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NATURAL FILTRATIONS OF SOME PLETHYSMS

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ABSTRACT. Let R be a commutative ring with unity and F a finite free R-module. For a nonnegative integer r, there exists a natural filtration of $S_r(S_2F)$ such that its associated graded module is isomorphic to $\sum_{\lambda\in\Gamma_r} L_\lambda F$, where Γ_r is the set of partitions such that $|\lambda| = 2r, \tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_k)$, each $\tilde{\lambda}_t$ is even. We call such filtrations plethysm formulas. We extend the above plethysm formula to the version of chain complexes. By plethysm formula we mean the composition of universally free functors. Let $\phi : G \to F$ be a morphism of finite free R-modules. We construct the natural decomposition of $S_r(S_2\phi)$, up to filtrations, whose associated graded complex is isomorphic to $\sum_{\lambda\in\Gamma_r} L_\lambda\phi$.

1. Introduction

The study of finite free resolutions, modular representations, invariant theory, and algebraic geometry over fields of positive characteristic has led to the characteristic-free representation theory of the general linear group. The representations of the general linear group over fields of arbitrary characteristics are constructed and investigated by many people in many different ways (See, for instance, [7, 8, 10, 14, 19]). Those constructions are nicely adapted in order to make sense over arbitrary commutative rings. Akin, Buchsbaum, and Weyman [4] have introduced and studied the Schur and Weyl modules parametrized by Young diagrams, which turn out to be the natural generalizations to

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commutative rings of the constructions of the classical representations of the general linear group given by I. Schur [18] and H. Weyl [20], respectively. In fact, the more general notion of Schur complexes of a complex in the category of finitely generated projective modules is defined; the Schur and Weyl modules result as special cases of the Schur complex, whose usefulness is abundant. For instance, Schur complexes play central roles in the resolutions of determinantal and pfaffian ideals [4, 11], and in the characteristic-free representation theory of the general linear group [2, 3, 12]. This forces us to further study Schur complexes. One way to study Schur complexes is to look for complex theoretic versions of classical character relations for the general linear group. The Grothendiek ring of the category of polynomial representations of the general linear group GL_n over an infinite field is canonically isomorphic to the ring of symmetric polynomials in n variables, the correspondence being obtained sending a representation to its formal character [10]. The formal character of the Schur module is the Schur function. D. E. Littlewood [15] introduced plethysm as an operation on symmetric functions. By means of the correspondence mentioned above we have plethysm on Schur and Weyl modules. In terms of representation theory, plethysm is generally known as the representation derived from a given representation by algebraic operations [9, 16].

Let R be a commutative ring with identity and let $\phi : G \longrightarrow F$ be a morphism of finitely generated free R-modules. We denote by $\Lambda F, SF$, and DF the exterior, the symmetric, and the divided power algebra of F, respectively. Over the last decade or so, plethysms such as $S(S_2F), S(\Lambda^2F)$, $S(F \otimes G), S_k\phi \otimes L_\lambda\phi$, and $\Lambda^k\phi \otimes L_\lambda\phi$ have been studied, mostly in connection with invariant theory, resolutions of determinantal ideals, and the characteristic-free representation theory of the general linear group [1, 5, 6, 12, 13, 17]. The main purpose of this article is to provide the natural filtrations of the plethysm on the Schur complex whose associated graded complex is isomorphic to the direct sum of the Schur complexes.

Section 2 covers the basic definitions and some of the important properties utilized in the main body of the paper. In section 3, we extend the plethysm formula to the version of chain complexes. We construct the natural decomposition of $S_r(S_2\phi)$, up to filtrations, whose associated graded complex is isomorphic to $\sum_{\lambda \in \Gamma_r} L_\lambda \phi$, where Γ_r is the set of partitions such that $|\lambda| = 2r, \tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_k)$, each $\tilde{\lambda}_i$ is even. Specializing our results, we obtain the plethysm formulas for Schur modules and Weyl modules.

2. Preliminaries

This chapter reviews the definitions and properties associated to Schur complexes. For the proofs of propositions, we refer to Akin et al. [4]. Throughout this chapter, R is a commutative ring with unity, F and G are finitely generated free R-modules of ranks m and n, respectively, and $\phi : G \to F$ is an R-module homomorphism. We denote by c_ϕ the element of $F \otimes G^*$ corresponding to the map ϕ under the canonical isomorphism $Hom_R(G, F) \cong F \otimes G^*$.

DEFINITION 2.1. A partition is a non-increasing sequence $\lambda=(\lambda_1, \cdots, \lambda_q)$ of non-negative integers. The weight of the partition, denoted by $|\lambda|$, is the sum of the terms of the sequence. The length of λ is the number of non-zero terms.

The diagram of λ, denoted by $\triangle_\lambda$, is the set of ordered pairs (i, j), ordered as in matrices (the i's increase downward and the j's increase from left to right), with $1 \le i \le$ (length of λ) and $1 \le j \le \lambda_i$. If λ is a partition, then the *transpose* $\tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_p)$ of λ is the partition whose j^{th} term $\tilde{\lambda}_j$ is the number of terms of λ which are greater than or equal to j. We introduce the lexicographic order to the set of partitions, i.e., for two partitions $\lambda = (\lambda_1, \cdots, \lambda_q)$ and $\mu = (\mu_1, \cdots, \mu_r)$, we say that λ is higher than μ, and write $\lambda > \mu$, if there exists i such that $\lambda_k = \mu_k$ for all $k < i$ and $\lambda_i > \mu_i$. $\lambda \ge \mu$ means $\lambda > \mu$ or $\lambda = \mu$. The partition μ is contained in λ (written $\mu \subseteq \lambda$) if $\mu_i \le \lambda_i$ for all i. For such a pair μ and λ, the skew-diagram $\triangle_{\lambda/\mu}$ is $\triangle_\lambda - \triangle_\mu$.

