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LOWER BOUNDS ON THE HOLOMORPHIC SECTIONAL CURVATURE OF THE BERGMAN METRIC ON LOCALLY CONVEX DOMAINS IN $\mathbb{C}^n$

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ABSTRACT. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r and let $z_0 \in b\Omega$ be a point of finite type. We also assume that Ω is convex in a neighborhood of z_0. Then we prove that all the holomorphic sectional curvatures of the Bergman metric of Ω are bounded below by a negative constant near z_0.

1. Introduction

Many questions about the complex function theory in $\mathbb{C}^n$ can be explored by examining an appropriate hermitian metric and its curvature tensor on the domain in question. Among these metrics the Bergman metric is one of the most important metrics, and the information about the lower bounds of the holomorphic sectional curvatures of the metric has been used to characterize domains of holomorphy [9]. However, the abstract nature of the metric makes it difficult to obtain informations about the curvature tensor except in some special cases.

It has been well known that for any bounded domain in $\mathbb{C}^n$, the holomorphic sectional curvature of the Bergman metric is less than or equal to 2. Using Fefferman's asymptotic expansion of the Bergman kernel, Klembeck [5] showed that for smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$, the holomorphic sectional curvatures of the Bergman metric approach $-\frac{4}{n+1}$, that of the ball, near the boundary. For weakly

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pseudoconvex domains, however, much less is known except some special cases [3,6]. In [6], J. McNeal showed that the holomorphic sectional curvatures of the Bergman metric for smoothly bounded pseudoconvex domains of finite type in C^2 are bounded below by a negative constant. He used the boundary behavior of the Bergman kernel function and its derivative estimates.

In this paper, we estimate a lower bound for the sectional curvature of the Bergman metric for some smoothly bounded pseudoconvex domains in $\mathbb{C}^n$ near a point of finite 1-type in the sense of D'Angelo [4]. The result is:

THEOREM 1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary. If $z_0 \in b\Omega$ is a point of finite 1-type in the sense of D'Angelo and Ω is convex in a neighborhood of z_0, then there is a neighborhood U of z_0 such that all the holomorphic sectional curvatures of the Bergman metric of Ω are bounded below by a negative constant in U.*

2. Convex Domains of Finite Type

In this section we investigate the local geometry of the locally convex domains of finite type and we get the estimates of the Bergman kernel function and its derivatives.

Roughly speaking, the Bergman metric is obtained by taking the second derivatives of the Bergman kernel function, and the sectional curvature of the Bergman metric is obtained by taking second derivatives of the Bergman metric. Therefore the lower bounds of the curvature tensor can be obtained from the derivative estimates of the Bergman kernel function. In the sequel, we assume that Ω is a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth defining function r.

Let $A(\Omega)$ be the set of holomorphic functions on Ω. The Bergman kernel $K_\Omega(z,\bar{z})$ for Ω is defined by

$$K_\Omega(z,\bar{z}) = \sup\{|f(z)|^2 : f \in A(\Omega),\ \|f\|_{L^2(\Omega)} \leq 1\},$$

and the Bergman metric is an (1,1) form defined by

$$\partial\bar{\partial}\log K(z,\bar{z}) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,\bar{z}) dz_i \wedge d\bar{z}_j. \tag{1}$$

Set, for $1 \le i,j \le n$ and $z \in \Omega$, $g_{ij}(z) = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,\bar{z})$. Then, if $Y = b_1\partial/\partial z_1 + \ldots + b_n\partial/\partial z_n$, it is well known [1] that the Bergman length of Y is given by

$$(2) \qquad B_\Omega(z;Y) = \left(\sum_{i,j=1}^{n} g_{ij}(z) b_i \bar{b}_j \right)^{\frac{1}{2}}.$$

Let $z_0 \in b\Omega$ be a point of finite 1-type in the sense of D'Angelo [4] and assume that $b\Omega$ is convex in a neighborhood U of z_0. Then a careful analysis of the local geometry of $b\Omega$ near z_0 [2,7,8] shows that, for each $p \in \Omega$ sufficiently close to z_0 and for each $\epsilon > 0$, one can construct special coordinates $z = (z_1, \ldots, z_n)$ centered at p, and positive numbers $\tau_1(p,\epsilon), \ldots, \tau_n(p,\epsilon)$ which are closely related with the type of z_0. That is, we have:

PROPOSITION 2.1. *After perhaps shrinking U, for every $q \in \Omega \cap U$ and every $\epsilon > 0$ sufficiently close to 0, there exist coordinates $(z_1, \cdots, z_n)$ centered at q, positive numbers $\tau_1(q,\epsilon), \cdots, \tau_n(q,\epsilon)$, and points $p_1, \cdots, p_n \in \{z : r(z) = \epsilon + r(q)\}$ such that, in the coordinates $(z_1, \cdots, z_n)$, the defining function r satisfies*

(i) *for $1 \le i \le n$,*

$$\frac{\tau_1(q,\epsilon)}{\tau_i(q,\epsilon)} \lesssim \left| \frac{\partial r}{\partial z_i}(p_i) \right| \lesssim \frac{\tau_1(q,\epsilon)}{\tau_i(q,\epsilon)},$$

(ii) *if $i < j$,*

$$\left| \frac{\partial r}{\partial z_i}(p_j) \right| \lesssim \frac{\tau_1(q,\epsilon)}{\tau_i(q,\epsilon)},$$

(iii) *if $i > j$,*

$$\left| \frac{\partial r}{\partial z_i}(p_j) \right| = 0.$$

Also if we define the polydisc

$$P_\epsilon(q) = \{z \in U : |z_1| < \tau_1(q,\epsilon), \cdots, |z_n| < \tau_n(q,\epsilon)\},$$

then there exists a constant $c > 0$ such that $cP_\epsilon(q) \subset \{z \in U : r(z) < \epsilon + r(q)\}$, where c is independent of ϵ and $q \in U \cap \Omega$.

Suppose that $q^1, q^2 \in U \cap \Omega$. Define

$$M(q^1, q^2) = \inf\{\epsilon > 0 \ : \ q^2 \in P_\epsilon(q^1)\}, \tag{3}$$

where $P_\epsilon(q^1)$ is constructed from the coordinates about q^1 as in Proposition 2.1.

Using the quantities $\tau_i(q, \epsilon)$, $i = 1, \dots, n$, we can estimate the Bergman kernel function and its derivatives [7,8].

THEOREM 2.2. *Suppose $\Omega \subset\subset \mathbb{C}^n$ is smoothly bounded and pseudoconvex. Let $z_0 \in b\Omega$ be a point of finite type and assume there is some neighborhood U of z_0 so that Ω is convex in U. There exists a neighborhood $V \subset\subset U$ so that if $q \in V \cap \Omega$,*

$$K_\Omega(q, q) \approx \prod_{i=1}^{n} \tau_i(q, \delta)^{-2}, \tag{4}$$

where $\delta = |r(q)|$.

THEOREM 2.3. *Let Ω, U and z_0 be as in* Theorem 2.2. *There exists a neighborhood $V \subset\subset U$ so that, for all multi-indices μ, ν, there exists a constant $C_{\mu\nu}$ such that for all $q^1, q^2 \in U \cap \Omega$*

$$|D^\mu \bar{D}^\nu K_\Omega(q^1, q^2)| \le C_{\mu\nu} \prod_{i=1}^{n} \tau_i(q^1, \delta)^{-2-\mu_i-\nu_i}, \tag{5}$$

where $\delta = (|r(q^1)| + |r(q^2)| + M(q^1, q^2))$, and where $M(q^1, q^2)$ is defined as in (3).

In terms of the quantities $\tau_1(q, \delta), \dots, \tau_n(q, \delta)$, for $\delta = |r(q)|$, the Bergman length of a vector field $Y = b_1\partial/\partial z_1 + \dots + b_n\partial/\partial z_n$ at $q \in U \cap \Omega$ is given by [2],

$$B_\Omega(q; Y) \approx \sum_{i=1}^{n} |b_i| \tau_i(q, |r(s)|)^{-1}. \tag{6}$$

3. Estimates of Sectional Curvature

The Riemann curvature tensor $(R(X,Y)\cdot Z, W)$ is used to define the sectional curvature, which plays an important role in the geometry of Riemann manifolds. At any $p \in M$ we denote by π a plane section, that is, a two-dimensional subspace of $T_p(M)$. Such a section is determined by any pair of mutually orthogonal unit vectors X, Y at p.

