구조동역학-열탄성학 연성문제의 유한요소 정식화 및 분류

The Finite Element Formulation and Its Classification of Dynamic Thermoelastic Problems of Solids

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요 지

본 논문은 구조물의 동역학 및 열탄성 연성문제 해석을 위한 통합된 유한요소법을 개발하는데 초점을 두고 있다. 첫째로, 열전도 방정식에 열변위라는 물리량을 도입하여 동역학의 운동 방정식과 유사하도록 유도한 후, 변분법과 일반좌표계를 이용하여 시간영역에서 정식화하였다. 둘째로, 두 방정식에 라플라스 변환을 동시에 도입하고, 공간변수만을 갖는 형상함수와 가중잔여법을 적용하여 유한요소식을 변환영역에서 표현하였다. 연성된 방정식을 문제의 특성에 따라서 분류하였고 정식화 과정을 검증하였다. 또한 수치해석 알고리듬이 갖는 수치 역변환의 정성적인 경향에 대하여 검토하였다.

핵심용어 : 열탄성, 구조 동역학, 전이 유한요소법, 변분법, 가중잔여법, 라플라스 변환, 수치 역변환

Abstract

This paper is for the first essential study on the development of unified finite element formulations for solving problems related to the dynamics/thermoelastics behavior of solids. In the first part of formulations, the finite element method is based on the introduction of a new quantity defined as heat displacement, which allows the heat conduction equations to be written in a form equivalent to the equation of motion, and the equations of coupled thermoelasticity to be written in a unified form. The equations obtained are used to express a variational formulation which, together with the concept of generalized coordinates, yields a set of differential equations with the time as an independent variable. Using the Laplace transform, the resulting finite element equations are described in the transform domain. In the second, the Laplace transform is applied to both the equation of heat conduction derived in the first part and the equations of motions and their corresponding boundary conditions, which is referred to the transformed equation. Selections of interpolation functions dependent on only the space variable and an application of the weighted residual method to the coupled equation result in the necessary finite element matrices in the transformed domain. Finally, to prove the validity of two approaches, a comparison with one finite element equation and the other is made term by term.

Keywords : *thermal elasticity, structural dynamics, transfinite element method, variational approach, weighted residual approach, laplace transform, numerical inversion*

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• 이 논문에 대한 토론을 2000년 6월 30일까지 본 학회에 보내주시면 2000년 9월호에 그 결과를 게재하겠습니다.

1. Introduction

1.1 The Conventional Approach

The thermal and structural modeling and their analyses are of practical importance to structural engineers concerned with problems related to temperature-induced displacements and the associated structural dynamic response. The basic theory for thermoelastic problems has been well established, as in the case of heat conduction, and analytical methods present difficulties for bodies of complex structural configurations or under complicated boundary conditions.[1] In the existing variational formulations equations of coupled thermoelasticity are treated as two separate equations with coupling terms appearing explicitly in the heat conduction equation.

1.2 Formulations of Coupled Thermoelastic Equations

By using definitions on mechanical quantities, heat displacement and heat strain,[2),3)] the equation of coupled thermoelasticity is written in such a form that the equation of motion for mechanics and the equation of equilibrium for the heat conduction are equivalent, with the coupling term appearing explicitly in the constitutive relations. Such formulations are appropriate since the behavior of a material is expressed through its constitutive relations, but not through the equation of motion nor equilibrium. A variational equation is next derived in the use of generalized coordinates and thus leads to a unified equation for the general case of coupled thermoelasticity.[4)] Accordingly, it can be used for deriving a finite element formulation for thermoelasticity problems.

1.3 Transfinite Formulations

Based on the variational approach, the thermal and structural finite element formulation for an investigation of the transient thermal and structural response are quite complex and very time-consuming for several reasons, principally the time-dependent nature of the problem. This fact is especially true when analyzing large complex structural configurations due to thermal effects, which require analyses to be carried out for a long duration, thereby escalating computational times and analysis cost. Furthermore, these analyses require step-by-step time-marching algorithms and estimations of time step.

In view of these conditions, the introduction of an effective methodology, referred as 'the transform finite element method,' will be presented and is a viable alternative to existing techniques.[5)] The Laplace transform finite element methodology and the Fourier transform finite element methodology are typical classifications of finite element formulations in the corresponding transformed domains. Such formulations are referred to as 'transfinite formulations,' and the corresponding finite elements are referred to as 'transfinite elements.' Although the associated study has focused on the simple problems, linear uncoupled cases, of structural or thermal dynamics, research currently under way has some difficulties in applying to nonlinear problems and cases for various kinds of thermal excitations.[6),7)]

A unique feature of the approach is the use of a common numerical methodology for each of the interdisciplinary areas, for example, thermal and structural areas, via transform methods in conjunction with the conventional finite element formulation such as a weighted residual scheme.[8)] The region under consideration is first idealized as a finite number of discrete elements for both the thermal and structural models. Therein, numerical computations for each element in the transform domain yield element matrices which are subsequently assembled in the conventional manner and then solved in the transform domain itself. To obtain the structural response due to thermal considerations, the solution is numerically inverted only in the final structural formulations at desired times of interest.[9] Recent work concentrates on the improvements of numerical stabilities and accuracies for the real time responses in the transformed domain.[10),11)]

The dynamic finite element equations obtained by a variational principle are converted to the transfinite formulation via the Laplace transform to check if that formulation makes the exact agreement in comparison with the resulting finite element equations from the transform finite element method. Accordingly, this paper will describe a detailed development of the unified finite element formulation for the thermal and structural dynamics as the first indispensable step to solve thermal elastics problems of solids.

