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REPRESENTATION OF OPERATOR SEMI-STABLE DISTRIBUTIONS

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ABSTRACT. For a linear operator Q from R^d into R^d, $\alpha > 0$ and $0 < b < 1$, the (Q, b, α)-semi-stability and the operator semi-stability of probability measures on R^d are defined. Characterization of (Q, b, α)-semi-stable Gaussian distribution is obtained and the relationship between the class of (Q, b, α)-semi-stable non-Gaussian distributions and that of operator semi-stable distributions is discussed.

1. Introduction

Let R^d be the d-dimensional Euclidean space. In the paper [2], we studied operator semi-stable processes on R^d, which are Lévy processes associated with operator semi-stable distributions. Under the condition of fullness, descriptions of operator semi-stable distributions on R^d were obtained by R. Jajte [4,5], W. Krakowiak [6], A. Łuczak [7,8], V. Chorny [3] and others. Here fullness means that the support of the distribution is not contained in any $(d-1)$-dimensional hyperplane in R^d.

Let $Aut(R^d)$ be the set of invertible linear operators from R^d onto R^d. Let $\{Y_n : n = 1, 2, \cdots\}$ be a sequence of i.i.d. (=independent identically distributed) random variables on R^d. In [4], R. Jajte investigated the weak limit of distributions of

$$A_n(Y_1 + Y_2 + \cdots + Y_{k_n}) + b_n, \tag{1.1}$$

where $A_n \in Aut(R^d)$, $b_n \in R^d$ and $\frac{k_{n+1}}{k_n} \to r$ with some $r \in [1, \infty)$. The limit distribution μ of (1.1) is called an *operator semi-stable* distribution. When the convergence of (1.1) holds with $b_n = 0$, we call μ

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a *strictly operator semi-stable* distribution. In this paper, we consider all operator semi-stable distributions on R^d without the assumption of fullness. Let $End(R^d)$ be the set of linear operators from R^d into R^d. The identity operator is denoted by I. For $B \in End(R^d)$ and $r > 0$, we define $r^B = \exp\{B \log r\} = \sum (B \log r)^n/n!$. For $T \in End(R^d)$, we write $(T\mu)(E) = \mu(T^{-1}(E))$. We denote the b-th convolution power of μ by μ^b. Let $M_+(R^d)$ be the class of linear operators on R^d all of whose eigenvalues have positive real parts.

Fix $\alpha > 0$ and $Q \in M_+(R^d)$. An infinitely divisible distribution μ on R^d is called operator semi-stable with exponent (Q, α) if there are a number $b \in (0,1)$ and a vector $c(b) \in R^d$ such that

$$\mu^{b^\alpha} = b^Q \mu * \delta_{c(b)}. \tag{1.2}$$

Here $\delta_{c(b)}$ is the delta distribution at $c(b)$. When (1.2) is satisfied, we call μ *(Q, b, α)-semi-stable.* It is called strictly operator semi-stable with exponent (Q, α) if there is $b \in (0,1)$ such that

$$\mu^{b^\alpha} = b^Q \mu. \tag{1.3}$$

When (1.3) is satisfied, we call μ *strictly (Q, b, α)-semi-stable.* The above definition of (Q, b, α)-semi-stable distribution is described without the assumption that μ is full. If μ is a (Q, b, α)-semi-stable distribution on R^d, then μ is an operator semi-stable distribution on R^d. But the converse is not true. The counterexamples are given at the end of this paper. The (Q, b, α) of a distribution satisfying (1.2) is not uniquely determined by μ. If μ is semi-stable with exponent α in the sense of [1], then μ is an operator semi-stable distribution with exponent (I, α). We note that μ is (Q, b, α)-semi-stable if and only if μ is $(\alpha^{-1}Q, b^\alpha, 1)$-semi-stable. The distribution satisfying (1.2) for every $b \in (0, \infty)$ is operator stable, which was introduced by M.Sharpe [13]. It is (Q, α)-stable in the sense of [12]. By introducing the terminology (Q, b, α), the relations between operator semi-stable distributions and semi-stable distributions become clearer. The characterization of full operator semi-stable distributions on R^d is investigated by many authors. But they did not treat the whole structure of Gaussian operator semi-stable distributions.

The main purpose of this paper is to obtain necessary and sufficient conditions for (Q, b, α)-semi-stable Gaussian distributions. The descriptions for full operator stable distribution were developed by many authors, but the complete characterization of Gaussian operator stable distributions is done by K. Sato [10] and K. Sato-M. Yamazato [12]. Our description of (Q, b, α)-semi-stable Gaussian distributions in this paper is an extension of the results in (Q, α)-stable case in [10, 12] to (Q, b, α)-semi-stable case.

In Section 2, we write some results and lemmas we use in the subsequent sections. In Section 3, we characterize (Q, b, α)-semi-stable Gaussian distributions, and in Section 4, we rewrite a necessary and sufficient condition for (Q, b, α)-semi-stable purely non-Gaussian distributions on R^d. Its necessity part is similar to that of [3]. Relations between (Q, b, α)-semi-stable distributions and operator semi-stable distributions are given in Section 5.

2. Preliminaries

For $x, y \in R^d$, we denote the Euclidean inner product of x and y by $\langle x, y\rangle$ and the Euclidean norm of x by $|x|$. Lévy shows that a distribution μ on R^d with characteristic function $\widehat{\mu}(z)$ is infinitely divisible if and only if $\widehat{\mu}(z)$ has form

$$\widehat{\mu}(z) = \exp\left\{ i\langle \gamma, z\rangle - \frac{1}{2}\langle Az, z\rangle + \int_{R^d} G(z, x)\nu(dx) \right\},$$

where $G(z, x) = e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle(1+|x|^2)^{-1}$, γ is a vector in R^d, A is a symmetric nonnegative definite operator and ν is a measure (called Lévy measure) on R^d satisfying $\nu(\{0\}) = 0$ and $\int |x|^2(1 + |x|^2)^{-1}\nu(dx) < \infty$. This representation is unique and called the Lévy representation (γ, A, ν). We call μ a purely non-Gaussian in the case of $A = 0$. If $\gamma = 0$ and $A = 0$, then we call μ a centered purely non-Gaussian. If $\gamma = 0$ and $\nu = 0$, then μ is called a centered Gaussian. We denote the adjoint of a linear operator T by T'.