DEFINITION 2.2. The symmetric algebra $S\phi$ of the morphism ϕ is the graded R-Hopf algebra $SF \otimes \Lambda G$ formed by taking the tensor product of the graded R-Hopf algebras SF and ΛG. We let $m_{S\phi} : S\phi \otimes S\phi \to S\phi$, $\Delta_{S\phi} : S\phi \to S\phi \otimes S\phi$, and $T_{S\phi} : S\phi \otimes S\phi \to S\phi \otimes S\phi$ denote the multiplication, comultiplication, and the twisting map of $S\phi$. We make

$S\phi$ into a complex as follows: let $(S\phi)_j = \sum_{i=0}^{\infty} S_iF \otimes \Lambda^j G$ be the j^{th} degree component of the complex and let the boundary map $\partial_{S\phi}$ be the collection $\{(\partial_{S\phi})_j\}$ where $(\partial_{S\phi})_j : (S\phi)_j \to (S\phi)_{j-1}$ is the R-map defined by the action $c_\phi \in SF \otimes \Lambda G^*$ on $SF \otimes \Lambda G$.

DEFINITION 2.3. By $S_k\phi$ we mean the subcomplex of $S\phi$ given by

$$0 \to \Lambda^k G \to F \otimes \Lambda^{k-1} G \to \cdots \to S_{k-j}F \otimes \Lambda^j G \to \cdots \to S_k F \to 0$$

where the j^{th} degree component $(S_k\phi)_j$ is $S_{k-j}F \otimes \Lambda^j G$.

DEFINITION 2.4. The exterior algebra $\Lambda\phi$ of the morphism ϕ is the bigraded R-Hopf algebra $\Lambda F \dot{\otimes} DG$ formed by taking the antisymmetric tensor product of the graded R-Hopf algebras ΛF and DG. We let $m_{\Lambda\phi} : \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi$, $\Delta_{\Lambda\phi} : \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$, and $T_{\Lambda\phi} : \Lambda\phi \otimes \Lambda\phi \to \Lambda\phi \otimes \Lambda\phi$ denote the multiplication, comultiplication, and the twisting map of $\Lambda\phi$. We make $\Lambda\phi$ into a complex as follows: let $(\Lambda\phi)_j = \sum_i \Lambda^i F \otimes D_j G$ be the j^{th} degree component of the complex and let the boundary map $\partial_{\Lambda\phi}$ be the collection $\{(\partial_{\Lambda\phi})_j\}$ where $(\partial_{\Lambda\phi})_j : (\Lambda\phi)_j \to (\Lambda\phi)_{j-1}$ is the R-map defined by the action of $c_\phi \in \Lambda F \dot{\otimes} SG^*$ on $\Lambda F \dot{\otimes} DG$.

DEFINITION 2.5. By $\Lambda^k\phi$ we mean the subcomplex of $\Lambda\phi$ given by

$$0 \to D_k G \to F \otimes D_{k-1} G \to \cdots \to \Lambda^{k-j} F \otimes D_j G \to \cdots \to \Lambda^k F \to 0$$

where the j^{th} degree component $(\Lambda^k\phi)_j$ is $\Lambda^{k-j} F \otimes D_j G$.

DEFINITION 2.6. Let $\mu = (\mu_1, \cdots, \mu_q)$ and $\lambda = (\lambda_1, \cdots, \lambda_q)$ be partitions such that $\mu \subseteq \lambda$. We define complexes

$$\Lambda_{\lambda/\mu}\phi = \Lambda^{\lambda_1-\mu_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_q-\mu_q}\phi, \; S_{\lambda/\mu}\phi = S_{\lambda_1-\mu_1}\phi \otimes \cdots \otimes S_{\lambda_q-\mu_q}\phi.$$

Let (a_{ij}) be the $q \times t$ matrix $(t = \lambda_1)$ whose entries a_{ij} are defined as follows: $a_{ij} = 1$ if $\mu_i + 1 \le j \le \lambda_i$, and $a_{ij} = 0$ otherwise, and define the map

$$d_{\lambda/\mu}\phi : \Lambda_{\lambda/\mu}\phi \to S_{\tilde{\lambda}/\tilde{\mu}}\phi$$

to be the composition

$$\Lambda_{\lambda/\mu}\phi \xrightarrow{\triangle\otimes\cdots\otimes\triangle}$$
$$\Lambda^{a_{11}}\phi\otimes\cdots\otimes\Lambda^{a_{1t}}\phi\cdot\otimes\cdots\otimes\Lambda^{a_{q1}}\phi\otimes\cdots\otimes\Lambda^{a_{qt}}\phi \xrightarrow{\cong}$$
$$S_{a_{11}}\phi\otimes\cdots\otimes S_{a_{q1}}\phi\otimes\cdots\otimes S_{a_{1t}}\phi\otimes\cdots\otimes S_{a_{qt}}\phi \xrightarrow{m\otimes\cdots\otimes m}$$
$$S_{\tilde\lambda_1-\tilde\mu_1}\phi\otimes\cdots\otimes S_{\tilde\lambda_t-\tilde\mu_t}\phi$$

where the first map is the diagonalization, the middle map is the isomorphism identifying $\Lambda^{a_{ij}}\phi$ with $S_{a_{ij}}\phi$, and the last map is the multiplication.

