NOTE ON CONTACT STRUCTURE
AND SYMPLECTIC STRUCTURE

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ABSTRACT. Let \((X, J)\) be a closed, connected almost complex four-manifold. Let \(X_1\) be the complement of an open disc in \(X\) and let \(\xi_1\) be the contact structure on the boundary \(\partial X_1\) which is compatible with a symplectic structure on \(X_1\). Then we show that \((X, J)\) is symplectic if and only if the contact structure \(\xi_1\) on \(\partial X_1\) is isomorphic to the standard contact structure on the 3-sphere \(S^3\) and \(\partial X_1\) is \(J\)-concave. Also we show that there is a contact structure \(\xi_0\) on \(S^2 \times S^1\) which is not strongly symplectically fillable but symplectically fillable, and that \((S^2 \times S^1, \sigma)\) has infinitely many non-diffeomorphic minimal fillings whose restrictions on \(S^2 \times S^1\) are \(\sigma\) where \(\sigma\) is the restriction of the standard symplectic structure on \(S^2 \times D^2\).

1. Introduction

Contact geometry has recently come to the foreground of low dimensional topology. Not only have there been striking advances in the understanding of contact structures on 3-manifolds, but there has been significant interplay with symplectic geometry and Seiberg-Witten theory. In 1971 Martinet [12] showed how to construct a contact structure on any 3-manifold. Later it became clear that contact structures fell into two distinct classes: tight and overtwisted.

It is the tight contact structures that carry significant geometric information. It was known that for any 3-manifold there are only finitely many elements in its second cohomology that can be realized by tight contact structures [7]. More recently Kronheimer and Mrowka [10] have

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shown that only finitely many homotopy types of plane fields can be realized by semi-fillable (and hence tight) contact structures.

We apply them to the simplest class of 3-manifolds. Recall lens spaces \( L(p, q) \) are 3-manifolds that can be written as the union of two solid tori, or in other words, lens spaces are Heegaard genus one manifolds. Recently Etnyre have shown that there is a unique tight contact structure on \( L(0, q) = S^1 \times S^2 \), \( L(1, q) = S^3 \), and \( L(2, q) = \mathbb{R}P^3 \).

The purpose of this paper is to introduce a relationship between the symplectic structure on a 4-manifold with boundary and the contact structure on its boundary. Let \((X, J)\) be a closed, connected almost complex 4-manifold. Let \(X_1\) be the complement of an open disc in \(X\) and let \(\xi_1\) be the contact structure on the boundary \(\partial X_1\) which is compatible with symplectic structure on \(X_1\). Then we show that \((X, J)\) is symplectic if and only if the contact structure \(\xi_1\) on \(\partial X_1\) is isomorphic to the standard contact structure on \(L(1, q) = S^3\) and \(\partial X_1\) is \(J\)-concave. Also in section 3, we show that there is an example that which is not strongly symplectically fillable but symplectically fillable contact structure on \(L(0, q) = S^1 \times S^2\) and we show that \(L(0, q) = S^1 \times S^2\) has infinitely many non-diffeomorphic minimal fillings.

2. Symplectic Structures on Almost Complex 4-Manifolds

2.1 Symplectic structures on open manifolds

Let \(X\) be a closed, connected, smooth 4-manifold with almost complex structure \(J\). Let \(g\) be a Riemannian metric on \(X\) on which \(J\) is an isometry. In this case we say that \(g\) is compatible with \(J\). The almost complex structure \(J\) and the metric \(g\) define a nondegenerate 2-form \(\omega'\) by \(\omega'(v_1, v_2) = g(Jv_1, v_2)\) for any \(v_1, v_2 \in TX\).

In this case the nondegenerate 2-form \(\omega'\) is called compatible with \(J\). In fact there is a one-to-one correspondence of nondegenerate 2-forms compatible with \(J\) and Riemannian metrics compatible with \(J\).

If \(X\) is not symplectic, then \(\omega'\) is not closed. Let \(N\) be a small neighborhood of a point \(p\) in \(X\) which is diffeomorphic to the standard open 4-disc \(D^4\). Then \(X_1 = X - N\) is an almost complex manifold with boundary \(\partial X_1\) and has a nondegenerate 2-form \(\omega_0'\) which is the restriction of \(\omega'\).
Note on contact structure and symplectic structure

A manifold is called open if each component is either non compact or has a non-empty boundary.