2. New Concept on Thermoelasticity

2.1 Heat Displacement and Heat Strain

First, consider an elastic medium subjected to pure heating. The medium is initially at a state of uniform temperature $T_o(x_i)$, where x_i is the Cartesian coordinate of a material particle and T_o the reference temperature. A dimensionless temperature change $\theta(x_i, t)$ is denoted as

$$\theta=\frac{1}{T_o}(T-T_o), \tag{1}$$

where $T(x_i, t)$ is the instantaneous temperature of the particle and t the time.[12)] The heat conduction equation with a heat source Q in a continuum has the form

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial\theta}{\partial x_j}\right)+\frac{Q}{T_o}=\rho c_v\frac{\partial\theta}{\partial t}, \tag{2}$$

where $K_{ij}=K_{ji}$ is the thermal conductivity tensor, ρ the constant mass density, c_v the specific heat at constant volume of material and $Q(x_i, t)$ the heat energy per unit volume. The heat displacement is defined as a vector field $H_i(x_i, t)$

$$\theta=\frac{\partial H_i}{\partial x_i}=H_{i,i}\,. \tag{3}$$

The vector field H_i has the dimension of displacement and equation (3) resembles the definition of a mechanical volume strain.

2.2 Heat Stress and Alternate Heat Equation

For a reversible process without mechanical deformation, the change in specific entropy is given by[13)]

$$\Delta s=\int_{T_o}^{T}\frac{c_v dT}{T}=c_v H_{i,i}\,. \tag{4}$$

A quantity $\breve{\sigma}$, the heat stress, is introduced by

$$\breve{\sigma}=\rho c_v T_o\theta=\rho T_o\Delta s\,. \tag{5}$$

The heat conduction can be written in the following form:

$$\frac{\partial}{\partial x_i}\left(\frac{K_{ij}}{\rho c_v T_o}\frac{\partial\breve{\sigma}}{\partial x_j}\right)+\frac{Q}{T_o}=\rho c_v\frac{\partial^2 H_i}{\partial x_i\partial t}\,. \tag{6}$$

In addition, a vector field $S_i(x_i, t)$ in terms of heat energy Q is introduced by

$$Q=\frac{\partial S_i}{\partial x_i}=S_{i,i}. \quad (7)$$

Equation (6) becomes

$$\frac{\partial \tilde{\sigma}}{\partial x_j}+C\lambda_{ij}S_i=C^2T_o\lambda_{ij}\frac{\partial H_i}{\partial t}, \quad (8)$$

where $\lambda_{ij}=(K_{ij})^{-1}$ is the thermal resistivity tensor and $C=\rho c_v$ the heat capacity per unit volume. Note that equation (8) is similar to the mechanical equation of motion with body force except that the variation of heat stress yields a diffusion effect instead of an inertia effect. This alternate formulation of a thermal process behaves itself like a damped mechanical system with the negligible mass inertia. The quantity $C\lambda_{ij}S_i$ occupies an equivalent position of a mechanical body force.

2.3 Coupled Thermoelastic Equation

To extend the concept of heat displacement to the formulation of coupled thermoelastic problems, the total mechanical strain e_{ij} is represented by the sum of the strain e_{ij}^M caused by mechanical forces and the strain e_{ij}^T by thermal expansion

$$e_{ij}=\frac{1}{2}(U_{i,j}+U_{j,i})=e_{ij}^M+e_{ij}^T. \quad (9)$$

If e_{ij}^T is written explicitly in terms of θ, the mechanical strain e_{ij}^M is expressed by

$$e_{ij}^M=\frac{1}{2}(U_{i,j}+U_{j,i})-\beta_{kl}T_o(C_{ijkl})^{-1}\theta \quad (10)$$

and $$\beta_{ij}=\alpha_{kl}C_{ijkl}, \quad (11)$$

where $U_i(x_i,t)$ is the mechanical displacement vector, β_{ij} the thermal moduli, α_{kl} the anisotropic coefficients of thermal expansion, and C_{ijkl} the elastic constants of the material. The mechanical stress-strain relationship has the following form:

$$\sigma_{ij}=C_{ijkl}e_{kl}^M=C_{ijkl}e_{kl}-\beta_{ij}T_o\theta. \quad (12)$$

Together with mechanical deformations, the change in specific entropy for small temperature variation should also be redefined. It can be obtained from the energy equation followed by

$$\Delta s=c_v\theta+\frac{\beta_{ij}e_{ij}}{\rho}. \quad (13)$$

Accordingly, the heat displacement vector $H_i(x_i,t)$ is redefined to include effects due to mechanical deformation

$$H_{i,i}=\theta+\frac{\beta_{ij}e_{ij}}{C}. \quad (14)$$

Compared to the equation (10), the above $H_{i,i}$ can be interpreted as the total heat strain and $\beta_{ij}e_{ij}/C$ as the strain induced by the mechanical deformation. The associated heat stress becomes

$$\tilde{\sigma}=CT_o\theta=CT_oH_{i,i}-T_o\beta_{ij}e_{ij}. \quad (15)$$

The second term $T_o\beta_{ij}e_{ij}$ in equation (15) represents a thermomechanical coupling effect.

An analysis including both the thermal expansion and the coupling term in equation (10) is usually referred as the coupled thermoelasticity. If only the thermal expansions are considered, while neglecting the effect of thermomechanical coupling, the analysis is then an uncoupled thermoelastic problem. Equations of motion and heat conduction are

$$\rho\ddot{U}_i=\sigma_{ij,j}+B_i \quad (16)$$

and $$C^2T_o\lambda_{ij}\dot{H}_i=\tilde{\sigma}_{,j}+C\lambda_{ij}S_i, \quad (17)$$

where ($\dot{}$) is the derivative with respect to time t, B_i the body force tensor and S_i the tensor field defined in equation (7).[3] Equations (16) and (17) have similar formulation forms. Thus, it is possible to formulate a unified variational equation which treats both the heat displacement and the mechanical displacement as elements of a generalized displacement vector.

