if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and, for any $x, y \in D$, $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2$$

for all $n = 1, 2, \cdots$.

REMARK 1.1. (1) If T is a nonexpansive mapping, then T is an asymptotically nonexpansive mapping with a constant sequence $\{k_n\}$ defined by $k_n = 1$ for all $n = 1, 2, \cdots$.

(2) If T is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ and $\lim_{n\to\infty} k_n = 1$, then T is a uniformly L-Lipschitzian mapping with the constant $L = \sup_{n\geq 1} k_n < \infty$.

(3) If T is an asymptotically nonexpansive mapping, then T is an asymptotically pseudo-contractive mapping. But the converse is not true in general.

This can be seen from the following example:

EXAMPLE 1.1. ([6]) Let $E = R$, $D = [0, 1]$ and the mapping $T : D \to D$ is defined by

$$Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$$

for all $x \in D$. We can prove that T is not Lipschitzian and so it is not asymptotically nonexpansive. Since $T \circ T = I$ and it is monotonically decreasing, it follows that

$$(x - y)(T^n x - T^n y) = \begin{cases} |x - y|^2, & \text{if } n \text{ is even,} \\ (x - y)(Tx - Ty) \leq 0 \leq |x - y|^2, & \text{if } n \text{ is odd.} \end{cases}$$

This implies that T is an asymptotically pseudo-contractive mapping with the constant sequence $\{1\}$.

DEFINITION 1.2. Let D be a nonempty convex subset of E, $T : D \to D$ be a mapping, $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$.

(1) The sequence $\{x_n\}$ defined by

$$(1.1)\qquad \begin{cases} x_0 \in D, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n \end{cases}$$

for all $n = 0, 1, 2, \cdots$ is called the *modified Ishikawa iterative sequence.*

(2) Taking $\beta_n = 0$ for all $n = 0, 1, 2, \cdots$ in (1.1), the sequence $\{x_n\}$ defined by

$$(1.2)\qquad \begin{cases} x_0 \in D, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n \end{cases}$$

for all $n = 0, 1, 2, \cdots$ is called the *modified Mann iterative sequence.*

The concept of an asymptotically nonexpansive mapping was introduced by Goebel-Kirk [4] in 1972, which was closely related to the theory of fixed points of mappings in Banach spaces. An early fundamental result due to Browder [1] states that, if E is a uniformly convex Banach space, D is a nonempty bounded closed convex subset of E and $T : D \to D$ is an asymptotically nonexpansive mapping, then T has a fixed point in D.

On the other hand, the concept of an asymptotically pseudo-contractive mapping was introduced by Schu [7] in 1991.

The iterative approximation problems for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings were studied extensively by Goebel-Kirk [4], Liu [5] and Schu [7] in Hilbert spaces and uniformly convex Banach spaces, respectively. The purpose of this paper is, by using a new iterative technique, to study the iterative approximative problem of fixed points for asymptotically nonexpansive mappings and asymptotically pseudo-contractive mappings in the setting of Banach spaces. The results represented in this paper extend and improve the main results in [4]-[5], [7] and the proof methods given in this paper is quite different from the methods given in [4]-[5], [7].

We first recall the following results for our main results:

LEMMA 1.1 ([2], [3]). *Let E be a real Banach space. Then, for any $x, y \in E$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$$

for all $j(x+y) \in J(x+y)$, where $J : E \to 2^{E^*}$ is the normalized duality mapping.

LEMMA 1.2 ([8]). *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences. If there exists a positive integer n_0 such that*

$$a_{n+1} \leq (1-t_n)a_n + b_n$$

for all $n \geq n_0$, where $0 \leq t_n < 1$, $\sum_{n=0}^{\infty} t_n = \infty$ and $b_n = o(t_n)$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

In this section, we give some iterative approximation theorems of fixed points for asymptotically pseudo-contraction mappings and asymptotically nonexpansive mappings. Let $F(T)$ denote the set of all fixed points of T.

THEOREM 2.1. *Let D be a nonempty closed convex subset of E, $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with a real sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and the constant $L \geq 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n \to 0,\ \beta_n \to 0\ (n \to \infty)$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be the modified Ishikawa iterative sequence defined by (1.1). If $F(T) \neq \emptyset$, $T(D)$ is bounded and, for any given $q \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ and $\phi(0) = 0$ such that

$$\langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle \leq k_n\|x_{n+1} - q\|^2 - \phi(\|x_{n+1} - q\|) \tag{2.1}$$

for all $n = 0, 1, 2, \cdots$, where $j(x_{n+1} - q) \in J(x_{n+1} - q)$ is such that

$$\langle T^n x_{n+1} - T^n q, j(x_{n+1} - q)\rangle \leq k_n\|x_{n+1} - q\|^2$$

for all $n = 0, 1, 2, \cdots$, then the sequence $\{x_n\}$ converges strongly to the fixed point q of T.