**Theorem 2.1 (Gromov).** Let $X$ be an open 4-manifold. Let $\omega \in \Omega^2(X)$ be a nondegenerate 2-form and let $a \in H^2(X; \mathbb{R})$. Then there is a smooth family of nondegenerate 2-forms $\omega_t$ on $X$ such that $\omega_0 = \omega$ and $\omega_1$ is a symplectic form which represents the class $a$.

If $a \in H^{2,1}(X; \mathbb{R})$ is a self-dual cohomology class then by the Gromov's Theorem there is a smooth family of nondegenerate forms $\omega_t$ on $X_1$ such that $\omega_0 = \omega_0'$ and $\omega_1$ is a self-dual symplectic form which represents the class $a$.

### 2.2 J-convexity

Let $(X, J)$ be an almost complex manifold of the real dimension 4 and $\Sigma$ be an oriented hypersurface in $X$ of the real codimension 1. Each tangent plane $T_x(\Sigma)$, $x \in \Sigma$, contains a unique complex line $\xi_x \subset T_x(\Sigma)$ which we will call a complex tangency to $\Sigma$ at $x$. The complex tangency is canonically oriented and, therefore, cooriented. Hence the tangent plane distribution $\xi$ on $\Sigma$ can be defined by an equation $\alpha = 0$ where the 1-form $\alpha$ is unique up to multiplication by the same positive factor. We say that $\Sigma$ is $J$-convex (or $J$-concave) if $d\alpha(v, Jv) > 0$ (or $d\alpha(v, Jv) < 0$) for any non-zero vector $v \in \xi_x$, $x \in \Sigma$. We use the word pseudo-convex or pseudo-concave when the almost complex structure $J$ is not specified.

Following Gromov (see [9]) we say that an almost complex manifold is tame if there exists a symplectic structure $\omega$ on $X$ such that the form $\omega(v, Jv)$, $v \in T(X)$, is positive definite.

The following theorem indicates that the topology of the $J$-convex boundary imposes very strong restrictions on the topology of the domain.

**Theorem 2.2 [7].** Let $(X, J)$ be a tame symplectic manifold and $\Omega \subset X$ be a domain bounded by a $J$-convex 3-sphere. Then for an almost complex structure $J'$ which is $C^\infty$-close to $J$ the manifold $(\Omega, J')$ is a 4-ball up to blowing up a few points. In particular, $\Omega$ is diffeomorphic to $D^4 \times k \mathbb{C}P^2$. 

183
2.3 Condition for symplectic manifold

Let \((X, J)\) be a closed, connected almost complex 4-manifold and \(g\) be a Riemannian metric on \(X\) on which \(J\) is an isometry. Then there is the nondegenerate 2-form \(\omega'\) on \(X\) which is compatible with \(J\). Let \(N\) be a small neighborhood of a point \(p\) in \(X\) which is diffeomorphic to the standard open 4-disc \(D^4\) and \(X_1 \equiv X - N\). Then \(X_1\) is an almost complex manifold with boundary \(\partial X_1 \simeq S^3\) and has a nondegenerate 2-form \(\omega_0\) which is the restriction of \(\omega'\). Then by the Gromov’s Theorem, there is a smooth family of nondegenerate forms \(\omega_t\) on \(X_1\) such that \(\omega_1\) is symplectic on \(X_1\). Let \(\xi_1 = T(\partial X_1) \cap J_1 T(\partial X_1)\), where \(J_1\) is a compatible almost complex structure with \(\omega_1\). Then \(\xi_1\) is a compatible contact structure on \(\partial X_1\) with \(\omega_1\). Let \(\omega = \sum_{i=1}^2 dx_i \wedge dy_i\) be the standard symplectic structure on \(\mathbb{R}^4\). Then the standard contact structure \(\xi_{st}\) on the 3-sphere \(S^3\) is given by the 1-form \(\frac{1}{2} \sum_{i=1}^2 (x_i dy_i - y_i dx_i)\). Now we are ready to prove the following theorem.