2.4 The Variational Approach

Consider a variation δU_i of the mechanical displacement U_i with a corresponding variation of strain δe_{ij} and a variation δH_i of the heat displacement H_i with a corresponding variation of heat strain $\delta\theta$. A variational form of equations of motion and heat conduction has the form

$$\int_{\Omega_n}(\sigma_{ij,j}+B_i-\rho\ddot{U}_i)\delta\ddot{U}_i d\Omega + \int_{\Omega_n}(\tilde{\sigma}_{,i}+C\lambda_{ij}S_j-C^2T_o\lambda_{ij}\dot{H}_j)\delta H_i d\Omega=0, \tag{18}$$

where Ω_n is a control volume. Employing the divergence theorem, the first term in equation (18) becomes

$$\begin{aligned}\int_{\Omega_n}(\sigma_{ij,j}+B_i-\rho\ddot{U}_i)\delta\ddot{U}_i d\Omega= & -\int_{\Omega_n}\rho\ddot{U}_i\delta U_i d\Omega-\int_{\Omega_n}\sigma_{ij}\delta e_{ij}d\Omega \\ & +\int_{\Gamma_n}\sigma_{ij}n_j\delta U_i d\Gamma+\int_{\Omega_n}B_i\delta U_i d\Omega,\end{aligned} \tag{19}$$

where n_i is the outward unit normal vector of the control surface Γ_n. Similarly, the second integral term in equation (18) becomes[10]

$$\begin{aligned}\int_{\Omega_n}(\tilde{\sigma}_{,i}+C\lambda_{ij}S_j-C^2T_o\lambda_{ij}\dot{H}_i)\delta H_i d\Omega= & -\int_{\Omega_n}C^2T_o\lambda_{ij}\dot{H}_i\delta H_i d\Omega \\ & -\int_{\Omega_n}CT_o\theta\delta\theta d\Omega \\ & -\int_{\Omega_n}T_o\beta_{ij}\theta\delta e_{ij}d\Omega+\int_{\Gamma_n}\tilde{\sigma}n_i\delta H_i d\Gamma \\ & +\int_{\Omega_n}C\lambda_{ij}S_j\delta H_i d\Omega.\end{aligned} \tag{20}$$

2.5 Energy Functions

The total energy function W is expressed as the product of stress and strain, then the function satisfies

$$W=\frac{1}{2}\sigma_{ij}e_{ij}+\frac{1}{2}\tilde{\sigma}\theta. \tag{21}$$

Alternately, writing W in terms of strain by substituting equations (12) and (15) into (21) yield

$$W=\frac{1}{2}C_{ijkl}e_{ij}e_{kl}+\frac{1}{2}CT_o\theta^2. \tag{22}$$

The potential function V defined by

$$V=\frac{1}{2}\int_{\Omega_n}(C_{ijkl}e_{ij}e_{kl}+CT_o\theta^2)d\Omega, \tag{23}$$

which for isothermal deformation($\theta=0$) has the physical meaning of a strain energy function, and for zero strain it reduces to a thermal potential. Taking the first variation of equation (23) yields

$$\delta V=\int_{\Omega_n}C_{ijkl}e_{ij}\delta e_{kl}d\Omega+\int_{\Omega_n}CT_o\delta\theta d\Omega. \tag{24}$$

Hence, equations (19) and (20) are substituted into equation (18) and then equation (24) is used in the resulting equation(18) to give

$$\begin{aligned}&\int_{\Omega_n}\rho\ddot{U}_i\delta U_i d\Omega+\int_{\Omega_n}C^2T_o\lambda_{ij}\dot{H}_j\delta H_i d\Omega+\delta V \\ &=\int_{\Gamma_n}(\sigma_{ij}n_j\delta U_i+\tilde{\sigma}n_i\delta H_i)d\Gamma \\ &+\int_{\Gamma_n}(B_i\delta U_i+C\lambda_{ij}S_j\delta H_i)d\Omega.\end{aligned} \tag{25}$$

Then, the equation (25) can be used as a variational form for coupled thermoelasticity problems.

2.6 Generalized Coordinates and Lagrangian Equations

An advantage of introductions of generalized coordinates into the variational principle has been well discussed in the classical mechanics. As an example of their applications, nodal displacements or nodal rotations may be regarded as generalized coordinates in the variation formulation of the finite element analysis. Mechanical and heat displacements can be represented by the following forms, respectively:

$$U_i = U_i(q_1, \ldots, q_n, x_i, t),$$
$$H_i = H_i(q_1, \ldots, q_n, x_i, t), \quad i=1,2,3, \tag{26a, b}$$

which are given functions of three space coordinates x_i's, time t and n parameters q_j's.[5] Parameters are unknown functions of time and will be considered as the generalized coordinates.

Then, variations of the field U_i, H_i and V with respect to generalized coordinates q_j's ($j=1, \cdots, n$) are of forms

$$\delta U_i = \frac{\partial U_i}{\partial q_j}\delta q_j,\ \delta H_i = \frac{\partial H_i}{\partial q_j}\delta q_j,$$
$$\delta V = \frac{\partial V}{\partial q_j}\delta q_j,\ i=1,2,3. \tag{27a, b, c}$$

Generalized forces F_i and Q_i have the following forms from right hand side terms in equation (25), respectively:

$$F_i\delta q_j = \left[\int_{\Gamma_n}\left(\sigma_{ij} n_j \frac{\partial U_i}{\partial q_j} + \tilde{\sigma} n_i \frac{\partial H_i}{\partial q_j}\right)d\Gamma\right]\delta q_j, \tag{28}$$

$$Q_i\delta q_j = \left[\int_{\Omega_n}\left(B_i \frac{\partial U_i}{\partial q_j} + C\lambda_{ij} S_j \frac{\partial H_i}{\partial q_j}\right)d\Omega\right]\delta q_j. \tag{29}$$