Proof. It follows from Lemma 1.1 and (1.1) that

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|(1-\alpha_n)(x_n - q) + \alpha_n(T^n y_n - q)\|^2 \\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle T^n y_n - q, j(x_{n+1} - q)\rangle \qquad (2.2)\\
&\le (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - q)\rangle \\
&\quad + 2\alpha_n\langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle.
\end{aligned}
$$

Now, we consider the third term on the right side of (2.2). By (2.1), we have

$$
\begin{aligned}
&2\alpha_n\langle T^n x_{n+1} - q, j(x_{n+1} - q)\rangle \qquad (2.3)\\
&\qquad \le 2\alpha_n[k_n\|x_{n+1} - q\|^2 - \phi(\|x_{n+1} - q\|)]
\end{aligned}
$$

Next, we consider the second term on the right side of (2.2). Since T is uniformly L-Lipschitzian, we have

$$
\begin{aligned}
&2\alpha_n\langle T^n y_n - T^n x_{n+1}, j(x_{n+1} - q)\rangle \qquad (2.4)\\
&\qquad \le 2\alpha_n L\|y_n - x_{n+1}\|\|x_{n+1} - q\|.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\|y_n - x_{n+1}\| \\
&= \|(1-\alpha_n)(x_n - y_n) + \alpha_n(T^n y_n - y_n)\| \\
&\le (1-\alpha_n)\beta_n\|T^n x_n - x_n\| + \alpha_n\|T^n y_n - q + q - y_n\| \\
&\le (1-\alpha_n)\beta_n\|T^n x_n - q + q - x_n\| + \alpha_n(1+L)\|y_n - q\| \qquad (2.5)\\
&\le (1-\alpha_n)\beta_n(1+L)\|x_n - q\| \\
&\quad + \alpha_n(1+L)\|(1-\beta_n)(x_n - q) + \beta_n(T^n x_n - q)\| \\
&\le (1-\alpha_n)\beta_n(1+L)\|x_n - q\| + \alpha_n(1+L)L\|x_n - q\| \\
&= d_n\|x_n - q\|,
\end{aligned}
$$

where $d_n = (1-\alpha_n)\beta_n(1+L) + \alpha_n(1+L)L$ and $d_n \to 0$ as $n \to \infty$ since α_n, $\beta_n \to 0$ as $n \to \infty$. Thus, substituting (2.3)$\sim$(2.5) into (2.2) and simplifying, we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
(2.6)\qquad &\le (1-\alpha_n)^2\|x_n-q\|^2 + 2\alpha_n L d_n \|x_n-q\|\|x_{n+1}-q\| \\
&\quad + 2\alpha_n k_n \|x_{n+1}-q\|^2 - 2\alpha_n \phi(\|x_{n+1}-q\|).
\end{aligned}$$

Since $2\|x_n-q\|\|x_{n+1}-q\| \le \|x_n-q\|^2 + \|x_{n+1}-q\|^2$, it follows from (2.6) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n L d_n}{1-2\alpha_n k_n-\alpha_n L d_n}\|x_n-q\|^2 \\
(2.7)\qquad &\quad - \frac{(2\alpha_n}{1-2\alpha_n k_n-\alpha_n L d_n}\phi(\|x_{n+1}-q\|).
\end{aligned}$$

Since $\alpha_n \to 0$, $\beta_n \to 0$ and $k_n \to 1$ as $n \to \infty$, there exists a positive integer n_0 such that $1-2\alpha_n k_n - \alpha_n L d_n > 0$ for all $n \ge n_0$. Hence, without loss of generality, we can assume that

$$1-2\alpha_n k_n - \alpha_n L d_n > 0$$

for all $n \ge 0$. Besides, if there exists a nonnegative integer m such that $x_m = q$, then we have

$$y_m = (1-\beta_m)x_m + \beta_m T^m x_m = q.$$

By induction, we can prove that $x_{m+i} = y_{m+i} = q$ for all $i \ge 0$ and so $x_n \to q$ as $n \to \infty$. The conclusion of Theorem 2.1 is proved. Hence, without loss of generality, we can assume that $x_n \ne q$ for all $n \ge 0$. Let