DEFINITION 2.7. Let $\lambda = (\lambda_1, \cdots, \lambda_q)$ and $\mu = (\mu_1, \cdots, \mu_q)$. For each $i = 1, \cdots, q$ let $p_i = \lambda_i - \mu_i$, $t_i = \mu_i - \mu_{i+1} + 1$, and for $i = 1, \cdots, q-1$ let

$$\square'_i : \sum_{\alpha=t_i}^{p_{i+1}} \Lambda^{p_i+\alpha}\phi\otimes\Lambda^{p_{i+1}-\alpha}\phi \to \Lambda^{p_i}\phi\otimes\Lambda^{p_{i+1}}\phi$$

be the map which for each α is the composite

$$\Lambda^{p_i+\alpha}\phi\otimes\Lambda^{p_{i+1}-\alpha}\phi \xrightarrow{\triangle\otimes 1} \Lambda^{p_i}\phi\otimes\Lambda^{\alpha}\phi\otimes\Lambda^{p_{i+1}-\alpha}\phi \xrightarrow{1\otimes m} \Lambda^{p_i}\phi\otimes\Lambda^{p_{i+1}}\phi$$

where $\triangle$ is the diagonalization and m is the multiplication. Thus for each $i = 1, \cdots, q-1$, we have maps $\square_i = 1\otimes\cdots\otimes\square'_i\otimes\cdots\otimes 1$ so

$$\square_i : \sum_{\alpha=t_i}^{p_{i+1}} \Lambda^{p_1}\phi\otimes\cdots\otimes\Lambda^{p_{i-1}}\phi\otimes\Lambda^{p_i+\alpha}\phi\otimes\Lambda^{p_{i+1}-\alpha}\phi\otimes$$
$$\Lambda^{p_{i+2}}\phi\otimes\cdots\otimes\Lambda^{p_q}\phi \to \Lambda^{p_1}\phi\otimes\cdots\otimes\Lambda^{p_q}\phi.$$

Next we define $\square_{\lambda/\mu}$ to be the sum $\sum_{i=1}^{q-1}\square_i$, so that

$$\square_{\lambda/\mu} : \sum_{i=1}^{q-1}\sum_{\alpha=t_i}^{p_{i+1}} \Lambda^{p_1}\phi\otimes\cdots\otimes\Lambda^{p_i+\alpha}\phi\otimes\Lambda^{p_{i+1}-\alpha}\phi\otimes\cdots\otimes\Lambda^{p_q}\phi$$
$$\to \Lambda^{p_1}\phi\otimes\cdots\otimes\Lambda^{p_q}\phi.$$

Denote the cokernel of $\square_{\lambda/\mu}$ by $\bar L_{\lambda/\mu}\phi$.

DEFINITION 2.8. A tableau of shape λ/μ with values in the set S is a function from $\triangle_{\lambda/\mu}$ to S. The set of all such tableaux is denoted by $Tab_{\lambda/\mu}(S)$. In particular, if S is the ordered basis $\Omega_F = \{f_1, \cdots, f_m\}$ of the free module F, then $T : \triangle_{\lambda/\mu} \to \Omega_F$ can be thought of assigning to each ordered pair of $\triangle_{\lambda/\mu}$ an element of f_i. Conversely, to each $T \in Tab_{\lambda/\mu}(\Omega_F)$ there corresponds an element $X_T \in \Lambda_{\lambda/\mu}F$.

Suppose we let $\{y_1, \cdots, y_n\}$ be a basis for G, and $\{x_1, \cdots, x_m\}$ a basis for F, and $S = \{y_1, \cdots, y_n\} \cup \{x_1, \cdots, x_m\}$. We shall totally order S by setting $y_i < x_j$, for all i, j, while maintaining the given orders among the $x's$ and $y's$.

DEFINITION 2.9. Let S be a totally ordered set, and let Y be a subset of S. A tableau $T \in Tab_{\lambda/\mu}(S)$ is said to be row-standard mod Y if each row of T is non-decreasing, and if, when repeats occur in a row, they occur only among elements of Y. A tableau is column-standard mod Y if each column is non-decreasing, and if, when repeats occur in a column, they occur only among elements in the complement of Y. T is standard mod Y if T is row-and column-standard mod Y.

THEOREM 2.10 (Standard basis theorem). *Let $\lambda = (\lambda_1, \cdots, \lambda_q)$, $\mu = (\mu_1, \cdots, \mu_q)$ be partitions with $\mu \subset \lambda$, and let $\phi : G \to F$ be a map of free modules. Let $Y = \{y_1, \cdots, y_n\}$, $X = \{x_1, \cdots, x_m\}$ be bases for G and F, and let $S = Y \cup X$ be totally ordered so that the orders of X and Y are preserved. Then $\{d_{\lambda/\mu}(Z_T) | T$ is a standard tableau mod Y in $Tab_{\lambda/\mu}(S)\}$ is a free basis for $L_{\lambda/\mu}\phi$, and the map $\theta_{\lambda/\mu}\phi : \bar{L}_{\lambda/\mu}\phi \to L_{\lambda/\mu}\phi$ is an isomorphism. Hence $L_{\lambda/\mu}\phi$ is universally free.*

Proof. See [4, Theorem V.1.10]. □

Suppose that $\phi_1 : G_1 \to F_1$ and $\phi_2 : G_2 \to F_2$ are maps of free modules, and let $\phi : G \to F$ be the direct sum of ϕ_1 and ϕ_2, with $G = G_1 \oplus G_2$ and $F = F_1 \oplus F_2$.

DEFINITION 2.11. Let $\phi = \phi_1 \oplus \phi_2$ be as above and let $\mu \subseteq \lambda$ be given partitions. If γ is any partition such that $\mu \subseteq \gamma \subseteq \lambda$, define subcomplexes $\mathcal{M}_\gamma(\Lambda_{\lambda/\mu}\phi)$, $\dot{\mathcal{M}}_\gamma(\Lambda_{\lambda/\mu}\phi)$ of $\Lambda_{\lambda/\mu}\phi$ and $\mathcal{M}_\gamma(L_{\lambda/\mu}\phi)$, $\dot{\mathcal{M}}_\gamma(L_{\lambda/\mu}\phi)$ of $L_{\lambda/\mu}\phi$ as follows:

(i) $\mathcal{M}_\gamma(\Lambda_{\lambda/\mu}\phi) = \text{Image}\,(\sum_{\mu\subseteq\sigma\subseteq\lambda,\ \gamma\leq\sigma} \Lambda_{\sigma/\mu}\phi_1 \otimes \Lambda_{\lambda/\sigma}\phi_2 \to \Lambda_{\lambda/\mu}\phi)$;
(ii) $\dot{\mathcal{M}}_\gamma(\Lambda_{\lambda/\mu}\phi) = \text{Image}\,(\sum_{\mu\subseteq\sigma\subseteq\lambda,\gamma<\sigma} \Lambda_{\sigma/\mu}\phi_1 \otimes \Lambda_{\lambda/\sigma}\phi_2 \to \Lambda_{\lambda/\mu}\phi)$;
(iii) $\mathcal{M}_\gamma(L_{\lambda/\mu}\phi) = d_{\lambda/\mu}(\mathcal{M}_\gamma(\Lambda_{\lambda/\mu}\phi))$;
(iv) $\dot{\mathcal{M}}_\gamma(L_{\lambda/\mu}\phi) = d_{\lambda/\mu}(\dot{\mathcal{M}}_\gamma(\Lambda_{\lambda/\mu}\phi))$.