PROPOSITION 2.1. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let μ be (Q, b, α)-semi-stable on R^d. If $T \in Aut(R^d)$, then $T\mu$ is (TQT^{-1}, b, α)-semi-stable on R^d.*

Proof. From the fact that $T^{-1}b^QT = b^{T^{-1}QT}$, we see that

$$\widehat{T\mu}(z)^{b^\alpha} = \widehat{\mu}(b^{Q'}T'z)e^{i\langle c(b),T'z\rangle} = \widehat{\mu}(T'b^{(TQT^{-1})'}z)e^{i\langle c(b),T'z\rangle}$$
$$= \widehat{T\mu}(b^{(TQT^{-1})'}z)e^{i\langle Tc(b),z\rangle}. \qquad \square$$

We fix $Q \in M_+(R^d)$. Let μ be an operator semi-stable distribution with exponent (Q, α). For a real symmetric nonnegative definite operator A, $\phi_A(z)$ stands for $\langle Az, z\rangle$ for $z \in C^d$. Here $\langle \quad \rangle$ denotes the Hermitian inner product on C^d. We write $(b^{nQ}\nu)(E) = \nu(b^{-nQ}E)$.

LEMMA 2.2. *Fix $b \in (0,1)$, $Q \in M_+(R^d)$ and $\alpha > 0$. Let μ be infinitely divisible on R^d with the Lévy representation (γ, A, ν). Then a necessary and sufficient condition for μ to be (Q,b,α)-semi-stable is that, for any integer n,*

$$\phi_A(b^{nQ'}z) = b^{n\alpha}\phi_A(z) \quad \text{for} \quad z \in C^d \tag{2.1}$$

and

$$(b^{nQ}\nu)(E) = b^{n\alpha}\nu(E) \quad \text{for} \quad E \in \mathcal{B}(R^d). \tag{2.2}$$

Proof. If μ is (Q,b,α)-semi-stable, then, iterating (1.2), we get, for any positive integer m,

$$\mu^{b^{m\alpha}} = b^{mQ}\mu * \delta(c(b^m)),$$

where $c(b^m) = b^\alpha c(b^{(m-1)}) + b^{(m-1)Q}c(b)$. Hence a necessary and sufficient condition for μ to be (Q,b,α)-semi-stable is that, for any positive integer m,

$$\phi_A(b^{mQ'}z) = b^{m\alpha}\phi_A(z) \quad \text{and} \quad (b^{mQ}\nu)(E) = b^{m\alpha}\nu(E).$$

From the facts that $\phi_A(z) = \phi_A((bb^{-1})^{Q'}z) = b^\alpha\phi_A(b^{-Q'}z)$ and $\nu(E) = (bb^{-1})^Q\nu(E) = b^\alpha b^{-Q}\nu(E)$, we see that

$$\phi_A(b^{-Q'}z) = b^{-\alpha}\phi_A(z) \quad \text{and} \quad (b^{-Q}\nu)(E) = b^{-\alpha}\nu(E),$$

which implies that, for any positive integer m,

$$\phi_A(b^{-mQ'}z) = b^{-m\alpha}\phi_A(z) \quad \text{and} \quad (b^{-mQ}\nu)(E) = b^{-m\alpha}\nu(E). \qquad \square$$

The following lemmas are known. Proofs are omitted.

LEMMA 2.3. (Lemma 6.3 in [10] and Lemma 3.1 in [12]). *Let* $z_0 \in C^d$. *If* A *is real symmetric nonnegative definite and* $\phi_A(z_0) = 0$, *then* $Az_0 = 0$.

LEMMA 2.4 (Lemma 5.7 in [10]). *If* $Q \in M_+(R^d)$, *then every* x *in* $R^d - \{0\}$ *is uniquely expressed as* $x = u^Q\xi$ *with* $\xi \in S = \{\xi \in R^d : |\xi| = 1, |u^Q\xi| > 1 \quad \text{for all} \quad u > 1\}$ *and* $u > 0$.

3. Gaussian operator semi-stable distributions

In the following Theorem 3.1, we obtain the characterization of (Q, b, α)-semi-stable Gaussian distributions on R^d. An example which shows that the class of Gaussian operator semi-stable distributions is strictly bigger than that of Gaussian operator stable distributions is given in a recent paper [9]. For $Q \in M_+(R^d)$, we write $B = b^Q$. For $x \in C^d$, $\overline{x}$ stands for the complex conjugate of x, that is, each component of $\overline{x}$ is the complex conjugate of the corresponding component of x. Let $\vartheta_1, \cdots, \vartheta_p$, be all distinct eigenvalues of b^Q. Let $f(\xi)$ be the minimal polynomial of b^Q with $f(\xi) = f_1(\xi)^{m_1} \cdots f_p(\xi)^{m_p}$, where $f_j(\xi) = \xi - \vartheta_j$ for $1 \le j \le p$. We denote the kernel of $(Q - \vartheta_j)^{m_j}$ in C^d by E_j, that is, E_j is the eigenspace of b^Q in the wide sense associated with the eigenvalue ϑ_j for $1 \le j \le p$. Denote by P_j the projector onto E_j in the decomposition

$$C^d = E_1 \oplus \cdots \oplus E_p. \tag{3.1}$$

Let

$$E'_j = \text{Kernel}(B' - \overline{\vartheta_j} I)^{m_j} \quad \text{in} \quad C^d \quad \text{for} \quad 1 \le j \le p.$$

Then we have

$$C^d = E'_1 \oplus \cdots \oplus E'_p. \tag{3.2}$$

We see that E'_j and E_k are orthogonal for $j \neq k$ and P'_j is the projector of C^d onto E'_j in the decomposition (3.2). Let $J = \{j : 1 \le j \le p, |\vartheta_j|^2 = b^\alpha\}$.