$$\sigma = \inf_{n\ge 0} \frac{\phi(\|x_{n+1}-q\|)}{\|x_{n+1}-q\|^2}.$$

1. If $\sigma > 0$. Taking $\gamma \in (0, \min\{1,\sigma\})$, it follows from (2.7) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \frac{(1-\alpha_n)^2+\alpha_n L d_n}{1-2\alpha_n k_n-\alpha_n L d_n}\|x_n-q\|^2 \\
&\quad - \frac{2\alpha_n\gamma}{1-2\alpha_n k_n-\alpha_n L d_n}\|x_{n+1}-q\|^2,
\end{aligned}$$

which implies that

$$\|x_{n+1}-q\|^2 \le \frac{(1-\alpha_n)^2+\alpha_n L d_n}{1-2\alpha_n k_n-\alpha_n L d_n+2\alpha_n\gamma}\|x_n-q\|^2. \tag{2.8}$$

Since $\alpha_n \to 0$, $d_n \to 0$ and $k_n \to 1$ as $n \to \infty$, there exists a positive integer n_1 such that, for all $n \ge n_1$,

$$\begin{aligned}&(1-\alpha_n)^2+\alpha_n L d_n-(1-\frac{\gamma}{2}\alpha_n)(1-2\alpha_n k_n-\alpha_n L d_n+2\alpha_n L)\\ =&\alpha_n\big[2(k_n-1)+\alpha_n+2Ld_n-\alpha_n k_n L-\frac{\gamma}{2}\alpha_n L d_n+\alpha_n\gamma^2-\frac{3}{2}\gamma\big]\\ \le&0.\end{aligned}$$

This implies that

$$\frac{(1-\alpha_n)^2+\alpha_n L d_n}{1-2\alpha_n k_n-\alpha_n L d_n+2\alpha_n\gamma} \le (1-\frac{\gamma}{2}\alpha_n)$$

for all $n \ge n_1$. Therefore, (2.8) can be written as follows:

$$\|x_{n+1}-q\|^2 \le (1-\frac{\gamma}{2}\alpha_n)\|x_n-q\|^2$$

for all $n \ge n_1$. Taking $a_n = \|x_n-q\|^2$, $b_n = 0$, $t_n = \frac{\gamma}{2}\alpha_n$, it follows from Lemma 1.2 that $a_n \to 0$, i.e., $x_n \to q$ as $n \to \infty$.

2. If $\sigma = 0$. By the strictly increasing property of ϕ and the definition of σ, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_n\}$ such that $\|x_{n_j+1} - q\| \to 0$ as $j \to \infty$. In fact, since $\delta = 0$, there exists a subsequence $\{x_{n_j+1}\}$ of $\{x_n\}$ such that

$$\frac{\phi(\|x_{n_j+1}-q\|)}{\|x_{n_j+1}-q\|^2} \to 0$$

as $n_j \to \infty$. Letting $M = \sup\{\|y\| : y \in T(D)\}$. For any $\epsilon > 0$, there exists a positive integer n_{j_0} such that

$$\frac{\phi(\|x_{n_j+1}-q\|)}{\|x_{n_j+1}-q\|^2} \le \frac{\phi^{-1}(\epsilon)}{M}$$

for $n_j \geq n_{j_0}$, and so $\phi(\|x_{n_j+1} - q\|) \leq \phi^{-1}(\epsilon)$, i.e., $\{x_{n_j+1} - q\| \leq \epsilon$ for $n_j \geq n_{j_0}$. Since ϵ is arbitrary, we have

$$\|x_{n_j+1} - q\| \to 0 \quad \text{as } n_j \to \infty.$$

Since $\alpha_n \to 0$, $k_n \to 1$ and $d_n \to 0$ as $n \to \infty$, for any given $\epsilon \in (0,1)$, there exists a positive integer n_{j_0} such that, for all $n \geq n_{j_0}$,

$$\begin{cases} \|x_{n_{j_0}+1} - q\| < \epsilon, \\ \alpha_n < \frac{1}{4}\phi(\epsilon), \ Ld_n < \frac{1}{4}\phi(\epsilon), \\ k_n < 1 + \frac{1}{4}\phi(\epsilon). \end{cases} \tag{2.9}$$