The maps indicated above are those obtained by tensoring the restriction maps

$$\Lambda^{\sigma_i-\mu_i}\phi_1 \otimes \Lambda^{\lambda_i-\sigma_i}\phi_2 \to \Lambda^{\lambda_i-\mu_i}(\phi_1 \oplus \phi_2)$$

of the natural isomorphism $\Lambda\phi_1 \dot{\otimes} \Lambda\phi_2 \cong \Lambda(\phi_1 \oplus \phi_2)$.

PROPOSITION 2.12. *The map*

$$\Lambda_{\gamma/\mu}\phi_1 \otimes \Lambda_{\lambda/\gamma}\phi_2 \to \mathcal{M}_\gamma(\Lambda_{\lambda/\mu}\phi)$$

induces a map

$$L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2 \to \mathcal{M}_\gamma(L_{\lambda/\mu}\phi)/\dot{\mathcal{M}}_\gamma(L_{\lambda/\mu}\phi).$$

Proof. See [4, Proposition V.1.12]. □

THEOREM 2.13. *The map*

$$L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2 \to \mathcal{M}_\gamma(L_{\lambda/\mu}\phi)/\dot{\mathcal{M}}_\gamma(L_{\lambda/\mu}\phi)$$

is an isomorphism. Hence, the complexes $\{\mathcal{M}_\gamma(L_{\lambda/\mu}\phi) | \mu \subseteq \gamma \subseteq \lambda\}$ *give a filtration of the complex* $L_{\lambda/\mu}\phi$ *whose associated graded complex is isomorphic to* $\sum_{\mu\subseteq\gamma\subseteq\lambda} L_{\gamma/\mu}\phi_1 \otimes L_{\lambda/\gamma}\phi_2$.

Proof. See [4, Theorem V.1.13]. □

COROLLARY 2.14. *Denote by* $(L_{\lambda/\mu}\phi)_j$ *the component in degree* j *of the complex* $L_{\lambda/\mu}\phi$. *There is a natural filtration on* $(L_{\lambda/\mu}\phi)_j$ *whose associated graded module is*

$$\sum_{\mu\subseteq\gamma\subseteq\lambda, |\lambda|-|\gamma|=j} L_{\gamma/\mu}F \otimes K_{\lambda/\gamma}G.$$

Proof. See [4, Corollary V.1.14]. □

To complete this section, we stress that Theorem 2.13 gives a natural filtration of complex, while Corollary 2.14 gives a filtration of modules.

REMARK 2.15.
(i) if G=0, then $L_{\lambda/\mu}\phi \cong L_{\lambda/\mu}F$;
(ii) if F=0, then $L_{\lambda/\mu}\phi \cong K_{\lambda/\mu}G$ in degree $|\lambda| - |\mu|$;
(iii) if $\lambda = (\lambda_1)$ and $\mu = 0$, then $L_{\lambda/\mu}\phi = \Lambda^{\lambda_1}\phi$;
(iv) if $\lambda = \underbrace{(1, \cdots, 1)}_{q}$ and $\mu = 0$, then $L_{\lambda/\mu}\phi = S_q\phi$.

Hence in the case of G=0 or F=0 we obtain the natural filtrations of Schur modules and Weyl modules.

3. Plethysm formulas

In this section we provide the plethysm formulas for Schur complex. As consequences, we recover the plethysm formulas for Schur modules [6, 13].

DEFINITION 3.1. Define a natural pairing $\langle , \rangle : \Lambda^p F \otimes D_p G \to \Lambda^p(F \otimes G)$ by induction on p. For $p = 1$, we define $\langle f, g\rangle = f \otimes g$. For $p > 1$ we define

$$\begin{aligned} &\left\langle f_1 \wedge \cdots \wedge f_p,\ g_1^{(\alpha_1)} \cdots g_t^{(\alpha_t)} \right\rangle \\ &= \sum_{i=1}^{t} \langle f_1, g_i\rangle \wedge \left\langle f_2 \wedge \cdots \wedge f_p,\ g_1^{(\alpha_1)} \cdots g_i^{(\alpha_i - 1)} \cdots g_t^{(\alpha_t)} \right\rangle \end{aligned}$$

where $\sum_{i=1}^t \alpha_i = p$ and $\alpha_i \geq 1$ for all i. Next we extend the above to a pairing $\langle , \rangle : \Lambda_\lambda F \otimes D_\lambda G \to \Lambda^k(F \otimes G)$, where $|\lambda| = k$. Let $\lambda = (\lambda_1, \cdots, \lambda_t)$ and define

$$\langle a_1 \otimes \cdots \otimes a_t, b_1 \otimes \cdots \otimes b_t\rangle = \langle a_1, b_1\rangle \wedge \cdots \wedge \langle a_t, b_t\rangle$$

where $a_i \in \Lambda^{\lambda_i} F,\ b_i \in D_{\lambda_i} G$.