**Theorem 2.3.** In the above notations, a closed, connected almost complex 4-manifold \((X, J)\) is symplectic if and only if the contact structure \(\xi_1\) on \(\partial X_1\) is isomorphic to the standard contact structure \(\xi_{st}\) on \(S^3\) and \(\partial X_1\) is \(J\)-concave.

**Proof.** Suppose that \(X\) is a closed, connected symplectic 4-manifold. Darboux’s theorem says that any symplectic form \(\omega\) on \(X\) is locally diffeomorphic to the standard symplectic form \(\omega_0 = \sum_{i=1}^2 dx_i \wedge dy_i\) on \(\mathbb{R}^4\). For any point \(p \in X\) there is a coordinate chart \(\phi : D^4(1 + \epsilon) \to X\) such that \(\phi(0) = p\) and \(\omega_0 = \phi^* \omega\) where \(D^4(1 + \epsilon)\) is a disc in \(\mathbb{R}^4\) with center 0 and radius \(1 + \epsilon\) for some small \(\epsilon > 0\). Then the restriction \(\phi : S^3 \to \phi(S^3)\) is a diffeomorphism. The contact structure \(\xi_1\) on \(\phi(S^3) = \partial(X_1 \equiv X - \phi(D^4(1)))\) compatible with the symplectic form \(\omega\) is isomorphic to the standard contact structure \(\xi_{st}\) on \(S^3\) since there is a unique fillable contact structure (up to isotopy) on \(S^3\).

Since \(\xi_1\) is a contact structure on \(\partial X_1 \simeq S^3\), there is a contact 1-form \(\alpha_1\) such that \(d\alpha_1|_{\xi_1} \neq 0\). If \(d\alpha_1|_{\xi_1} > 0\), then \(\partial X_1\) is \(J\)-convex. Since \(X_1\) is minimal, by Theorem 2.2, \(X_1 \simeq D^4\) and

\[X = \partial X_1 \cup D^4 = \partial D^4 \cup D^4 \simeq S^4.\]
Note on contact structure and symplectic structure

Since $S^4$ cannot have any almost complex structure, this contradicts
the assumption. Therefore $d\alpha|_{\xi_1} < 0$. Hence $\partial X_1$ is $J$-concave.

Conversely, suppose that $(X_1, \omega_1)$ is a symplectic 4-manifold with
boundary $\partial X_1$ on which the compatible contact structure $\xi_1$ with $\omega_1$ is
isomorphic to the standard contact structure $\xi_{st}$ on $S^3$.

Let $U_1$ be a small collar neighborhood of $\partial X_1$ in $X_1$. Let $\phi : U_1 \to
D^4$ be diffeomorphic onto its image $\phi(U_1)$ such that $\phi(\partial X_1) = S^3$ and
$\phi^* (\xi_{st}) = \xi_1$. Let $\theta_{st}$ be the 1-form on $S^3$ whose kernel is $\xi_{st}$ and let $\theta'_1$
be the 1-form on $\partial X_1$ whose kernel is $\xi_1$. Then $\phi^* \theta_{st} = f \theta'_1$, where $f$
is a negative function on $\partial X_1$.

Extend $\theta'_1$ and $\theta_{st}$ on their collared neighborhoods respectively. Then
$\phi^* \omega_0 = \phi^* d\theta_{st} = d\phi^* (\theta_{st}) = d(f \theta'_1)$. Since $d\omega_1 = 0$, $(d\omega_1)|_{U_1} = d(\omega_1)|_{U_1}$ = 0. Since $\partial X_1$ is a strong deformation retract of $U_1$ and $\partial X_1$
is diffeomorphic to $S^3$, $\omega_1$ is exact on $U_1$.