By using the procedure of Lagrangian mechanics, the first term in equation (25) can be written in terms of the kinetic energy functions as follows:

$$\int_{\Omega_n}\rho\, \ddot{U}_i \delta U_i d\Omega = \int_{\Omega_n}\rho\, \ddot{U}_i \frac{\partial U_i}{\partial q_j}\delta q_j d\Omega. \tag{30}$$

From the next characteristics of the differential calculus,

$$\ddot{U}_i \frac{\partial U_i}{\partial q_j} = \frac{d}{dt}\left(\dot{U}_i \frac{\partial U_i}{\partial q_j}\right) - \dot{U}_i \frac{d}{dt}\left(\frac{\partial U_i}{\partial q_j}\right)$$
$$= \frac{d}{dt}\left(\dot{U}_i \frac{\partial \dot{U}_i}{\partial \dot{q}_j}\right) - \dot{U}_i \frac{\partial \dot{U}_i}{\partial q_j}, \tag{31}$$

the kinematic energy expressed as

$$K = \frac{1}{2}\int_{\Omega_n}\rho\, \dot{U}_i \dot{U}_i d\Omega, \tag{32}$$

and the equation (30) becomes

$$\int_{\Omega_n}\rho\, \ddot{U}_i \delta U_i d\Omega = \left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right) - \frac{\partial K}{\partial q_i}\right]\delta q_j. \tag{33}$$

Similarly, if the relation, $\partial \dot{H}_i / \partial \dot{q}_j = \partial H_i / \partial q_j$, is utilized, then the second term in equation (25)

$$\delta D = \frac{\partial D}{\partial \dot{q}_k}\delta q_k = \int_{\Omega_n} C^2 T_o \lambda_{ij} \dot{H}_j \frac{\partial \dot{H}_i}{\partial \dot{q}_k}\delta q_k d\Omega, \tag{34}$$

where $D = \frac{1}{2}\int_{\Omega_n} C^2 T_o \lambda_{ij} \dot{H}_i \dot{H}_j d\Omega.$ (35)

Together with $\delta V = (\partial V / \partial q_j)\delta q_j$, substituting equations (28), (29), (33) and (34) into equation (25) yields

$$\left[\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_j}\right)-\frac{\partial K}{\partial q_j}+\frac{\partial D}{\partial \dot{q}_j}+\frac{\partial V}{\partial q_j}\right]\delta q_j = (F_i + Q_i)\delta q_j \tag{36}$$

The equation (36) represents the Lagrangian equation of the thermoelastic problem. It constitutes a system of n differential equations corresponding to n generalized coordinates $q_j(t)$'s.

2.7 A Finite Element Approach to the Variational Equation

Consider a special case in which displacements can be represented by a linear combination of the generalized coordinates,

$$U_k(x_l, t) = \sum_{i=1}^{n} q_i(t) f_{ik}(x_l),$$
$$H_k(x_l, t) = \sum_{i=1}^{n} p_i(t) g_{ik}(x_l), \tag{37a, b}$$

where $p_i(t)$ and $q_i(t)$ are n generalized coordinates, $f_{ik}(x)$ and $g_{ik}(x)$ are interpolation functions(i=1, ⋯, n, k=l=1, 2, 3), respectively, therefore equation (37a, b) may be viewed as a general form of finite element formulation. Corresponding strains are

$$e_{kl} = \frac{1}{2} q_i(f_{ik,l} + f_{il,k}),$$
$$\theta = p_i g_{ik,k} - \frac{\beta_{kl}}{C} e_{kl}. \tag{38a, b}$$

The potential energy function V can then be written expressed as

$$V = \frac{1}{2} a_{ij} q_i q_j - b_{ij} q_i p_j + \frac{1}{2} c_{ij} p_i p_j, \tag{39}$$

where

$$a_{ij} = a_{ji} = \frac{1}{4}\int_{\Omega_n}\left(C_{ijkl} + \frac{T_o}{C}\beta_{kl}\beta_{mn}\right)(f_{ik,l} + f_{il,k})(f_{jm,n} + f_{jn,m})d\Omega, \tag{40}$$

$$b_{ij} = \frac{1}{2}\int_{\Omega_n} T_o \beta_{kl}(f_{ik,l} + f_{il,k}) g_{jm,m} d\Omega, \tag{41}$$

$$c_{ij} = c_{ji} = \int_{\Omega_n} C\, T_o\, g_{ik,k}\, g_{jm,m} d\Omega. \tag{42}$$

The dissipation and kinetic energy functions are

$$D = \frac{1}{2} d_{ij} \dot{p}_i \dot{p}_j, \quad K = \frac{1}{2} m_{ij} \dot{q}_i \dot{q}_j, \tag{43a, b}$$

where

$$d_{ij} = d_{ji} = \int_{\Omega_n} C^2 T_o \lambda_{kl} g_{ik} g_{jl} d\Omega,$$
$$m_{ij} = m_{ji} = \int_{\Omega_n} \rho f_{ik} f_{jk} d\Omega. \tag{44a, b}$$