THEOREM 3.1. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let μ be infinitely divisible on R^d with the Lévy representation $(\gamma, A, 0)$. Then a necessary and sufficient condition for μ to be (Q, b, α)-semi-stable is that*

$$AP_j' = 0 \quad \text{for all} \quad j \notin J, \tag{3.3}$$

$$(B - \vartheta_j)AP_j' = 0 \quad \text{for all} \quad j \in J. \tag{3.4}$$

We will use the following lemma in the proof of Theorem 3.1. The proof is given in [10].

LEMMA 3.2 (Lemma 6.4 in [10] and Remark 3.1 in [12]). *Let A be real symmetric nonnegative definite. Then*

$$(B - \vartheta_j)AP_j' = 0 \quad \text{for} \quad 1 \le j \le p$$

if and only if

$$P_k AP_j' = 0 \quad \text{for} \quad j \neq k, \tag{3.5}$$

$$A(B' - \overline{\vartheta_j})P_j' = 0 \quad \text{for} \quad 1 \le j \le p. \tag{3.6}$$

Proof of Theorem 3.1. Suppose that μ is a (Q, b, α)-semi-stable distribution with Lévy representation $(\gamma, A, 0)$. Then we assert that, for any positive integer m and $z_0 \in C^d$,

$$(B' - \overline{\vartheta_j})^m z_0 = 0 \quad \text{implies} \quad A(B' - \overline{\vartheta_j})z_0 = 0. \tag{3.7}$$

For the proof of (3.7), we use induction in m. For $m = 1$, (3.7) is trivial. Suppose that (3.7) is true for $m - 1$ in place of m, and assume $(B' - \overline{\vartheta_j})^m z_0 = 0$. Let us write $\overline{\vartheta_j}^{-k}(B' - \overline{\vartheta_j})^k z_0 = z_k$ for each nonnegative integer k. Since $(B' - \overline{\vartheta_j})^m z_k = (B' - \overline{\vartheta_j})^{m-1}(B' - \overline{\vartheta_j})z_k = 0$ for $k \ge 0$, we have $A(B' - \overline{\vartheta_j})^2 z_k = 0$ for $k \ge 0$ by the induction hypothesis. Hence we see that, for $n = 1, 2, \cdots$,

$$AB'^n z_0 = \overline{\vartheta_j}^n A[z_0 + nz_1]$$

and

$$\phi_A(B'^n z_0) = |\vartheta_j|^{2n}[\phi_A(z_0) + 2nRe\langle Az_0, z_1\rangle + n^2\phi_A(z_1)].$$

We write

$$\Delta(n) = 2nRe\langle Az_0, z_1\rangle + n^2\phi_A(z_1).$$

Noticing that by Lemma 2.2

$$\phi_A(B'^n z_0) = b^{\alpha n}\phi_A(z_0),$$

we see that $b^{\alpha n}\phi_A(z_0) = |\vartheta_j|^{2n}[\phi_A(z_0)+\Delta(n)]$. We consider three cases: $b^\alpha = |\vartheta_j|^2$, $b^\alpha < |\vartheta_j|^2$ and $b^\alpha > |\vartheta_j|^2$.

(1) $b^\alpha = |\vartheta_j|^2$. In this case, we have that $\Delta(n) = 0$. If $\phi_A(z_1) \neq 0$, then we have that $\Delta(n) \to \infty$ as $n \to \infty$, which is a contradiction. Thus, $\phi_A(z_1) = 0$, from which follows $Az_1 = 0$ by Lemma 2.3.

(2) $b^\alpha < |\vartheta_j|^2$. In this case, we have that

$$\left(\frac{b^\alpha}{|\vartheta_j|^2}\right)^n \phi_A(z_0) = \phi_A(z_0) + \Delta(n).$$

Letting $n \to \infty$, we get that $(\frac{b^\alpha}{|\vartheta_j|^2})^n \to 0$. This leads to the fact that $\Delta(n) \to -\phi_A(z_0)$ as $n \to \infty$. But we have $\Delta(n) \to \infty$ as $n \to \infty$ if $\phi_A(z_1) \neq 0$. Hence, we see that $Az_1 = 0$.

(3) $b^\alpha > |\vartheta_j|^2$. In this case, we have that

$$\left(\frac{|\vartheta_j|^2}{b^\alpha}\right)^n [\phi_A(z_0) + \Delta(n)] = \phi_A(z_0).$$

Since $(\frac{|\vartheta_j|^2}{b^\alpha})^n n^2 \to 0$ as $n \to \infty$, we see that $\phi_A(z_0) = 0$. Hence, we have that $\Delta(n) = 0$. Thus, by the same method as (1), we see that $Az_1 = 0$.

Now we have proved that (3.7) is true. Let $z \in E'_j$. From (3.7) we see that

$$\begin{aligned}\phi_A(B'z) &= \langle AB'z, B'z\rangle = \langle A\overline{\vartheta_j}z, B'z\rangle = \overline{\vartheta_j}\langle Az, B'z\rangle \\ &= \overline{\vartheta_j}\langle z, A\overline{\vartheta_j}z\rangle = |\vartheta_j|^2\phi_A(z).\end{aligned}$$

If $j \notin J$, then, by (2.1), $\phi_A(z) = 0$, which is (3.3).