DEFINITION 3.2. For a positive integer r, a set of partitions Γ_r is defined as

$$\Gamma_r = \{\lambda : \text{partition} | |\lambda| = 2r;\ \text{when } \tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_k),\ \text{each } \tilde{\lambda}_i \text{ is even}\}$$

DEFINITION 3.3. For a positive integer k, δ_k is defined to be the map

$$\delta_k : \Lambda^k F \otimes \Lambda^k F \to S_k(S_2F)$$

where, if we let $f_1, \cdots, f_k, g_1, \cdots, g_k$ be elements of F, δ_k sends $f_1 \wedge \cdots \wedge f_k \otimes g_1 \wedge \cdots \wedge g_k$ to

$$\sum_\sigma (sgn\ \sigma)(f_1 \times g_{\sigma(1)}) \circ \cdots \circ (f_k \times g_{\sigma(k)})$$

where the sum is over all permutations. ($\times$ is a multiplication in S_2F, and $\circ$ is a multiplication in $S_k(S_2F)$). Moreover, for a partition $\lambda = (\lambda_1, \lambda_1, \cdots, \lambda_t, \lambda_t)$ contained in Γ_r, the following composite map is denoted by δ_λ:

$$\begin{aligned} \Lambda_\lambda F &= (\Lambda^{\lambda_1} F \otimes \Lambda^{\lambda_1} F) \otimes \cdots \otimes (\Lambda^{\lambda_t} F \otimes \Lambda^{\lambda_t} F) \\ &\xrightarrow{\delta_{\lambda_1} \otimes \cdots \otimes \delta_{\lambda_t}} S_{\lambda_1}(S_2F) \otimes \cdots \otimes S_{\lambda_t}(S_2F) \\ &\xrightarrow{m} S_r(S_2F). \end{aligned}$$

DEFINITION 3.4. For a positive integer r, consider the composite map

$$D_rG \otimes D_rG \xrightarrow{d'_{(r,r)}} \Lambda^2 G \otimes \cdots \otimes \Lambda^2 G \xrightarrow{m} D_r(\Lambda^2 G)$$

where m is the multiplication in $D(\Lambda^2 G)$ and $d'_{(r,r)}$ is the co-Schur map associated to the partition (r, r). Define δ'_r to be $(1/r!)(m \cdot d'_{(r,r)})$. For a partition $\lambda = (\lambda_1, \lambda_1, \cdots, \lambda_t, \lambda_t)$ contained in Γ_r, the following composite map is denoted by δ'_λ :

$$\begin{aligned} D_\lambda G &= (D_{\lambda_1} G \otimes D_{\lambda_1} G) \otimes \cdots \otimes (D_{\lambda_t} G \otimes D_{\lambda_t} G) \\ &\xrightarrow{\delta'_{\lambda_1} \otimes \cdots \otimes \delta'_{\lambda_t}} D_{\lambda_1}(\Lambda^2 G) \otimes \cdots \otimes D_{\lambda_t}(\Lambda^2 G) \\ &\xrightarrow{m} D_r(\Lambda^2 G). \end{aligned}$$

Notice that $Im\,(m \cdot d'_{(r,r)})$ has $r!$ factors and the map δ'_r makes sense.

DEFINITION 3.5. For a positive integer l, δ_l'' is defined to be the map

$$\delta_l'' : \Lambda^l\phi \otimes \Lambda^l\phi \to S_l(S_2\phi)$$

where the $(2l-m)$th degree component of $\Lambda^l\phi \otimes \Lambda^l\phi$,

$$\sum_{r=l-m}^{l} \Lambda^{l-r}F \otimes D_rG \otimes \Lambda^{l-(2l-m-r)}F \otimes D_{2l-m-r}G$$

maps to

$$\sum_{r=l-m}^{[l-\frac{m}{2}]} S_{m+r-l}(S_2F) \otimes \Lambda^{2l-m-2r}(F \otimes G) \otimes D_r(\Lambda^2G)$$

of the same degree component of $S_l(S_2\phi)$. The map indicated above is that obtained by the compositions of the following maps in the case of $l-r \geq m+r-l$;

$$\Lambda^{l-r}F \otimes D_rG \otimes \Lambda^{m+r-l}F \otimes D_{2l-m-r}G$$
$$\xrightarrow{T} \Lambda^{l-r}F \otimes \Lambda^{m+r-l}F \otimes D_rG \otimes D_{2l-m-r}G$$
$$\xrightarrow{\triangle\otimes 1\otimes 1\otimes\triangle} \Lambda^{m+r-l}F \otimes \Lambda^{2l-m-2r}F \otimes \Lambda^{m+r-l}F \otimes D_rG \otimes D_rG \otimes D_{2l-m-2r}G$$
$$\xrightarrow{T} \Lambda^{m+r-l}F \otimes \Lambda^{m+r-l}F \otimes \Lambda^{2l-m-2r}F \otimes D_{2l-m-2r}G \otimes D_rG \otimes D_rG$$
$$\xrightarrow{\delta\otimes<,>\otimes\delta'} S_{m+r-l}(S_2F) \otimes \Lambda^{2l-m-2r}(F \otimes G) \otimes D_r(\Lambda^2G)$$

where T is the twisting map, $\triangle$ is the diagonalization, $\langle , \rangle$ is the natural pairing, and δ, δ' are the maps defined as above.

When $l-r < m+r-l$, it maps to

$$\sum_{r=l-\frac{m}{2}(m \text{ is even})}^{l} S_{l-r}(S_2F) \otimes \Lambda^{2r+m-2l}(F \otimes G) \otimes D_{2l-m-r}(\Lambda^2G)$$

or

$$\sum_{r=[l-\frac{m}{2}+1](m \text{ is odd})}^{l} S_{l-r}(S_2F) \otimes \Lambda^{2r+m-2l}(F \otimes G) \otimes D_{2l-m-r}(\Lambda^2G).$$

For a partition $\lambda = (\lambda_1, \lambda_1, \cdots, \lambda_t, \lambda_t)$ contained in Γ_r, the following composite map is denoted by δ''_λ:

$$\begin{aligned}\Lambda_\lambda \phi &= \Lambda^{\lambda_1}\phi \otimes \Lambda^{\lambda_1}\phi \otimes \cdots \otimes \Lambda^{\lambda_t}\phi \otimes \Lambda^{\lambda_t}\phi \\ &\xrightarrow{\delta''_{\lambda_1} \otimes \cdots \otimes \delta''_{\lambda_t}} S_{\lambda_1}(S_2\phi) \otimes \cdots \otimes S_{\lambda_t}(S_2\phi) \\ &\xrightarrow{m} S_r(S_2\phi).\end{aligned}$$