The variational equation (36) becomes in a matrix form

$$\begin{bmatrix} m_{ij} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \ddot{q}_j \\ \ddot{p}_j \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d_{ij} \end{bmatrix}\begin{bmatrix} \dot{q}_j \\ \dot{p}_j \end{bmatrix} + \begin{bmatrix} a_{ij} & -b_{ij} \\ -b_{ij} & c_{ij} \end{bmatrix}\begin{bmatrix} q_j \\ p_j \end{bmatrix} = \begin{bmatrix} SF_i + BF_i \\ TF_i + HF_i \end{bmatrix} \tag{45}$$

where SF_i is the mechanical surface force, BF_i the mechanical body force corresponding to q_i, TF_i and HF_i are equivalent thermal forces for surface temperature changes and heat sources, respectively. Each explicit form from equations (28), (29) and (37a, b) is

$$SF_i = \int_{\Gamma_n} \sigma_{kl} n_l f_{ik} d\Gamma, \quad BF_i = \int_{\Omega_n} B_k f_{ik} d\Omega, \tag{46a, b}$$

and

$$TF_i = \int_{\Gamma_n} \tilde{\sigma} n_k g_{ik} d\Gamma,$$
$$HF_i = \int_{\Omega_n} C \lambda_{kl} S_l g_{ik} d\Omega. \tag{46c, d}$$

If q_j and p_j are defined as nodal displacements, equation (45) represents a general formulation for finite elements. Interpolation functions f_{ik} and g_{ik} may be assumed to be the same function if discretizations for the mechanical field and the thermal field are identical. Furthermore, if b_{ij} is a symmetric tensor, then the generalized stiffness matrix, the third term in equation (45), is also symmetric. Equation (45) therefore represents a generalized dynamic system with both the mechanical deformation and heat displacement treated as generalized coordinates. Also note that the element b_{ij} characterizes both the thermal expansion and the thermomechanical coupling effects which play an interactive role between the mechanical field and the thermal field.

3. The Transfinite Element Formulation

3.1 Summary of Thermoelastic Equations

Coupled thermoelastic equations are summarized for an application of a weighted residual method which yields the weak form for following governing equations:

on the thermal field,

equation of motion;

$$C^2 T_o \lambda_{ij} \ddot{H}_i = \tilde{\sigma}_{,j} + C \lambda_{ij} S_i \quad [0,\ t] \text{ on } \Omega \qquad (17)$$

constitutive equations;

displacement-strain;

$$\theta = H_{i,i} - \frac{\beta_{ij}}{C} e_{ij} \qquad [0,\ t] \text{ on } \Omega \qquad (14)$$

strain-stress;

$$\tilde{\sigma} = C\, T_o \theta \qquad [0,\ t] \text{ on } \Omega \qquad (15)$$

boundary conditions;

$$H_i = H_{ig} \qquad [0,\ t] \text{ on } \Gamma_g \qquad (47a)$$

$$\tilde{\sigma}\, n_i = t_{ih} \qquad [0,\ t] \text{ on } \Gamma_h \qquad (47b)$$

initial condition;

$$H_i(x_j, 0) = H_{io} \qquad \text{on } \Omega \qquad (47c)$$

on the structural field,

equation of motion;

$$\rho \ddot{U}_i = \sigma_{ij,j} + B_i \qquad [0,\ t] \text{ on } \Gamma_g \qquad (16)$$

constitutive equations;

displacement-strain;

$$e_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) \qquad [0,\ t] \text{ on } \Omega \qquad (9)$$

strain-stress;

$$\sigma_{ij} = C_{ijkl}\, e_{kl} - \beta_{ij} T_o \theta \ [0,\ t] \text{ on } \Omega \qquad (12)$$

boundary conditions;

$$U_i = U_{ib}[0,\ t] \qquad \text{on } \Gamma_u \qquad (48a)$$

$$\sigma_{ij}\, n_j = t_{ib}\ [0,\ t] \qquad \text{on } \Gamma_\sigma \qquad (48b)$$

initial conditions;

$$U_i(x_j, 0) = U_{io} \text{ on } \Omega \qquad (48c)$$

$$\dot{U}_i(x_j, 0) = \dot{U}_{io} \text{ on } \Omega \qquad (48d)$$

3.2 The Laplace Transform of Thermoelastic Equations

The principal advantage of the Laplace transform method applied to both the transient thermal analysis and the structural dynamic analysis is that the time parameter t can be eliminated, thereby resulting into the finite element formulation on a type of steady problem. The next equation (49) is the fundamental formula commonly used in conjunction with the Laplace transform method,

$$\bar{f}(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \qquad (49)$$

where $f(t)$ is any real function of time t and $\bar{f}(s)$ the Laplace transform of $f(t)$. After dividing the physical domain into finite elements appropriate for a given problem, the first step to the thermal and the structural analysis involves an application of the Laplace transform in the time domain to equations of motion and their corresponding boundary and initial conditions, and thus leads to the following transfinite element equations:

on the thermal field,
equation of motion;

$$\bar{\sigma}_{,j}+\lambda_{ij}\bar{Q}_i=\alpha\lambda_{ij}\bar{H}_i \quad \text{on } \Omega \tag{50}$$

constitutive equations;
displacement-strain;

$$\bar{\theta}=\bar{H}_{i,i}-\frac{\beta_{ij}}{C}\bar{e}_{ij} \quad \text{on } \Omega \tag{51}$$

strain-stress;

$$\bar{\sigma}=C\,T_o\bar{\theta} \quad \text{on } \Omega \tag{52}$$

boundary conditions;

$$\bar{H}_i=\bar{H}_{ig} \quad \text{on } \Gamma_g \tag{53a}$$

$$\bar{\sigma}\,n_i=\bar{t}_{ih} \quad \text{on } \Gamma_h \tag{53b}$$

on the structural field,
equation of motion:

$$\bar{\sigma}_{ij,j}+\bar{B}_i=\gamma\bar{U}_i \quad \text{on } \Omega \tag{54}$$

constitutive equations;
displacement-strain;

$$\bar{e}_{ij}=\frac{1}{2}(\bar{U}_{i,j}+\bar{U}_{j,i}) \quad \text{on } \Omega \tag{55}$$

strain-stress;

$$\bar{\sigma}_{ij}=C_{ijkl}\bar{e}_{kl}-\beta_{ij}T_o\bar{\theta} \text{ on } \Omega \tag{56}$$

boundary conditions:

$$\bar{U}_i=\bar{U}_{ib}\text{on } \Gamma_u \tag{57a}$$

$$\bar{\sigma}_{ij}\,n_j=\bar{t}_{ib}\text{on } \Gamma_\sigma \tag{57b}$$

where

$$\bar{Q}_i=C\bar{S}_i+\frac{\alpha}{s}H_{io},\quad \alpha=C^2\,T_o s, \tag{58a, b}$$

$$\bar{B}_i=B_i+\frac{\gamma}{s}U_{io}+\frac{\gamma}{s^2}\dot{U}_{io},\quad \gamma=\rho s^2. \tag{58c, d}$$