Suppose that $z \in E'_j$, $w \in E'_k$ and $j \neq k$. If $j \notin J$ or $k \notin J$, then $\langle Az, w\rangle = 0$ by (3,3). Let us show that $\langle Az, w\rangle = 0$ when $j \in J$ and $k \in J$. Using (2.1) and (3.7), we get

$$\begin{aligned}\phi_A(B'^n(z+w)) &= b^{\alpha n}\phi_A(z+w)\\ &= b^{\alpha n}\phi_A(z) + b^{\alpha n}\phi_A(w) + 2b^{\alpha n}Re\langle Az, w\rangle\end{aligned}$$

and

$$\phi_A(B'^n(z+w)) = b^{\alpha n}\phi_A(z) + b^{\alpha n}\phi_A(w) + 2Re\overline{\vartheta_j}^n{\vartheta_k}^n\langle Az, w\rangle.$$

Hence, we see that $Re\overline{\vartheta_j}^n{\vartheta_k}^n\langle Az, w\rangle = b^{\alpha n}Re\langle Az, w\rangle$ for $n = 1, 2, \cdots$. Thus, we get $Re\langle Az, w\rangle = 0$. We also get $Im\langle Az, w\rangle = Re\langle Az, w\rangle = 0$. Hence $\langle Az, w\rangle = 0$. Now we have (3.4). In fact, if $z \in E'_j$, $j \in J$, and $w \in C^d$, then

$$\begin{aligned}\langle (B - \vartheta_j)Az, w\rangle &= \langle Az, (B' - \overline{\vartheta_j})w\rangle = \langle Az, (B' - \overline{\vartheta_j})P'_j w\rangle\\ &= \langle z, A(B' - \overline{\vartheta_j})P'_j w\rangle = 0\end{aligned}$$

by (3.7).

Conversely, suppose that A satisfies (3.3) and (3.4). From Lemma 3.2, we see that (3.5) and (3.6) hold. Thus by (3.3) and (3.5), we see that

$$\phi_A(B'z) = \phi_A\left(\sum_{j=1}^{p} P'_j B'z\right) = \sum_{j=1}^{p}\phi_A(P'_j B'z) = \sum_{j\in J}\phi_A(P'_j B'z).$$

By (3.6) and by $B'P'_j = P'_j B'P'_j$, we have that

$$\phi_A(P'_j B'z) = \phi_A(B'P'_j z) = |\vartheta_j|^2\phi_A(P'_j z) = b^\alpha\phi_A(P'_j z)$$

for $j \in J$. Hence $\phi_A(B'z) = b^\alpha\phi_A(z)$. The proof is complete. □

4. Purely non-Gaussian operator semi-stable distributions

We begin with some notation which follows [10, 11, 12]. We fix an arbitrary $Q \in M_+(R^d)$. Let $\sigma_j = \alpha_j + i\beta_j$, $1 \le j \le q+2r$, be all distinct eigenvalues of Q, where α_j and β_j are real numbers such that $\beta_j = 0$ for $1 \le j \le q$, $\beta_j \ne 0$ for $q+1 \le j \le q+2r$, and $\alpha_j + i\beta_j = \alpha_{j+r} - i\beta_{j+r}$ for $q+1 \le j \le q+r$. Here q and r are allowed to be zero. We note that $p \le q+2r$ and the set $\{\vartheta_1, \cdots, \vartheta_p\}$ coincides with the set $\{b^{\sigma_1}, \cdots, b^{\sigma_{q+2r}}\}$, where $b^{\sigma_j} = b^{\sigma_k}$ if $\beta_j = \beta_k + 2n\pi$ with some integer n. Let $g(\xi)$ be the minimal polynomial of Q with $g(\xi) = g_1(\xi)^{n_1} \cdots g_{q+r}(\xi)^{n_{q+r}}$, where $g_j(\xi) = \xi - \alpha_j$ for $1 \le j \le q$, $g_j(\xi) = (\xi - \alpha_j)^2 + \beta_j{}^2$ for $q+1 \le j \le q+r$ and n_j, $1 \le j \le q+r$ are positive integers with $\sum_{j=1}^{q+r} n_j \le d$. Let W_j be the kernel of $g_j(Q)^{n_j}$ in R^d, $1 \le j \le q+r$. The projector onto W_j in the direct sum decomposition

$$R^d = W_1 \oplus \cdots \oplus W_{q+r}$$

is written as U_j. We denote the kernel of $(Q - \sigma_j)^{n_j}$ in C^d, $1 \le j \le q+2r$, by V_j, that is, V_j is the eigenspace of Q in the wide sense associated with the eigenvalue σ_j for $1 \le j \le q+2r$. Denote by T_j the projector onto V_j in the decomposition

$$C^d = V_1 \oplus \cdots \oplus V_{q+2r}. \tag{4.1}$$

We set $J(\alpha) = \{j : 1 \le j \le q+2r, \alpha_j = \frac{\alpha}{2}\}$, $K(\alpha) = \{j : 1 \le j \le q+r,\ \alpha_j > \frac{\alpha}{2}\}$, $W_{K(\alpha)} = \bigoplus_{j \in K(\alpha)} W_j$ and $S_{K(\alpha)} = \{\xi \in W_{K(\alpha)} : |\xi| = 1, |u^Q \xi| > 1 \quad \text{for all} \quad u > 1\}$. We write for $x \ne 0$ in R^d, $\alpha(x) = \min\{\alpha_j : 1 \le j \le q+2r, T_j x \ne 0\}$, and for j such that $T_j x \ne 0$, we set $n(x,j) = \max\{n \ge 0 : (Q - \sigma_j)^n T_j x \ne 0\}$. For $x \ne 0$ in R^d, we denote $n(x) = \max\{n(x,j) : 1 \le j \le q+2r, U_j x \ne 0, \alpha_j = \alpha(x)\}$, and $N = \max\{n_j : 1 \le j \le q+2r\}$.