Next, we prove that

$$\|x_{n_{j_0}+i} - q\| \leq \epsilon \tag{2.10}$$

for all $i \geq 1$. Indeed, for $i = 1$, the conclusion follows from (2.9). For $i = 2$, if

$$\|x_{n_{j_0}+2} - q\| > \epsilon, \tag{2.11}$$

then, by the strictly increasing property of ϕ, we have

$$\phi(\|x_{n_{j_0}+2} - q\|) > \phi(\epsilon).$$

Letting

$$h_n = \frac{1}{1 - 2\alpha_n k_n - \alpha_n L d_n}.$$

Then the first term on the right side of (2.7) can be written as follows:

$$\begin{aligned} &\frac{(1-\alpha_n)^2 + \alpha_n L d_n}{1 - 2\alpha_n k_n - \alpha_n L d_n}\|x_n - q\|^2 \\ &\quad = \|x_n - q\|^2 + h_n \alpha_n(\alpha_n - 2 + 2Ld_n + 2k_n)\|x_n - q\|^2. \end{aligned}$$

It follows from (2.7) and (2.9), we have

$$\begin{aligned}
&\|x_{n_{j_0}+2} - q\|^2 \\
&\quad \leq \|x_{n_{j_0}+1} - q\|^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}[(\alpha_{n_{j_0}+1} - 2 + 2Ld_{n_{j_0}+1} \\
&\qquad + 2k_{n_{j_0}+1})\|x_{n_{j_0}+1} - q\|^2 - 2\phi(\epsilon)] \\
&\quad \leq \epsilon^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\left[\frac{\phi(\epsilon)}{4} - 2 + \frac{\phi(\epsilon)}{2} + 2\Big(1 + \frac{1}{4}\phi(\epsilon)\Big)\epsilon^2 - 2\phi(\epsilon)\right] \\
&\quad \leq \epsilon^2 + h_{n_{j_0}+1}\alpha_{n_{j_0}+1}\left[\frac{5}{4}\phi(\epsilon) - 2\phi(\epsilon)\right] \\
&\quad \leq \epsilon^2,
\end{aligned}$$

which contradicts (2.11) and so we have

$$\|x_{n_{j_0}+2} - q\|^2 \leq \epsilon.$$

By induction, we can prove that (2.10) is true. By the arbitrariness of $\epsilon \in (0,1)$, we have $x_n \to q$ as $n \to \infty$. This completes the proof. □

From Theorem 2.1, we can obtain the following result:

THEOREM 2.2. *Let D be a nonempty closed convex subset of E and $T : D \to D$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$ satisfying the conditions* (i) *and* (ii) *in* Theorem 2.1. *Let $\{x_n\}$ be the modified Ishikawa iterative sequence defined by* (1.1). *If $F(T) \neq \emptyset$ and, for any given $q \in F(T)$, there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ and $\phi(0) = 0$ such that the condition* (2.1) *in* Theorem 2.1 *is satisfied. Then the sequence $\{x_n\}$ converges strongly to the fixed point q of T.*

Proof. Since T is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ in $[1,\infty)$ and $\lim_{n\to\infty} k_n = 1$, T is a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with the sequence $\{k_n\}$ in $[1,\infty)$, $\lim_{n\to\infty} k_n = 1$ and the constant $L = \sup_{n\geq 1} k_n$. Therefore, the conclusion can be obtained from Theorem 2.1 immediately. □

THEOREM 2.3. *Let D be a nonempty closed convex subset of E and $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudo-contractive mapping with a sequence $\{k_n\}$ in $[1, \infty)$ such that*

$$\lim_{n \to \infty} k_n = 1.$$

Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:

(i) $\alpha_n \to 0$ *as* $n \to \infty$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Let $\{x_n\}$ be the modified Mann iterative sequence defined by (1.2). *If $F(T) \neq \emptyset$ and, for any given $q \in F(T)$, there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ and $\phi(0) = 0$ such that the condition* (2.1) *in* Theorem 2.1 *is satisfied. Then the sequence $\{x_n\}$ converges strongly to the fixed point q of T.*

Proof. Taking $\beta_n = 0$ for all $n = 0, 1, 2, \cdots$ in Theorem 2.1, the conclusion can be obtained immediately. □

REMARK 2.1. Theorems 2.1, 2.2 and 2.3 extend and improve the main results in Goebel-Kirk [4], Liu [5] and Schu [7]. Moreover, the methods of proof given in this paper are more simple and quite different from the proof methods given in [4]-[6].

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