3.3 The Weighted Residual Approach

Equations (50), (53b), (54) and (57b) are introduced into the weighted residual statement[6)]

$$\begin{aligned}&\int_{\Omega_n} w_i(\gamma\bar{U}_i-\bar{\sigma}_{ij,j}-\bar{B}_i)d\Omega\\&+\sum_{i=1}^{n_{sd}}\int_{\Gamma_{\sigma i}}\hat{w}_i(\bar{\sigma}_{ij}\,n_j-\bar{t}_{ib})d\Gamma\\&+\int_{\Omega_n} y_i(\alpha\lambda_{ij}\bar{H}_i-\bar{\sigma}_{,j}-\lambda_{ij}\bar{Q}_i)d\Omega\\&+\sum_{i=1}^{n_{sd}}\int_{\Gamma_{hi}}\hat{y}_i(\bar{\sigma}\,n_i-\bar{t}_{ih})d\Gamma=0,\end{aligned} \tag{59}$$

where n_{sd} is the number of space dimension and w_i, $\hat{w}_i$, y_i and $\hat{y}_i$ are chosen to be arbitrary weighting functions which satisfy

$$w_i=\hat{w}_i=0 \;\; on\,\Gamma_{ui}, \quad y_i=\hat{y}_i=0 \;\; on\,\Gamma_{gi}. \tag{60a, b}$$

An application of the divergence theorem to the first term and the third term in equation (59) with choices of weighting functions, $w_i=\hat{w}_i$ and $y_i=\hat{y}_i$, leads to

$$\begin{aligned}&\int_{\Omega_n} w_i\gamma\bar{U}_i d\Omega+\int_{\Omega_n} w_{i,j}\bar{\sigma}_{ij}d\Omega\\&-\int_{\Omega_n} w_i\bar{B}_i d\Omega-\sum_{i=1}^{n_{sd}}\int_{\Gamma_{\sigma i}} w_i\bar{t}_{ib}d\Gamma\\&+\int_{\Omega_n} y_i\alpha\bar{H}_i d\Omega+\int_{\Omega_n} y_{i,j}\bar{\sigma}d\Omega\\&-\int_{\Omega_n} y_i\lambda_{ij}\bar{Q}_i d\Omega-\sum_{i=1}^{n_{sd}}\int_{\Gamma_{hi}} y_i\bar{t}_{ih}d\Gamma=0.\end{aligned} \tag{61}$$

In the use of equations (51), (52), (55) and (56), the second term and the sixth in equation (61) can be written as

$$\begin{aligned}w_{i,j}\bar{\sigma}_{ij} &= w_{(i,j)}C_{ijkl}\bar{U}_{(k,l)}+w_{(i,j)}T_o\beta_{ij}\bar{H}_{i,i}\\&+w_{(i,j)}\frac{T_o}{C}\beta_{ij}\bar{U}_{(i,j)},\end{aligned} \tag{62a}$$

$$y_{i,j}\bar{\sigma}=y_{(i,j)}C\,T_o\bar{H}_{i,i}-y_{(i,j)}T_o\beta_{ij}\bar{U}_{(i,j)}. \tag{62b}$$

Note that if notations (,) and [,] are referred to symmetric and antisymmetric parts of the tensor, respectively, then relations $w_{i,j}=w_{(i,j)}+w_{[i,j]}$, $w_{(i,j)}=(w_{i,j}+w_{j,i})/2$, and $w_{[i,j]}=(w_{i,j}-w_{j,i})/2$ are possible. Because of symmetries of C_{ijkl} and β_{ij},

$$w_{[i,j]}\, C_{ijkl}\, \overline{U}_{(k,l)} = 0,\quad w_{[i,j]}\, \overline{U}_{(i,j)}=0,\quad w_{[i,j]}\, \beta_{ij}\, \overline{H}_{i,i}=0 \qquad (63a,\ b,\ c)$$

which are also absolutely true for $y_{i,j}$.

For finite element formulations, equation (61) with substitutions of equations (62a, b) can be of the vector form

$$\begin{aligned}&\int_{\Omega_n}(\nabla \boldsymbol{w})^T \boldsymbol{C}(\nabla \overline{\boldsymbol{U}})d\Omega \\ &+\int_{\Omega_n}(\nabla \boldsymbol{w})^T\, T_o \boldsymbol{\beta}\left(1-\frac{1}{C}\right)(\nabla \overline{\boldsymbol{H}})d\Omega \\ &+\int_{\Omega_n}\boldsymbol{w}^T(\gamma \overline{\boldsymbol{U}}-\overline{\boldsymbol{B}})d\Omega-\sum_{i=1}^{n_{sf}}\int_{\Gamma_{\sigma i}}\boldsymbol{w}^T\, \overline{\boldsymbol{t}}_b d\Gamma \\ &+\int_{\Omega_n}(\nabla \boldsymbol{y})^T\, T_o(C\nabla \overline{\boldsymbol{H}} \\ &-\boldsymbol{\beta}\nabla \overline{\boldsymbol{U}})d\Omega+\int_{\Omega_n}\boldsymbol{y}^T \alpha\boldsymbol{\lambda}\overline{\boldsymbol{H}}d\Omega-\int_{\Omega_n}\boldsymbol{y}^T\boldsymbol{\lambda}\overline{\boldsymbol{Q}}d\Omega \\ &-\sum_{i=1}^{n_{sf}}\int_{\Gamma_{hi}}\boldsymbol{y}\, \overline{\boldsymbol{t}}_h d\Gamma=0,\end{aligned} \qquad (64)$$