The following theorem characterizes the class of all (Q, b, α)-semi-stable purely non-Gaussian distributions without assuming fullness. The first necessary and sufficient condition for a purely non-Gaussian distribution on R^d to be (Q, b, α)-semi-stable was obtained in [7,8]. But, from the results in [7,8], it is not easy to find the relations between the

Lévy measure of operator semi-stable distributions and that of operator stable distributions. With this consideration, we rewrite the Lévy measure of a (Q, b, α)-semi-stable distribution in a form similar to that of the Lévy measure of an operator stable distribution. Our description of the Lévy measure of a (Q, b, α)- semi-stable distribution in the case of $Q = I$ is that of a semi-stable distribution in [1]. Let $R_+ = (0, \infty)$, the open half line. Denote the support of a measure ρ by $Spt\ \rho$. The indicator function of E is denoted by $I_E(x)$.

THEOREM 4.1. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let μ be infinitely divisible on R^d with the Lévy representation $(\gamma, 0, \nu)$. Then a necessary and sufficient condition for μ to be (Q, b, α)-semi-stable is that*

$$\nu(E) = \int_{S_{K(\alpha)}} \lambda(d\xi) \int_0^\infty I_E(u^Q \xi) d\left\{ \frac{-H_\xi(u)}{u^\alpha} \right\} \tag{4.2}$$

for all Borel sets $E \subset R^d$, where λ is a finite measure on $S_{K(\alpha)}$, $\frac{H_\xi(u)}{u^\alpha}$ is nonincreasing in u, $H_\xi(u)$ is right-continuous in u and measurable in ξ, $H_\xi(1) = 1$ and $H_\xi(bu) = H_\xi(u)$ for any u and ξ. If μ is (Q, b, α)-semi-stable, then the measure λ is unique and the function $H_\xi(u)$ is unique for λ-almost every $\xi \in S_{K(\alpha)}$. For any finite measure λ on $S_{K(\alpha)}$ and for any function H_ξ satisfying the conditions above, there exists a (Q, b, α)-semi-stable purely non-Gaussian distribution μ with the Lévy measure ν of (4.2).

Since $W_{K(\alpha)}$ is Q-invariant, using Lemma 2.4, we see that any point $x \neq 0$ in $W_{K(\alpha)}$ has unique expression $x = u^Q \xi$ with $\xi \in S_{K(\alpha)}$ and $u > 0$. From Lemma 4.1 in [11] (see Lemma 5.1 in [12] or Lemma 5.6 in [10]), we see that there is C_1 such that

$$|u^Q \xi| \leq C_1 u^{\alpha(\xi)} |\log u|^{N-1} \quad \text{for} \quad 0 < u \leq 1/e. \tag{4.3}$$

Put $h(u) = \frac{u^2}{1+u^2}$. Then, by (4.3), there is C_2 such that

$$h(|u^Q \xi|) \leq C_2 u^{2\alpha(\xi)} |\log u|^{2N-2} \quad \text{for} \quad 0 < u \leq 1/e. \tag{4.4}$$

Here C_1 and C_2 are constants independent of u and ξ.

LEMMA 4.2. *If μ is (Q, b, α)-semi-stable, purely non-Gaussian with Lévy measure ν, then*

$$Spt\ \nu \subset W_{K(\alpha)}.$$

Proof. Define a finite measure ν' by $\nu'(E) = \int_E h(|x|)\nu(dx)$ for $E \in \mathcal{B}(R^d)$. Let n be a positive integer such that $0 < b^n < \frac{1}{e}$. By Lemma 2.2, we obtain that

$$\nu'(b^{nQ}E) = b^{-n\alpha} \int h(|b^{nQ}x|) I_E(x)\nu(dx).$$

Using Lemma 4.1 in [11], we see that there is a positive function $b_0(x)$ for $x \neq 0$ in R^d such that

$$b^{-n\alpha} \int h(|b^{nQ}x|) I_E(x)\nu(dx) \geq b^{-n\alpha} \int h(b_0(x) b^{n\alpha(x)}|x|) I_E(x)\nu(dx).$$

Let $x_0 \notin W_{K(\alpha)}$. Choose a bounded open neighborhood E of x_0 such that $\alpha(x) \leq \frac{\alpha}{2}$ for $x \in E$. By Fatou's lemma we have

$$\begin{aligned} &\liminf_{n\to\infty} \nu'(b^{nQ}E) \\ &\geq \int \liminf_{n\to\infty} b^{-n\alpha} h(b_0(x) b^{n\alpha(x)}|x|) I_E(x)\nu(dx). \end{aligned}$$

Let E_1 be the set of $x \in E$ such that $\alpha(x) < \frac{\alpha}{2}$, and E_2 be the set of $x \in E$ such that $\alpha(x) = \frac{\alpha}{2}$. Then,

$$\liminf_{n\to\infty} b^{-n\alpha} h(b_0(x) b^{n\alpha(x)}|x|) = \begin{cases} \infty & \text{for} \quad x \in E_1 \\ b_0(x)^2|x|^2 & \text{for} \quad x \in E_2. \end{cases}$$

Hence, we see that $\nu(E_1) = 0$. By (4.4), we have that

$$|b^{nQ}x| \leq C_1 b^{n\frac{\alpha}{2}} |\log b^n|^{N-1}|x| \quad \text{for} \quad x \in E_2,$$

if $b^n \leq 1/e$. This leads to $\liminf_{n\to\infty} \nu'(b^{nQ}E_2) = \nu'(\{0\}) = 0$. Since

$$\liminf_{n\to\infty} \nu'(b^{nQ}E_2) \geq \int b_0(x)^2|x|^2 I_{E_2}(x)\nu(dx),$$

we get $\nu(E_2) = 0$. Hence $\nu(E) = 0$, which means that $x_0 \notin Spt\ \nu$. □