DEFINITION 3.1. The sectional curvature $K(\pi)$ of the section π with orthonormal basis X, Y is defined as

$$K(\pi) = -R(X,Y,X,Y) = -(R(X,Y)\cdot X, Y).$$

We now ready to prove Theorem 1. For the time being, $q \in V \cap \Omega$ will be fixed and we will denote K_Ω by K and derivatives of K_Ω with subscripts, e.g., $\partial^2/\partial z_i \partial \bar{z}_j K_\Omega = K_{i\bar{j}}$. Set $\tau_i(q, |r(q)|) = \tau_i$, $i = 1, \dots, n$. If we compare the coefficients in (2) and (6), we obtain from (4) and (5) that

$$|g_{ii}(q)| \approx \tau_i^{-2}, \quad |g_{ij}(q)| \lesssim \tau_i^{-1}\tau_j^{-1}. \tag{7}$$

Set $G = (g_{ij})_{1\le i,j\le n}$ and let $P = (P_{jk})$ be a unitary matrix such that $P^*GP = D$ where D is a diagonal matrix whose entries are positive eigenvalues of G.

For $x \in \mathbb{C}^n$ with $|x| = 1$, set $b = Px$. Then by (2) and (6) one obtains that

$$\begin{aligned} b^*Gb = x^*Dx &= \lambda_1|x_1|^2 + \cdots + \lambda_n|x_n|^2 \\ &\approx |b_1|^2\tau_1^{-2} + \cdots + |b_n|^2\tau_n^{-2}. \end{aligned} \tag{8}$$

Set $x = x^k = (0, \cdots, 0, 1, 0, 0, \cdots, 0)$, where 1 is in the k-th place. Then (8) implies that

$$\lambda_k \approx \sum_j |P_{jk}|^2\tau_j^{-2},$$

and hence

$$\det G = \lambda_1 \cdots \lambda_n \approx \prod_{k=1}^{n} \Big(\sum_j |P_{jk}|^2 \tau_j^{-2} \Big). \tag{9}$$

LEMMA 3.2. det $G \approx \prod_{k=1}^{n} \tau_k^{-2}$.

Proof. Let us fix $q \in V \cap \Omega$ for a moment. From (9) one obtains that det $G \lesssim \Pi_{k=1}^{n} \tau_k^{-2}$. Let's prove the reverse inequality. Set $a_1 = \max\{|P_{1j}|^2; j = 1, 2, \cdots, n\}$. Note that P is a unitary matrix. Therefore $a_1 \geq \frac{1}{n}$. Without loss of generality we may assume that $a_1 = |P_{11}|^2$ and

$$|P_{n2}|^2 \leq |P_{n3}|^2 \leq \cdots \leq |P_{nn}|^2.$$

Since $|P_{11}|^2 \geq \frac{1}{n}$, we have $|P_{n1}|^2 \leq \frac{n-1}{n}$ and hence $|P_{nn}|^2 \geq \frac{1}{n(n-1)}$. Since $|P_{n2}|^2 \leq \frac{1}{n-1}$, we have $\sum_{k=1}^{n-1} |P_{k2}|^2 \geq 1 - \frac{1}{n-1}$. Assuming that $n \geq 3$, we have

$$|P_{k_2 2}|^2 \geq \frac{1}{(n-1)^2},$$

for some $k_2, 1 \leq k_2 \leq n-1$. We may assume that $k_2 = 2$. Therefore we have

$$\prod_{k=1}^{n}\Big(\sum_j |P_{jk}|^2 \tau_j^{-2}\Big) \geq \Big(\frac{1}{n}\tau_1^{-2}\Big)\Big(\frac{1}{(n-1)^2}\tau_2^{-2}\Big)\prod_{k=3}^{n}\Big(\sum_j |P_{jk}|^2 \tau_j^{-2}\Big).$$