There is a 1-form $\theta_0$ on $U_1$ such that $d\theta_0 = \omega_1$ and $\phi^* \omega_0 - \omega_1 =
d(f \theta'_1) - d\theta_0 = d(f \theta'_1 - \theta_0)$. Let $\theta_1 = f \theta'_1$. Since the cotangent
bundle on $U_1$ is trivial there is a smooth 1-parameter family of 1-form $\theta_t$ joining
$\theta_0$ and $\theta_1$. Thus we may extend smooth 1-parameter family $d\theta_t = \omega_t$
joining $\omega_1$ to $\phi^* \omega_0$. So by attaching $D^4$ to $X_1$ via $\phi$ we have a symplectic
4-manifold $X = X_1 \cup_{\phi} D^4$. \qed

**Corollary 2.4.** Let $X$ be a closed, connected smooth 4-manifold
with almost complex structure. According to the notations of the above
Theorem if $X$ is not symplectic, then the boundary $(\partial X_1, \xi_1)$ and the
3-sphere $(S^3, \xi_{st})$ are diffeomorphic but not contactomorphic.

3. Fillable Contact Structure

In this chapter, we introduce some definitions on the boundary of a 4-
manifold. Using this, we have some results about symplectic manifolds
with contact-type boundaries. A contact structure on a 3-dimensional
manifold $M$ is a 2-plane field $\xi$ in $TM$ which is nowhere integrable. It
determines an orientation which must agree with the given one on $M^3$.

$(M^3, \xi)$ is called to be symplectically fillable if $M$ bounds a compact,
symplectic 4-manifold $(X^4, \omega)$ such that $\omega|_{\xi} \neq 0$. And $(M^3, \xi)$ is called
to be strongly symplectically fillable if $M$ bounds a compact, symplectic
4-manifold \((X^4, \omega)\) such that \(\omega|_{\xi} \neq 0\) and there exists a vector field \(V\) near \(M\), which is outward pointing and transverse to \(M\) at \(M\) and has the property that its flow expands \(\omega\), i.e., \(\mathcal{L}_V \omega = \omega\). If \((M^3, \xi)\) is strongly symplectically fillable, then \((M^3, \xi)\) is a contact-type boundary of a symplectic 4-manifold \((X, \omega)\).

It is known that if \((M^3, \xi)\) is strongly symplectically fillable, then it is symplectically fillable. But the converse is not known yet. Hence we will construct a contact structure which is symplectically fillable but not strongly symplectically fillable.

Consider the symplectic manifold \((S^2, \omega_0)\). The standard symplectic form \(\omega_0\) on \(S^2 = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}\) is given by

\[
\omega_0 = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}
\]

in the usual coordinates \(x + iy\) on \(\mathbb{C}\). Let \(M = S^2 \times S^1\) and let \(\alpha = ydx + d\theta\) be a 1-form on \(M = S^2 \times S^1\), where \(e^{i\theta}\) is the coordinate on \(S^1\). Let \(\xi_0 = \text{Ker} \alpha\). Then \(\xi_0\) is a contact structure on \(M\) and \(\alpha\) is a contact form on \(M = S^2 \times S^1\).

Consider the symplectic 4-manifold \((S^2 \times D^2, \omega)\) where \(\omega = \omega_0 \oplus \omega^*\), \(\omega_0\) is the standard symplectic form on \(S^2\) and \(\omega^*\) is the standard symplectic form on \(D^2\). Then \(\partial(S^2 \times D^2) \cong S^2 \times S^1\) and \(\omega|_{\xi_0} \neq 0\). Therefore \((S^2 \times S^1, \xi_0)\) is symplectically fillable.

**Lemma 3.1.** \((S^2 \times S^1, \xi_0)\) is symplectically fillable and there is no \(\omega\)-tame almost complex structure \(J\) on \(S^2 \times D^2\) such that \(S^2 \times S^1\) is \(J\)-convex.

**Proof.** Suppose not. That is, there is an \(\omega\)-tame almost complex structure \(J\) on \((S^2 \times D^2, \omega)\) such that \(S^2 \times S^1\) is \(J\)-convex. Hence \((S^2 \times D^2, \omega)\) has an \(\omega\)-tame almost complex structure \(J\) with the \(J\)-convex boundary. By the following Theorem 3.2, \(S^2 \times \{p\}\) in \(S^2 \times S^1\) is \(\partial(C^2 \times D^2)\) bounds an embedded ball \(B^3\) in \(S^2 \times D^2\). This is impossible.

\[\Box\]

**Theorem 3.2 [7].** Let \((X, \omega)\) be a symplectic 4-manifold and \(J\) be an \(\omega\)-tame almost complex structure and \(\partial X\) is a \(J\)-convex boundary.
Any closed surface $\Sigma \subset \partial X$ different from $S^2$ satisfies the inequality

$$\chi(\Sigma) \leq -|c_1(X)(\Sigma)|.$$ 

If $M$ is diffeomorphic to $S^2$, then it can be filled by holomorphic disc. In particular, it bounds an embedded ball $B^3 \subset X$.