Proof of Theorem 4.1. Suppose that μ is (Q, b, α)-semi-stable. For any $B \in \mathcal{B}(S_K(\alpha))$, define $\lambda(B) = \nu(\{u^Q\xi : \xi \in B, u > 1\})$ and $N(s, B) = \nu(\{u^Q\xi : \xi \in B, u > s\})$. Then for any positive real number r, we can choose integer m such that $r > b^m > 0$, so

$$0 \leq N(r, B) \leq N(b^m, B) = b^{-m\alpha}\lambda(B).$$

Hence $N(r, B)$ is absolutely continuous with respect to λ. Thus for each positive real number r, there is a nonnegative measurable function $N_\xi(r)$ of ξ such that

$$N(r, B) = \int_B N_\xi(r)\lambda(d\xi), \quad B \in \mathcal{B}(S_{K(\alpha)}).$$

Here $N_\xi(r)$ is uniquely defined for λ-almost every ξ. We can take $N_\xi(r)$ nonincreasing right-continuous in r. For $E = \{u^Q\xi : \xi \in B, u \in F\}$ with $F \in \mathcal{B}(R_+)$, we obtain

$$\nu(E) = -\int_B \lambda(d\xi) \int_F dN_\xi(u).$$

From the fact that $\nu(b^{-Q}\{u^Q\xi : \xi \in B, u \in (s, \infty)\}) = \nu(\{u^Q\xi : \xi \in B, u \in (b^{-1}s, \infty)\}) = b^\alpha \nu(\{u^Q\xi : \xi \in B, u \in (s, \infty)\})$, we see that $N_\xi(bu) = b^{-\alpha}N_\xi(u)$. Putting $N_\xi(u) = H_\xi(u)u^{-\alpha}$, we see that $\frac{H_\xi(u)}{u^\alpha}$ is nonincreasing in u, $H_\xi(u)$ is right-continuous in u and measurable in ξ, $H_\xi(1) = 1$ and $H_\xi(bu) = H_\xi(u)$ for any u and ξ. Since $\mathcal{B}(W_{K(\alpha)})$ is generated by sets E of the above form, we get (4.2) for all $E \in \mathcal{B}(W_{K(\alpha)})$, which shows (4.2) by Lemma 4.2.

Conversely, assume that λ is a finite measure on $S_{K(\alpha)}$ and define a measure ν on R^d by (4.2). Let $\alpha^+ = \min\{\alpha_j : 1 \leq j \leq q + 2r, \alpha_j > \frac{\alpha}{2}\}$ and let $\alpha^{++} = \max\{\alpha_j : 1 \leq j \leq q + 2r, \alpha_j > \frac{\alpha}{2}\}$. Then $\alpha^+ \leq \alpha(\xi) \leq \alpha^{++}$ for $\xi \in S_{K(\alpha)}$. Let M be the positive integer satisfying $1 \leq e^{-1}b^{-M} < b^{-M}$. For any $\xi \in S_{K(\alpha)}$, we have the following. By (4.4), we see that

$$\begin{aligned}
&\int_0^{e^{-1}} h(|u^Q\xi|)d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\} \\
&\leq C_2 \sum_{n=0}^{\infty} \int_{e^{-1}b^{n+1}}^{e^{-1}b^n} u^{2\alpha(\xi)}|\log u|^{2N-2} d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\}.
\end{aligned}$$

For $n = 0, 1, \cdots$, we have that

$$\int_{e^{-1}b^{n+1}}^{e^{-1}b^{n}} u^{2\alpha(\xi)}|\log u|^{2N-2} d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\}$$
$$\leq |(n+1)\log b - 1|^{2N-2} \int_{e^{-1}b^{n+1}}^{e^{-1}b^{n}} u^{2\alpha(\xi)} d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\},$$

because $|\log u| \leq |\log(e^{-1}b^{n+1})| = |(n+1)\log b - 1|$ for $e^{-1}b^{n+1} \leq u < e^{-1}b^{n}$. Letting $u = b^{n+1+M}v$, we obtain that

$$\int_{e^{-1}b^{n+1}}^{e^{-1}b^{n}} u^{2\alpha(\xi)} d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\}$$
$$= b^{2(n+1+M)(\alpha(\xi)-\frac{\alpha}{2})} \int_{e^{-1}b^{-M}}^{e^{-1}b^{-M-1}} v^{2\alpha(\xi)} d\left\{\frac{-H_\xi(v)}{v^\alpha}\right\}.$$

Since $\int_1^\infty d\left\{\frac{-H_\xi(v)}{v^\alpha}\right\} = H_\xi(1) = 1$, we have that

$$b^{2(n+1+M)(\alpha(\xi)-\frac{\alpha}{2})} \int_{e^{-1}b^{-M}}^{e^{-1}b^{-M-1}} v^{2\alpha(\xi)} d\left\{\frac{-H_\xi(v)}{v^\alpha}\right\}$$
$$\leq b^{2(n+1+M)(\alpha^+-\frac{\alpha}{2})} \int_{1}^{b^{-M-1}} v^{2\alpha(\xi)} d\left\{\frac{-H_\xi(v)}{v^\alpha}\right\}$$
$$\leq b^{2(n+1+M)(\alpha^+-\frac{\alpha}{2})} b^{-2(M+1)\alpha(\xi)} \int_{1}^{\infty} d\left\{\frac{-H_\xi(v)}{v^\alpha}\right\}$$
$$\leq b^{2(n+1+M)(\alpha^+-\frac{\alpha}{2})-2(M+1)\alpha^{++}}.$$