Continuing, we have $|P_{nn-1}|^2 \leq \frac{1}{2}$ and hence

$$|P_{k_{n-1}n-1}|^2 \leq \frac{1}{2(n-1)},$$

for some k_{n-1}, $1 \leq k_{n-1} \leq n-1$. So

$$\det\, G \geq c_n' \tau_1^{-2}\tau_2^{-2}\cdots\tau_{n-1}^{-2}\Big(\sum_j |P_{jn}|^2 \tau_j^{-2}\Big) \geq c_n \prod_{k=1}^{n} \tau_k^{-2},$$

because $|P_{nn}|^2 \geq \frac{1}{n(n-1)}$. Here c_n is independent of δ and q. □

Set $G^{-1} = (g^{pq})$. Then Lemma 3.2 gives us that

$$|g^{pq}| \lesssim \tau_p \tau_q. \tag{10}$$

Then the components of the Riemann curvature tensor, R^{Ω}, for the Bergman metric are locally defined by

$$\begin{aligned}
R^{\Omega}_{i\bar{j}k\bar{l}} &= -\frac{\partial^4}{\partial z_i \partial \bar{z}_j \partial z_k \partial \bar{z}_l} \log K \\
&\quad + \sum_{p,q=1}^{n} g^{pq} \frac{\partial^3}{\partial z_i \partial z_k \partial \bar{z}_p} \log K \frac{\partial^3}{\partial \bar{z}_j \partial \bar{z}_l \partial z_q} \log K \\
&= g_{ij} g_{kl} + g_{il} g_{jk} - \frac{1}{K^2}(K K_{i\bar{j}k\bar{l}} - K_{ik} K_{\bar{j}\bar{l}}) \\
&\quad + \frac{1}{K^4} \sum_{p,q=1}^{n} g^{pq} (K K_{ik\bar{p}} - K_{ik} K_{\bar{p}})(K K_{\bar{j}\bar{l}q} - K_{\bar{j}\bar{l}} K_q).
\end{aligned}$$

Set $\triangle = (\tau_1, \cdots, \tau_n)$, and $\triangle^{\alpha} = \tau_1^{\alpha_1} \cdots \tau_n^{\alpha_n}$, for $\alpha = (\alpha_1, \cdots, \alpha_n)$. We now estimate all the terms on the right hand side of the above equation. By (4) and (5), We have

$$\left| \frac{1}{K^2}(K K_{ijk\bar{l}}) \right| \lesssim \prod_{d=1}^{n} \tau_d^2 \prod_{d=1}^{n} \tau_d^{-2} \triangle^{-\alpha} = \triangle^{-\alpha}, \tag{11}$$

where α_i denotes the number of i and $\bar{i}, 1 \leq i \leq n$, appearing in the subscripts of K. So $\triangle^{-\alpha} = \tau_i^{-1} \tau_j^{-1} \tau_k^{-1} \tau_l^{-1}$ Similarly, we have

$$\left| \frac{1}{K^2} K_{ik} K_{\bar{j}\bar{l}} \right| \lesssim \triangle^{-\alpha}.$$

For the terms in the summation, let T denote any of the terms in the sum. Combining (4), (5) and (10), and by the method similar as above, we get

$$|T(q)| \leq C' \triangle^{-\alpha}.$$

For the estimates of $g_{ij} g_{kl}'$s, we use the estimates (7) and we get

$$|R^{\Omega}_{i\bar{j}k\bar{l}}| \leq C \triangle^{-\alpha}. \tag{12}$$

Let $Y = \sum_{k=1}^{n} b_i \frac{\partial}{\partial \zeta_i}$ be a holomorphic tangent vector with unit length,

$$\sum g_{ij} b_i \bar{b}_j = 1 \tag{13}$$

Then the holomorphic sectional curvature determined by Y is defined by

$$S(Y) = \sum R^{\Omega}_{i\bar{j}k\bar{l}} b_i \bar{b}_j b_k \bar{b}_l.$$

Note that (13) and the growth conditions (6) and (7) give us

$$|b_j(q)| \approx \tau_j, \tag{14}$$

unless $|b_j(q)| = 0$. Hence $|b_i \bar{b}_j b_k \bar{b}_l| \lesssim \triangle^{\alpha}$. In either case (12) and (14) give us

$$|S(Y)(q)| \leq C,$$

where the constant C is again independent of $q \in V \cap \Omega$ and $\delta = |r(q)|$. This completes the proof of Theorem 1.

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