where β and λ also represent vector quantities. If interpolation functions are assumed by

$$\boldsymbol{w}=\boldsymbol{N}\boldsymbol{d},\quad \overline{\boldsymbol{U}}=\boldsymbol{N}\,\boldsymbol{q}_n,\quad \boldsymbol{y}=\boldsymbol{L}\boldsymbol{u},\quad \overline{\boldsymbol{H}}=\boldsymbol{L}\,\boldsymbol{p}_n, \qquad (65a,\ b,\ c,\ d)$$

where $\boldsymbol{p}_n$ and $\boldsymbol{q}_n$ are nodal heat displacements and nodal mechanical displacements at their elements, respectively. Shape functions in equation (65a, b, c, d) are substituted into equation (64) and $\boldsymbol{p}_n$, $\boldsymbol{q}_n$, $\boldsymbol{d}$ and $\boldsymbol{u}$ are pulled out of integrals to give the following matrix equation:

$$\begin{bmatrix}\overline{\boldsymbol{m}}+\overline{\boldsymbol{a}} & -\overline{\boldsymbol{b}}_T \\ -\overline{\boldsymbol{b}} & \overline{\boldsymbol{c}}+\overline{\boldsymbol{d}}\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_n \\ \boldsymbol{p}_n\end{bmatrix}=\begin{bmatrix}\overline{\boldsymbol{F}}_S+\overline{\boldsymbol{F}}_B \\ \overline{\boldsymbol{F}}_T+\overline{\boldsymbol{F}}_H\end{bmatrix} \qquad (66)$$

where $\overline{\boldsymbol{m}}=\int_{\Omega_n}\boldsymbol{N}^T\gamma\boldsymbol{N}d\Omega,$ (67a)

$$\overline{\boldsymbol{a}}=\int_{\Omega_n}(\nabla \boldsymbol{N})^T\left(-\frac{T_o}{C}+\boldsymbol{C}\right)(\nabla \boldsymbol{N})d\Omega, \qquad (67b)$$

$$\overline{\boldsymbol{b}}=\int_{\Omega_n}(\nabla \boldsymbol{L})^T\, T_o\boldsymbol{\beta}(\nabla \boldsymbol{N})d\Omega, \qquad (67c)$$

$$\overline{\boldsymbol{c}}=\int_{\Omega_n}(\nabla \boldsymbol{L})^T C\, T_o(\nabla \boldsymbol{L})d\Omega, \qquad (67d)$$

$$\overline{\boldsymbol{d}}=\int_{\Omega_n}\boldsymbol{L}^T\alpha\boldsymbol{\lambda}\boldsymbol{L}d\Omega, \qquad (67e)$$

$$\overline{\boldsymbol{F}}_S=\sum_{i=1}^{n_{sf}}\int_{\Gamma_{\sigma i}}\boldsymbol{N}^T\,\overline{\boldsymbol{t}}_b d\Gamma, \qquad (67f)$$

$$\overline{\boldsymbol{F}}_B=\int_{\Omega_n}\boldsymbol{N}^T\overline{\boldsymbol{B}}d\Omega, \qquad (67g)$$

$$\overline{\boldsymbol{F}}_T=\sum_{i=1}^{n_{sf}}\int_{\Gamma_{hi}}\boldsymbol{L}^T\,\overline{\boldsymbol{t}}_h d\Gamma, \qquad (67h)$$

and $\overline{\boldsymbol{F}}_H=\int_{\Omega_n}\boldsymbol{L}^T\boldsymbol{\lambda}\overline{\boldsymbol{Q}}d\Omega.$ (67i)

Equation (66) is the finite element equation for coupled thermal and mechanical analysis in the Laplace transform domain and has an element stiffness matrix and an element force vector.

3.4 Assembled Finite Element Equation

Next, Boolean operators, $\boldsymbol{A}_n$ and $\boldsymbol{B}_n$, are substituted into interpolation functions to assemble the global system such that

$$\boldsymbol{q}_n=\boldsymbol{A}_n\boldsymbol{q},\quad \overline{\boldsymbol{U}}=\boldsymbol{N}\boldsymbol{A}_n\boldsymbol{q},\quad \boldsymbol{p}_n=\boldsymbol{A}_n\boldsymbol{p},\quad \overline{\boldsymbol{H}}=\boldsymbol{L}\boldsymbol{B}_n\boldsymbol{p}. \qquad (68a,\ b,\ c,\ d)$$

Accordingly, weighting functions become

$$\boldsymbol{w}=\boldsymbol{N}\boldsymbol{A}_n\boldsymbol{d},\quad \nabla\boldsymbol{w}=(\nabla\boldsymbol{N})\boldsymbol{A}_n,\quad \boldsymbol{y}=\boldsymbol{L}\boldsymbol{B}_n\boldsymbol{u},\quad \nabla\boldsymbol{y}=(\nabla\boldsymbol{L})\boldsymbol{B}_n. \qquad (69a,\ b,\ c,\ d)$$

The assembled algebraic equation results into

$$\begin{bmatrix} \sum_{i=1}^{n_{el}} A_n^T(\bar{m}+\bar{a})B_n & -\sum_{i=1}^{n_{el}} A_n^T \bar{b}^T B_n \\ -\sum_{i=1}^{n_{el}} B_n^T \bar{b} A_n & \sum_{i=1}^{n_{el}} B_n^T(\bar{c}+\bar{d})B_n \end{bmatrix} \begin{bmatrix} q_n \\ p_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_{el}} A_n^T(\bar{F}_S+\bar{F}_B) \\ \sum_{i=1}^{n_{el}} B_n^T(\bar{F}_T+\bar{F}_H) \end{bmatrix} \tag{70}$$

where n_{el} is the number of elements in the transform domain.