Hence

$$\int_{S_{K(\alpha)}} \lambda(d\xi) \int_0^{e^{-1}} h(|u^Q\xi|) d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\}$$
$$\leq C_2\lambda(S_{K(\alpha)}) \sum_{n=0}^{\infty} |(n+1)\log b - 1|^{2N-2} b^{2(n+1+M)(\alpha^+-\frac{\alpha}{2})-2(M+1)\alpha^{++}}$$
$$< \infty,$$

since $\alpha^+ - \frac{\alpha}{2} > 0$. Since $h(\cdot) \leq 1$, we see that

$$\int_{e^{-1}}^{\infty} h(|u^Q\xi|) d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\} \leq \int_{e^{-1}}^{\infty} d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\}.$$

Using the fact that

$$\sup_{u>0} H_\xi(u) = \sup_{1\leq u<b^{-1}} H_\xi(u) = \sup_{1\leq u<b^{-1}} u^\alpha \frac{H_\xi(u)}{u^\alpha} \leq b^{-\alpha} H_\xi(1) = b^{-\alpha}$$

and

$$\inf_{u>0} H_\xi(u) = \inf_{1\leq u<b^{-1}} H_\xi(u) = \inf_{1\leq u<b^{-1}} u^\alpha \frac{H_\xi(u)}{u^\alpha} \geq b^\alpha H_\xi(b^{-1}) = b^\alpha,$$

we obtain that $\lim_{u\to\infty} -\frac{H_\xi(u)}{u^\alpha} = 0$. It follows that

$$\int_{e^{-1}}^{\infty} d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\} = \frac{H_\xi(e^{-1})}{e^{-\alpha}} \leq e^\alpha b^{-\alpha}.$$

Hence

$$\int_{S_{K(\alpha)}} \lambda(d\xi) \int_{e^{-1}}^{\infty} h(|u^Q\xi|) d\left\{\frac{-H_\xi(u)}{u^\alpha}\right\} \leq e^\alpha b^{-\alpha} \lambda(S_{K(\alpha)}) < \infty.$$

Therefore

$$\int_{R^d} h(|x|)\nu(dx) < \infty.$$

Hence ν is the Lévy measure of a purely non-Gaussian infinitely divisible distribution μ. It is easy to see that ν satisfies (2.2). Thus, μ is (Q, b, α)-semi-stable. □

Let μ be (Q, b, α)-semi-stable and centered purely non-Gaussian with Lévy measure ν. Let W_μ, W_ν be the smallest linear subspaces that contain *Spt* μ, *Spt* ν, respectively. From Lemma 4.2, we see that W_ν is a linear subspace of $W_{K(\alpha)}$. Using Lemma 5.2 and Theorem 5.2 in [12], we get the following remark.

REMARK 4.3. Suppose that μ is (Q, b, α)-semi-stable and centered purely non-Gaussian with Lévy measure ν. Then $W_\mu = W_\nu$, μ is full in W_μ and W_μ is b^Q-invariant.

REMARK 4.4. Suppose that μ is a (Q, b, α)-semi-stable distribution on R^2 with Lévy representation $(0, 0, \nu)$. If the subspace W_ν is contained in $R = \{x = (x_i)_{i=1,2} : x_2 = 0\}$, then μ is a semi-stable distribution with some exponent $\tilde{\alpha}$ on R in the sense of [1].

5. Relations between (Q, b, α)-semi-stable distributions and operator semi-stable distributions

R. Jajte in the Theorem of [4] described that, if μ is full, then a necessary and sufficient condition for μ to be an operator semi-stable distribution is that it is infinitely divisible and there exist a number $a \in (0, 1)$, a vector $c_0 \in R^d$, and $A \in Aut(R^d)$ such that

$$\mu^a = A\mu * \delta_{c_0}. \tag{5.1}$$

In [3], V. Chorny pointed out that the relation (5.1) was equivalent to

$$\mu^b = b^Q \mu * \delta_{c_1}$$

with some $b \in (0, 1)$, $Q \in M_+(R^d)$ and $c_1 \in R^d$. This distribution is a $(Q, b, 1)$-semi-stable distribution.

The following Remarks 5.1 and 5.2 for operator semi-stable distributions are given in R. Jajte [4].

REMARK 5.1. Fix $b \in (0, 1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let μ be (Q, b, α)-semi-stable on R^d. Then μ is an operator semi-stable distribution.

REMARK 5.2. If μ is a full operator semi-stable distribution on R^d, then μ is (Q, b, α)-semi-stable on R^d with some $b \in (0, 1)$, $\alpha > 0$ and $Q \in M_+(R^d)$.

PROPOSITION 5.3. *If μ is an operator semi-stable distribution on R^d and $T \in End(R^d)$, then $T\mu$ is an operator semi-stable distribution.*

Proof. We choose $T_n \in Aut(R^d)$ such that $T_n \to T$. By the definition of an operator semi-stable distribution, there are $A_n \in Aut(R^d)$ and $a_n \in R^d$ such that

$$\lim_{n\to\infty} A_n\mu^{k_n} * \delta_{a_n} = \mu,$$

where $k_n^{-1}k_{n+1} \to r$ for some $r \in [1,\infty)$. Hence, we have that

$$\lim_{n\to\infty} T_nA_n\mu^{k_n} * \delta_{T_na_n} = T\mu.$$

This shows that $T\mu$ is an operator semi-stable distribution. □

In [10,14], there are examples of operator stable distributions that are not (Q,α)-stable. Modifying it, we get the following examples. These will show that the converse of Remark 5.1 is not true without the condition of fullness.