DEFINITION 3.6. For a partition λ contained in Γ_r, the subcomplexes $\mathbf{M}_\lambda$ and $\dot{\mathbf{M}}_\lambda$ of $S_r(S_2\phi)$ are defined as

$$\mathbf{M}_\lambda = \sum_{\mu \in \Gamma_r,\ \mu \geq \lambda} Im\ \delta''_\mu\ ,\ \dot{\mathbf{M}}_\lambda = \sum_{\mu \in \Gamma_r,\ \mu > \lambda} Im\ \delta''_\mu .$$

Note that $\mathbf{M}_{(1,\cdots,1)} = S_r(S_2\phi)$, so $\{\mathbf{M}_\lambda\}_{\lambda \in \Gamma_r}$ is a filtration of $S_r(S_2\phi)$

THEOREM 3.7. *For an arbitrary commutative ring R and an arbitrary non-negative integer r, $\{\mathbf{M}_\lambda\}_{\lambda \in \Gamma_r}$ is a filtration of $S_r(S_2\phi)$ such that its associated graded complex is $\sum_{\lambda \in \Gamma_r} L_\lambda \phi$.*

Proof. We have only to show that for any partition λ in Γ_r, $\mathbf{M}_\lambda / \dot{\mathbf{M}}_\lambda$ is isomorphic to $L_\lambda \phi$. Consider the commutative diagram

$$\begin{array}{ccc} \Lambda_\lambda \phi & \xrightarrow{\delta_\lambda''} & \mathbf{M}_\lambda \subseteq S_r(S_2\phi) \\ {\scriptstyle d_\lambda}\downarrow & \searrow^{\phi_\lambda} & \downarrow{\scriptstyle \rho_\lambda} \\ L_\lambda \phi & & \mathbf{M}_\lambda / \dot{\mathbf{M}}_\lambda \end{array}$$

where ρ_λ is the projection and ϕ_λ is the composite map $\rho_\lambda \circ \delta''_\lambda$. To construct the isomorphism $L_\lambda \phi \cong \mathbf{M}_\lambda / \dot{\mathbf{M}}_\lambda$, first we will prove that Ker d_λ is contained in Ker ϕ_λ. In short, we have to show that $\phi_\lambda \circ \square_\lambda = 0$ for each partition λ contained in Γ_r. Let $\lambda = (a_1, a_2, \cdots, a_{2d})$, where $a_1 = a_2 = \lambda_1, \cdots, a_{2d-1} = a_{2d} = \lambda_d$. Then the map $\square_\lambda$ is

$$\sum_{t=1}^{2d-1} \sum_{\nu=1}^{a_{t+1}} \Lambda^{a_1}\phi \otimes \cdots \otimes \Lambda^{a_{t-1}}\phi \otimes \Lambda^{a_t+\nu}\phi \otimes \Lambda^{a_{t+1}-\nu}\phi \otimes \cdots \otimes \Lambda^{a_{2d}}\phi$$
$$\xrightarrow{\square_\lambda = \sum_{t=1}^{2d-1} \sum_{\nu=1}^{a_{t+1}} 1^{\otimes(t-1)} \otimes \square_\nu \otimes 1^{\otimes(2d-t-1)}} \Lambda_\lambda \phi .$$

In order to prove $\phi_\lambda \circ \square_\lambda = 0$, we have only to show that $\phi_\lambda \circ \{1^{\otimes(t-1)} \otimes \square_\nu \otimes 1^{\otimes(2d-t-1)}\} = 0$ for every t and ν. Consider the following two cases:

Case1. Suppose that t is odd. Let $i = \frac{t+1}{2}$. Then $1 \leq \nu \leq \lambda_i$. In this case, $1^{\otimes(t-1)} \otimes \square_\nu \otimes 1^{\otimes(2d-t-1)}$ is the map

$$\Lambda^{\lambda_1}\phi \otimes \Lambda^{\lambda_1}\phi \otimes \cdots \otimes (\Lambda^{\lambda_i+\nu}\phi \otimes \Lambda^{\lambda_i-\nu}\phi) \otimes \cdots \otimes \Lambda^{\lambda_d}\phi \otimes \Lambda^{\lambda_d}\phi$$
$$\xrightarrow{1^{\otimes(t-1)}\otimes\square_\nu\otimes 1^{\otimes(2d-t-1)}} \Lambda_\lambda\phi.$$

The fact that the next composite map

$$\Lambda^{\lambda_i+\nu}\phi \otimes \Lambda^{\lambda_i-\nu}\phi \xrightarrow{\square_\nu} \Lambda^{\lambda_i}\phi \otimes \Lambda^{\lambda_i}\phi \xrightarrow{\delta''_{\lambda_i}} S_{\lambda_i}(S_2\phi)$$

is equal to the 0-morphism implies that

$$\phi_\lambda \circ \{1^{\otimes(t-1)} \otimes \square_\nu \otimes 1^{\otimes(2d-t-1)}\} = 0.$$