Let $\text{Fill}^a(\xi_0)$ be the set of all symplectic fillings of $(S^2 \times S^1, \xi_0)$ and $\text{Fill}^{a,-}(\xi_0)$ the set of all strongly symplectic fillings of $(S^2 \times S^1, \xi_0)$. Hence if $(X, \omega) \in \text{Fill}^a(\xi_0)$, then $(X, \omega)$ is a compact symplectic 4-manifold with $\partial X \simeq S^2 \times S^1$ and $\omega|_{\xi_0} \neq 0$. Similarly if $(X, \omega) \in \text{Fill}^{a,-}(\xi_0)$, then $(X, \omega)$ is a compact symplectic 4-manifold with a contact type boundary $(S^2 \times S^1, \xi_0)$. Also $\text{Fill}^a(\xi_0) \supset \text{Fill}^{a,-}(\xi_0)$.

**Remark.** $(S^2 \times D^2, \omega) \not\in \text{Fill}^{a,-}(\xi_0)$. If not, $\omega|_{\partial(S^2 \times D^2)} = d\alpha$. Since $(S^2 \times D^2, \omega) \in \text{Fill}^a(\xi_0)$, $\omega|_{\xi_0} \neq 0$. Hence we can choose an almost complex structure $J$ on $S^2 \times D^2$ has a $J$-convex boundary $(S^2 \times S^1, \xi_0)$. This contradicts the above Lemma 3.1.

By Lemma 3.1, $(S^2 \times D^2, \omega) \in \text{Fill}^a(\xi_0)$. If $(X, \omega) \in \text{Fill}^a(\xi_0)$, then $(X, \omega)$ is a compact symplectic 4-manifold with $\partial X \simeq S^2 \times S^1$ and $\omega|_{\xi_0} \neq 0$. Then we can choose an almost complex structure $J \in \mathcal{J}_r(S^2 \times D^2, \omega)$ such that $\xi_0$ is $J$-invariant. Hence $\omega|_{\xi_0} > 0$. Since $\alpha$ is a contact form on $\xi_0$, $d\alpha|_{\xi_0} \neq 0$. If $d\alpha|_{\xi_0} < 0$, then $(X, \omega) \not\in \text{Fill}^{a,-}(\xi_0)$. If not, that is, $(X, \omega) \in \text{Fill}^{a,-}(\xi_0)$, $\omega|_{\partial X} = d\alpha$. Since $\omega|_{\xi_0} > 0$ and $d\alpha|_{\xi_0} < 0$, this is impossible. If $d\alpha|_{\xi_0} > 0$, then $(X, \omega)$ has a $J$-convex boundary $(S^2 \times S^1, \xi_0)$. Let $F_p = S^2 \times \{p\}$ be a sphere in a $J$-convex boundary $(S^2 \times S^1, \xi_0)$, for all $p \in S^1$. Then by Theorem 3.2, $F_p$ bounds an embedded ball $B^3$ in $X$. Hence $B^3 \times S^1$ is embedded in $(X, \omega)$. Let $Y \equiv X \cup_{\partial X} (B^3 \times S^1)$. Then $Y$ is a closed 4-manifold and $Y$ contains a closed 4-manifold $S^3 \times S^1$. Hence this is impossible unless $X \cong B^3 \times S^1$. Therefore the only candidate which is an element of $\text{Fill}^{a,-}(\xi_0)$ is $B^3 \times S^1$. Note that if $(X, \omega) \in \text{Fill}^{a,-}(\xi_0)$, then the first Chern class $c_1(X)$ restricted to $\partial X \simeq S^2 \times S^1$ coincides with the Euler class $e(\xi_0)$ of the bundle $\xi_0$. Since $(S^2 \times D^2, \omega) \in \text{Fill}^a(\xi_0)$,