EXAMPLE 5.4. Let $d = 2$, $Q = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & 2 \end{pmatrix}$ and $\xi_0 = 2^{-\frac{1}{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then $u^Q = \begin{pmatrix} u^{\frac{3}{2}} & 0 \\ 0 & u^2 \end{pmatrix}$ and $u^Q\xi_0 = 2^{-\frac{1}{2}}\begin{pmatrix} u^{\frac{3}{2}} \\ -u^2 \end{pmatrix}$. Fix $b \in (0,1)$ and $\alpha \in (0,2)$. We choose a positive number C_0 such that $C_0 = (|\frac{2\pi}{\log b}| + 1 < 1$. Let

$$H_{\xi_0}(u) = C_0 \cos\left(\frac{2\pi}{\log b}\log u\right) + 1.$$

Then the function $H_{\xi_0}(u)$ satisfies the conditions in Theorem 4.1. We consider the (Q,b,α)- semi-stable distribution μ having the Lévy representation $(0,0,\nu)$ with

$$\nu(E) = \int_0^\infty I_E(u^Q\xi_0)d\left\{\frac{-H_{\xi_0}(u)}{u^\alpha}\right\}.$$

This shows that μ is an operator semi-stable distribution by Remark 5.1. Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then, by Proposition 5.3, $T\mu$ is an operator semi-stable distribution. We have, for some positive real number s,

$$Spt\ T\nu = \{x = (x_i)_{i=1,2} : x_1 \le s, x_2 = 0\}.$$

Suppose that $T\mu$ is a $(\tilde{Q},\tilde{b},\tilde{\alpha})$ -semi-stable distribution on R^2 with some $\tilde{b}$, $\tilde{\alpha}$ and $\tilde{Q} \in M_+(R^2)$. Then, by Remark 4.4, $T\mu$ is regarded as a semi-stable distribution with some exponent on R in the sense of [1]. But, if the support of the Lévy measure of a semi-stable distribution on R is not contained in $(-\infty,0]$, then it must be unbounded to both directions. So we get a contradiction. Thus we conclude that there are no numbers $\tilde{b}$, $\tilde{\alpha}$ and $\tilde{Q} \in M_+(R^2)$ such that $T\mu$ is a $(\tilde{Q},\tilde{b},\tilde{\alpha})$-semi-stable distribution on R^2.

EXAMPLE 5.5. Let $d=2$. Let Q,T,ξ_0 be as in Example 5.4. Fix $\alpha \in (0,2)$. Let n be an integer. Consider a (Q,b,α)-semi-stable distribution μ having Lévy representation $(0,0,\nu)$ with

$$\nu(E) = \int_0^\infty I_E(u^Q\xi_0)d\left\{\frac{-H_{\xi_0}(u)}{u^\alpha}\right\},$$

where $\frac{-H_{\xi_0}(u)}{u^\alpha} = \sum_{b^{-n}>u} b^{n\alpha}$. We see that

$$Spt\ T\nu = \{x=(x_i)_{i=1,2} : x_1 = 2^{-\frac{1}{2}}(b^n)^{\frac{3}{2}} - 2^{-\frac{1}{2}}(b^n)^2, x_2 = 0\}.$$

By a similar argument to the previous example, we can show that $T\mu$ is an operator semi-stable distribution on R^2. But, we can not find numbers $\tilde{b}$, $\tilde{\alpha}$ and $\tilde{Q} \in M_+(R^2)$ such that $T\mu$ is a $(\tilde{Q},\tilde{b},\tilde{\alpha})$-semi-stable distribution on R^2.

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References

[1] Choi, G. S., *Criteria for recurrence and transience of semistable processes*, Nagoya Math. J. **134** (1994), 91-106.

[2] Choi, G. S. and Sato, K., *Recurrence and transience of operator semi-stable processes*, Proc. Japan Acad. **71 Ser A** (1995), 87-89.

[3] Chorny, V., *Operator-semistable distributions on R^d* (in Russian), Teor. Verojatnost. i Primenen. **31** (1986), 795-797; English translation, Theory Probab. Appl. **31** (1986), 703-705.

[4] Jajte, R., *Semi-stable probability measures on $\mathbf{R}^N$*, Studia Math. **61** (1977), 29-39.

[5] ______, *Semi-stable measures*, in "Probability Theory", Banach Center Publications, Vol. 5, Polish Scientific Publ., Warsaw, 1979, pp. 141-150.

[6] Krakowiak, W., *Operator semi-stable probability measures on Banach spaces*, Colloq. Math. **43** (1980), 351-363.

[7] Luczak, A., *Operator semi-stable probability measures on R^d*, Colloq. Math. **45** (1981), 287-300; Corrigenda, **52** (1987), 167-169

[8] ______, *Operator semi-stable Lévy measures on finite dimensional vector spaces* Probab. Theory. Fields **90** (1991), 317-340.

[9] Maejima, M., Sato, K. and Watanabe, T., *Completely operator semi-selfdecomposable distributions*, Tokyo J. Math., to appear.

[10] Sato, K., *Lectures on multivariate infinitely divisible distributions and operator-stable processes*, Technical Report Series, Lab. Res. Statist. Probab. Carleton Univ. and Univ Ottawa, No. 54, 1985.

[11] ______, *Strictly operator-stable distributions*, J. Multivar. Anal. **22** (1987), 278-295.

[12] Sato, K. and Yamazato, M., *Completely operator-selfdecomposable distributions and operator-stable distributions*, Nagoya. Math. J. **97** (1985), 71-94.

[13] Sharpe, M., *Operator-stable probability distributions on vector groups*, Trans. Amer. Math. Soc. **136** (1969), 51-65.

[14] Yamazato, M., *OL distributions on Euclidean space*, Teor. Veroyatnost i Primenen **29** (1984), 3-18.

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