3.5 Comparisons of Finite Element Equations

The dynamic equation (45), via an introduction of generalized coordinates and a variational approach, is needed to change both the space domain and the time to only one space domain through the Laplace transform, and is compared with equation (66) by the Laplace transform and then by a weighted residual method.

Using the following Laplace transform:

$$L(\dot{x}) = sL(x) - x(0) = s\bar{x} - x_{io} \tag{71a}$$

$$L(\ddot{x}) = s^2L(x) - sx(0) - \dot{x}(0) = s^2\bar{x} - s\,x_{io} - \dot{x}_{io} \tag{71b}$$

where x_{io} and $(dx/dt)_{io}$ are initial conditions, the equation (45) becomes the coupled thermoelasticity problem expressed by

$$\begin{bmatrix} s^2 m_{ij}+a_{ij} & -b_{ij} \\ -b_{ij} & c_{ij}+s\,d_{ij} \end{bmatrix} \begin{bmatrix} \bar{q}_j \\ \bar{p}_j \end{bmatrix} = \begin{bmatrix} SF_i+BF_i+m_{ij}(s\,q_{io}+\dot{q}_{io}) \\ TF_i+HF_i+p_{io} \end{bmatrix}. \tag{72}$$

Equations (66) and (72) turn out to be exactly the same if comparison of matrices is performed term by term, and a consideration is taken into boundries $\Gamma_n=\Gamma_\sigma$ in SF_i and $\Gamma_n=\Gamma_h$ in TF_i, respectively.

3.6 Types of Thermoelastic Problem in the Laplace Domain

For a particular case of the uncoupled thermoelasticity, the temperature variation due to coupling mechanical deformation is negligibly small and the b_{ij} term in the equation (72) can be ignored. If the wave effects are small, the inertia term m_{ij} is neglected in the first equation in equation (72), then these two equations give the solution for the quasistatic behavior of coupled or uncoupled thermoelasticity.

For an coupled quasistatic thermoelasticity, its problem is governed by

$$\begin{bmatrix} a_{ij} & -b_{ij} \\ -b_{ij} & c_{ij}+s\,d_{ij} \end{bmatrix} \begin{bmatrix} \bar{q}_j \\ \bar{p}_j \end{bmatrix} = \begin{bmatrix} SF_i+BF_i \\ TF_i+HF_i+p_{io} \end{bmatrix}. \tag{73}$$

For an uncoupled quasistatic thermoelasticity, its problem is governed by

$$\begin{bmatrix} a_{ij} & 0 \\ 0 & c_{ij}+s\,d_{ij} \end{bmatrix} \begin{bmatrix} \bar{q}_j \\ \bar{p}_j \end{bmatrix} = \begin{bmatrix} SF_i+BF_i \\ TF_i+HF_i+p_{io} \end{bmatrix}. \tag{74}$$

3.7 Numerical Implementations

Durbin's algorithm, one of numerical transform inversion methods, has been in existence for quite some time. It was originally used in applied mathematics and later applied modal response studies and related problems. The numerical inversion algorithm adopting both Fourier sine and cosine transforms presents relatively moderate accuracy and efficiency. The error bound on the inverse of a function is independent of time being exponential in time. In addition, from a viewpoint of qualitative trend the corresponding trigonometric series obtained for the function in the transform domain has been valid for the entire period of the series until recently. The accuracy of the inversion algorithm increases with the total number of terms used in the series. When the last time point is slightly increased, the results for the previous last time point are as accurate as those during the total transient. However, this algorithm tends to produce less accurate results at the very last point of the duration.

For obtaining the real time response of a transformed function at desired times of interest, the corresponding summation series formulated for the function currently need a certain multiplier which is directly proportional to the exponential of the desired time of interest and inversely proportional to the exponential to the total duration of the transient. Hence, it will be further research to make numerical results more stable and reliable for the implementation of this inversion algorithm.

4. Conclusions

This paper describes an useful and alternative methodology based on the transfinite element formulation for interfacing the interdisciplinary areas of heat transfer and structural dynamics. The introduction of new quantity, defined as heat displacement, is needed for a unified expression of equations of structural dynamics and thermoelasticity, and the associated thermal constitutive relations are newly established. Based on an analogy between the heat displacement and mechanical displacement, a variational principle is developed in the use of generalized coordinates which represent heat and mechanical displacements to derive finite element equations for the general case of coupled thermoelasticity.

The development of a transfinite element approach for the unified thermal and structural analysis is presented using the Laplace transform. Fundamental concepts of the approach and details of the transfinite formulations are described for combined heat transfer and structural analysis. The dynamic finite element equation obtained by a variational approach demonstrates the exact agreement of results from the transfinite element method in conjunction with a weighted residual scheme. Particular cases of the thermoelastic problem are classified in the Laplace domain in terms of the coupled problem and the coupled or the uncoupled quasistatic problem.

It is strongly anticipated for a further work that after developing the finite element formulation in the transform domain, system equations will be then solved in the transform domain itself. Therein, the solution response will be obtained by employing an inverse numerical transform, which is an important step in the solution methodology. Nevertheless, the approach presented provides significant features and concepts for combined thermal and structural models and offers a significant potential for extensions to other interdisciplinary areas as well.

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(접수일자 : 1999. 5. 11)