$$e(\xi_0) = c_1(T(S^2 \times D^2))|_{S^2 \times S^1} \neq 0 \in H^2(S^2 \times S^1).$$
If there is a symplectic form $\omega'$ on $B^3 \times S^1$ such that $(B^3 \times S^1, \omega') \in \text{Fill}^a(\xi_0)$, then $e(\xi_0) = c_1(T(B^3 \times S^1)|_{\partial(B^3 \times S^1)})$. But this is impossible because $c_1(T(B^3 \times S^1)) = 0$ in $H^2(B^3 \times S^1) = \{0\}$. Hence $\text{Fill}^{a,a}(\xi_0) = \emptyset$. Therefore we have the following proposition.

**Proposition 3.3.** There is a contact structure $\xi_0$ on $S^2 \times S^1$ which is not strongly symplectically fillable but symplectically fillable.

Consider an oriented 3-dimensional manifold $M$ with closed 2-form $\sigma$. We will say that $(M^3, \sigma)$ has contact type if there is a positively oriented contact form $\alpha$ on $M$ such that $d\alpha = \sigma$. Following Eliashberg [7], we say that the symplectic manifold $(Z, \omega)$ fills $(M, \sigma)$ if there is a diffeomorphism $f : \partial X \to M$ such that $f^*\sigma = \omega|_{\partial Z}$. Further, the filling $(Z, \omega)$ is said to be minimal if $Z$ contains no exceptional spheres in its interior. In [13], McDuff show that the lens space $L_{p,1}$, all have minimal symplectic fillings and if $p \neq 4$, minimal fillings $(Z, \omega)$ of $(L_{p,1}, \sigma)$ are unique up to diffeomorphism, and up to symplectomorphism if one fixes the cohomology class $[\omega]$. However $(L_4, \sigma)$ has exactly two nondiffeomorphic minimal fillings.

Fix the symplectic form $\omega = \omega_0 \oplus \omega^*$ on $S^2 \times D^2$. Here $\omega_0$ is the standard symplectic form on $S^2$ and $\omega^*$ is the standard symplectic form on $D^2$. Let $\sigma = \omega|_{S^2 \times S^1}$. Then $\sigma$ is a closed 2-form on $S^2 \times S^1$. Let $\omega_g$ be the standard symplectic form on $\Sigma_g$. Then $(S^2 \times \Sigma_g, \tilde{\omega} = \omega_0 \oplus \omega_g)$ is a symplectic 4-manifold. Choose a point $p$ in $\Sigma_g$. Then there is a neighborhood $\mathcal{N}(p)$ in $\Sigma_g$ such that $\omega_g|_{\mathcal{N}(p)} = \omega^*$. Therefore

$$\tilde{\omega}|_{S^2 \times \mathcal{N}(p)} = \omega_0 \oplus \omega^*.$$

Let $Z_g = (S^2 \times \Sigma_g) \setminus (S^2 \times \mathcal{N}(p))$ and $\tilde{\omega}_g = \tilde{\omega}|_{Z_g}$. Then $(Z_g, \tilde{\omega}_g)$ are symplectic 4-manifolds with $\partial Z_g \simeq S^2 \times S^1$ and

$$\tilde{\omega}_g|_{S^2 \times S^1} = \omega_0 \oplus \omega^*|_{S^2 \times S^1} = \sigma.$$

Hence $(Z_g, \tilde{\omega}_g)$ are minimal symplectic fillings of $(S^2 \times S^1, \sigma)$. Therefore we have the following proposition.

**Proposition 3.4.** $(S^2 \times S^1, \sigma)$ has infinitely many non-diffeomorphic minimal fillings whose restrictions on the boundary are $\sigma$.

**Remark.** The above $(Z_g, \tilde{\omega}_g) \in \text{Fill}^a(\xi_0)$, but $(Z_g, \tilde{\omega}_g) \notin \text{Fill}^{a,a}(\xi_0)$. 

188
Note on contact structure and symplectic structure

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