So we want to show that

$$\Lambda^{\lambda_i+\nu}\phi \otimes \Lambda^{\lambda_i-\nu}\phi \xrightarrow{\square_\nu} \Lambda^{\lambda_i}\phi \otimes \Lambda^{\lambda_i}\phi \xrightarrow{\delta''_{\lambda_i}} S_{\lambda_i}(S_2\phi)$$

is the 0-morphism. We proceed by induction on λ_i.
If $\lambda_i = 1$, then $\nu = 1$. We know that $\Lambda^2\phi \to \Lambda^1\phi \otimes \Lambda^1\phi \to S_2\phi$ is zero.
Suppose $\lambda_i > 1$. If $\nu = 1$, the diagram is commutative:

$$\begin{array}{ccccc}
\Lambda^{\lambda_i+1}\phi \otimes \Lambda^{\lambda_i-1}\phi & \xrightarrow{\square} & \Lambda^{\lambda_i}\phi \otimes \Lambda^{\lambda_i}\phi & \xrightarrow{\delta''} & S_{\lambda_i}(S_2\phi) \\
\downarrow{\scriptstyle \Delta\otimes 1} & & & & \uparrow \\
\Lambda^2\phi\otimes\Lambda^{\lambda_i-1}\phi\otimes\Lambda^{\lambda_i-1}\phi & & & \xrightarrow{\Delta\otimes 1} & \Lambda^1\phi\otimes\Lambda^1\phi \\
 & & & & \otimes\Lambda^{\lambda_i-1}\phi\otimes\Lambda^{\lambda_i-1}\phi
\end{array}$$

Hence $\delta''_{\lambda_i} \circ \square_\nu = 0$. In the case that $\nu > 1$, the diagram is commutative:

$$\begin{array}{ccccc}
\Lambda^{\lambda_i+\nu}\phi \otimes \Lambda^{\lambda_i-\nu}\phi & \xrightarrow{\square} & \Lambda^{\lambda_i}\phi \otimes \Lambda^{\lambda_i}\phi & \xrightarrow{\delta''} & S_{\lambda_i}(S_2\phi) \\
\downarrow & & & & \uparrow \\
\Lambda^2\phi\otimes\Lambda^{\lambda_i+\nu-2}\phi\otimes\Lambda^{\lambda_i-\nu}\phi & & & \xrightarrow{\Delta\otimes\square} & \Lambda^1\phi\otimes\Lambda^1\phi \\
 & & & & \otimes\Lambda^{\lambda_i-1}\phi\otimes\Lambda^{\lambda_i-1}\phi
\end{array}$$

Hence $\delta''_{\lambda_i} \circ \square_\nu = 0$.

Case 2. Suppose that t is even. We show that the image of the composite map

$$\Lambda^a\phi \otimes \Lambda^{a+\nu}\phi \otimes \Lambda^{b-\nu}\phi \otimes \Lambda^b\phi \xrightarrow{1\otimes\square\otimes 1} \Lambda_\mu\phi \xrightarrow{\delta''} \mathbf{M}_\mu$$

where $\mu = (a,a,b,b)$ and $1 \leq \nu \leq b \leq a$, is contained in $\dot{\mathbf{M}}_\mu$. We know that the diagram

$$\begin{array}{ccc}
\Lambda^a\phi \otimes \Lambda^{a+\nu}\phi \otimes \Lambda^{b-\nu}\phi \otimes \Lambda^b\phi & \xrightarrow{1\otimes\square\otimes 1} & \Lambda^a\phi \otimes \Lambda^a\phi \otimes \Lambda^b\phi \otimes \Lambda^b\phi \\
\tilde{\square}\downarrow & & \delta''\downarrow \\
\Lambda^{a+\nu}\phi \otimes \Lambda^{a+\nu}\phi \otimes \Lambda^{b-\nu}\phi \otimes \Lambda^{b-\nu}\phi & \xrightarrow{\delta''} & S_{a+b}(S_2\phi)
\end{array}$$

is commutative.
So we have

$$\begin{aligned}
&\delta''(1\otimes\square\otimes 1)(\Lambda^a\phi \otimes \Lambda^{a+\nu}\phi \otimes \Lambda^{b-\nu}\phi \otimes \Lambda^b\phi) \\
&= \delta''(\tilde{\square}(\Lambda^a\phi \otimes \Lambda^{a+\nu}\phi \otimes \Lambda^{b-\nu}\phi \otimes \Lambda^b\phi)) \subseteqq \dot{\mathbf{M}}_\mu.
\end{aligned}$$

Hence, we can construct the map ι_λ such that the following diagram is commutative:

$$\begin{array}{ccc}
\Lambda_\lambda\phi & \xrightarrow{\delta''_\lambda} & \mathbf{M}_\lambda \subseteqq S_r(S_2\phi) \\
d_\lambda\downarrow & \searrow^{\phi_\lambda} & \downarrow\rho_\lambda \\
L_\lambda\phi & \overset{\iota_\lambda}{\dashrightarrow} & \mathbf{M}_\lambda/\dot{\mathbf{M}}_\lambda
\end{array}$$

ι_λ is surjective for each λ contained in Γ_r. Because $\sum_{\lambda\in\Gamma_r} L_\lambda\phi$ and $S_r(S_2\phi)$ are free complexes of the same rank (See Lemma 3.8 below) and by the universal freeness described in Theorem 2.10, the ranks of both complexes are independent of the coefficient ring R, so the $\iota'_\lambda s$ must be isomorphisms for each λ. $\square$

LEMMA 3.8. *rank* $(\sum_{\lambda\in\Gamma_r} L_\lambda\phi) =$ *rank* $S_r(S_2\phi)$.

Proof. Suppose that $\phi : G \to F$ is a R-homomorphism of finitely generated free modules of rank G=m and rank F=n. Then it follows from [4, 11] that we have $S_2\phi : 0 \to \Lambda^2 G \to F\otimes G \to S_2 F \to 0$, $S(S_2\phi) = S(S_2F)\otimes\Lambda(F\otimes G)\otimes D(\Lambda^2 G)$ and thus $S_r(S_2\phi) = \sum_{p+q+s=r} S_p(S_2F)\otimes\Lambda^q(F\otimes G)\otimes D_s(\Lambda^2G)$. Hence

$$\mathrm{rank}S_r(S_2\phi) = \sum_{p+q+s=r} \mathrm{rank}\{S_p(S_2F)\otimes\Lambda^q(F\otimes G)\otimes D_s(\Lambda^2G)\}.$$

But the plethysm formulas [16] and Lemma I.4.10 of [11] give us rank $S_p(S_2F) = \sum_{\mu\in\Gamma_p} s_{\tilde{\mu}}(1_n) = \sum_{\mu\in\Gamma_p}$ rank $L_\mu F$ (Here $s_{\tilde{\mu}}(1_n)$ is the value of Schur function in n variables $s_{\tilde{\mu}}(x_1,\cdots,x_n)$ at $x_1=\cdots=x_n=1$). Also,

$$\mathrm{rank}\Lambda^q(F\otimes G) = \sum_{|\sigma|=q}\mathrm{rank}L_\sigma F\otimes K_\sigma G = \sum_{|\sigma|=q} s_{\tilde{\sigma}}(1_n)\ s_\sigma(1_m)$$

and

$$\mathrm{rank}D_s(\Lambda^2G) = \sum_{\rho\in\Gamma_s} s_\rho(1_m) = \sum_{\rho\in\Gamma_s}\mathrm{rank}K_\rho G.$$

So

$$\begin{aligned}&\mathrm{rank}S_r(S_2\phi)\\ &= \sum_{p+q+s=r}\sum_{\mu\in\Gamma_p} s_{\tilde{\mu}}(1_n)\sum_{|\sigma|=q} s_{\tilde{\sigma}}(1_n)\ s_\sigma(1_m)\sum_{\rho\in\Gamma_s} s_\rho(1_m).\end{aligned}$$

On the other hand,

$$\mathrm{rank}L_\lambda\phi = \sum_{\gamma\subseteqq\lambda}\mathrm{rank}L_\gamma F\otimes K_{\lambda/\gamma}G = \sum_{\gamma\subseteqq\lambda} s_{\tilde{\gamma}}(1_n)\ s_{\lambda/\gamma}(1_m).$$

We have only to show the following claim:

CLAIM.

$$\sum_{\lambda\in\Gamma_r}\sum_{\gamma\subseteqq\lambda} s_{\tilde\gamma}(1_n)\ s_{\lambda/\gamma}(1_m)$$
$$=\sum_{p+q+s=r}\ \sum_{\mu\in\Gamma_p} s_{\tilde\mu}(1_n)\sum_{|\sigma|=q} s_{\tilde\sigma}(1_n)\ s_\sigma(1_m)\sum_{\rho\in\Gamma_s} s_\rho(1_m)$$

Proof of the claim.
$\sum_{\tilde\lambda:\ even} s_{\lambda/\gamma}=\prod_{i<j}(1-x_ix_j)^{-1}\ \sum_{\tilde\mu:\ even} s_{\gamma/\mu}$
But $\prod_{i<j}(1-x_ix_j)^{-1}=\sum_{\tilde\rho:\ even} s_\rho$.
Hence

$$\sum_{\tilde\lambda:\ even} s_{\lambda/\gamma}(1_m)=\sum_{\tilde\rho:\ even} s_\rho(1_m)\sum_{\tilde\mu:\ even} s_{\gamma/\mu}(1_m)$$

So

$$\sum_{\tilde\lambda:\ even} s_{\lambda/\gamma}(1_m)s_{\tilde\gamma}(1_n)=\sum_{\tilde\rho:\ even} s_\rho(1_m)\sum_{\tilde\mu:\ even} s_{\gamma/\mu}(1_m)s_{\tilde\gamma}(1_n)$$
$$=\sum_{\tilde\rho:\ even} s_\rho(1_m)\sum_{\tilde\mu:\ even}\sum_\sigma s_\sigma(1_m)C^\gamma_{\mu\sigma}\ s_{\tilde\gamma}(1_n)$$

(Here $s_{\gamma/\mu}=\sum_\sigma\ C^\gamma_{\mu\sigma}\ s_\sigma$ where $s_\mu s_\sigma=\sum_\gamma C^\gamma_{\mu\sigma}\ s_\gamma$.)
Therefore,

$$\sum_{\gamma\subseteqq\lambda}\sum_{\tilde\lambda:\ even} s_{\lambda/\gamma}(1_m)s_{\tilde\gamma}(1_n)$$
$$=\sum_{\tilde\rho:\ even} s_\rho(1_m)\sum_{\tilde\mu:\ even}\sum_\sigma s_\sigma(1_m)\sum_{\gamma\subseteqq\lambda} C^\gamma_{\mu\sigma}\ s_{\tilde\gamma}(1_n)$$
$$=\sum_{\tilde\rho:\ even} s_\rho(1_m)\sum_{\tilde\mu:\ even}\sum_\sigma s_\sigma(1_m)s_{\tilde\mu}(1_n)s_{\tilde\sigma}(1_n)$$
$$=\sum_{\tilde\mu:\ even} s_{\tilde\mu}(1_n)\sum_\sigma s_\sigma(1_m)\ s_{\tilde\sigma}(1_n)\sum_{\tilde\rho:\ even} s_\rho(1_m).$$

This completes the proof. □

As special cases of the Theorem 3.7 in terms of Remark 2.15, we obtain the plethysm formulas [1, 6, 13].

COROLLARY 3.9. *Let R be any commutative ring with unity, and let F denote any finitely generated free R-module. Then one has the isomorphisms up to filtrations:*
(a) $S_r(S_2F) \cong \sum_{\lambda\in\Gamma_r} L_\lambda F$
(b) $S_r(\Lambda^2F) \cong \sum_{\lambda\in\Gamma_r} K_\lambda F$

Using the contravariant duality to the plethysms of Corollary 3.5 we have

COROLLARY 3.10. *Let R be any commutative ring with unity, and let F denote any finitely generated free R-module. Then one has the isomorphisms up to filtrations:*
(a) $D_r(D_2F) \cong \sum_{|\lambda|=r} K_{2\lambda}F$
(b) $D_r(\Lambda^2F) \cong \sum_{|\lambda|=r} K_{\tilde{2\lambda}}F$
where $2\lambda = 2(\lambda_1, \lambda_2, \cdots, \lambda_q) = (2\lambda_1, 2\lambda_2, \cdots, 2\lambda_q)$.

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