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Tests of Statistical Hypotheses for Gaussian Fuzzy Population

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Abstract. In this paper, tests of statistical hypotheses for the expectation and variance of Gaussian fuzzy population are discussed.

Key Words : *Fuzzy numbers, Gaussian Fuzzy population , Statistical Hypotheses .*

1. INTRODUCTION

In man–machine control and communication systems, the theory of probability alone is unsuitable for the evaluation of system reliability. This is because the key elements are not numbers, but labels of fuzzy sets. Ordinarily imprecision and indeterminacy are considered to be statistical random characteristics and are taken into account by classical methods of probability theory. In real situations, a frequent source of imprecision is not only the presence of randomness, but inexactness due to subjective factors. In order to represent relationships between the randomness and inexactness, Puri and Ralescu (1986) introduced the concept of a fuzzy random variable. Since then, there has been much attentions for fuzzy statistical inferences. Casals et al. (1986, 1989) discussed statistical hypotheses testing based on a model represented by fuzzy sets and Schnatter (1992) generalized statistical methods to fuzzy data by using the concept of fuzzy sample mean and fuzzy sample variance. Also, Korner (2000) suggested test hypotheses for the expectation of a fuzzy random variable and Grzegorzewski (2000) discussed fuzzy test for statistical hypotheses with vague data.

In this paper, we deal with tests of statistical hypotheses for the expectation and variance of Gaussian fuzzy population. Section 2 is devoted to describe some basic concepts of fuzzy numbers. The main results and some examples are given in Section 3.

2. PRELIMINARIES

Let $F(R)$ be the family of fuzzy number $\tilde{u} : R \longrightarrow [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0,1]$.

(3) $\tilde{u}$ is upper semicontinuous.

(4) supp $\tilde{u} = cl\{x \in R : \tilde{u}(x) > 0\}$ is compact.

For $x \in R$, if we denote the indicator function of $\{x\}$ by $\hat{x}$, then $\hat{x} \in F(R)$ for all $x \in R$. For a fuzzy set $\tilde{u}$, the α-level set of $\tilde{u}$ is defined by

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1 \\ supp\ \tilde{u}, & \alpha = 0. \end{cases}$$

Then it follows that $L_1 \tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0,1]$.

Theorem 2.1 For $\tilde{u} \in F(R)$, if we denote $L_\alpha \tilde{u} = [u^1(\alpha), u^2(\alpha)]$, then the followings hold.

(1) u^1 is a bounded increasing function on $[0,1]$.

(2) u^2 is a bounded decreasing function on $[0,1]$.

(3) $u^1(1) \leq u^2(1)$.

(4) u^1 and u^2 are left continuous on $(0,1]$ and right continuous at 0.

(5) If v^1 and v^2 satisfy the above (1) – (4), then there exists a unique $\tilde{v} \in F(R)$ such that $L_\alpha \tilde{v} = [v^1(\alpha), v^2(\alpha)]$ for all $\alpha \in [0,1]$.

Proof. See Goetschel and Voxman (1986). □

For $\tilde{u}, \tilde{v} \in F(R)$ and $\lambda \in R$, the addition and scalar multiplication are defined as usual;

$$(\tilde{u} \oplus \tilde{v})(z) = \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0 \\ \hat{0}, & \lambda = 0. \end{cases}$$

Then it follows that

$$L_\alpha(\tilde{u} \oplus \tilde{v}) = L_\alpha\tilde{u} \oplus L_\alpha\tilde{v} = [u^1(\alpha) + v^1(\alpha), u^2(\alpha) + v^2(\alpha)],$$

$$L_\alpha(\lambda\tilde{u}) = \lambda L_\alpha\tilde{u} = \begin{cases} [\lambda u^1(\alpha), \lambda u^2(\alpha)], & \text{if } \lambda \geq 0 \\ [\lambda u^2(\alpha), \lambda u^1(\alpha)], & \text{if } \lambda < 0. \end{cases}$$

Now we define the metric d on $F(R)$ by

$$d(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha\tilde{u}, L_\alpha\tilde{v}),$$

where h is the Hausdorff metric defined as

$$h(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u^1(\alpha) - v^1(\alpha)|, |u^2(\alpha) - v^2(\alpha)|).$$

Also, the norm $\|\tilde{u}\|$ of fuzzy number $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d(\tilde{u}, \hat{0}) = \max(|u^1(0)|, |u^2(0)|).$$

From the results of Kaleva (1987), we see that

(1) $(F(R), d)$ is complete but not separable,

(2) $d(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = d(\tilde{u}, \tilde{v})$ for all $\tilde{u}, \tilde{v}, \tilde{w} \in F(R)$,

(3) $d(\lambda\tilde{u}, \lambda\tilde{v}) = |\lambda| d(\tilde{u}, \tilde{v})$.

Let $\bar{C}[0,1]$ be the class of all real-valued functions f on $[0,1]$ such that f is left continuous on $(0,1]$ and has right limit on $[0,1)$, especially f is right continuous at 0. Then $\bar{C}[0,1]$ is a Banach space with the norm $\|f\| = \sup_{\alpha\in[0,1]} |f(\alpha)|$ and so is $\bar{C}[0,1] \times \bar{C}[0,1]$ with norm $\|(f,g)\| = \max(\|f\|, \|g\|)$.

Theorem 2.2 Define $j : F(R) \to \bar{C}[0,1] \times \bar{C}[0,1]$ by $j(\tilde{u}) = (u^1, u^2)$. Then the followings hold;

(1) $j(F(R))$ is a closed convex cone with vertex 0 in $\bar{C}[0,1] \times \bar{C}[0,1]$.

(2) $j(\lambda\tilde{u} \oplus \mu\tilde{v}) = \lambda j(\tilde{u}) + \mu j(\tilde{v})$ for $\tilde{u}, \tilde{v} \in F(R)$ and $\lambda \geq 0, \mu \geq 0$.

(3) $d(\tilde{u}, \tilde{v}) = \|j(\tilde{u}) - j(\tilde{v})\|$.

Proof. Cong-Xin and Ming (1991). □

3. MAIN RESULTS

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable (for short, f.r.v.) if $j(\tilde{X})$ is a random element of $\bar{C}[0,1] \times \bar{C}[0,1]$, where j is the embedding function defined in Theorem 2.2. A f.r.v. $\tilde{X}$ is called Gaussian if $j(\tilde{X})$ is Gaussian. If $E\|\tilde{X}\| < \infty$, the expectation of $\tilde{X}$ is defined by the fuzzy number $E(\tilde{X})$ satisfying $L_\alpha E(\tilde{X}) = [EX^1(\alpha), EX^2(\alpha)]$.

Theorem 3.1 $\tilde{X}$ is a Gaussian f.r.v. if and only if $\tilde{X} = E(\tilde{X}) \oplus \hat{\xi}$, where ξ is a real valued Gaussian r.v. with mean zero.

Proof. See Feng (2000). □

The variance of $\tilde{X}$ is defined by

$$Var(\tilde{X}) = E(d^2(\tilde{X}, E\tilde{X})).$$

Let $\tilde{\mu} = E\tilde{X}$ and $\sigma^2 = Var(\tilde{X})$. If $\tilde{X}$ is a Gaussian f.r.v. with $\tilde{X} = \tilde{\mu} \oplus \hat{\xi}$, then it follows that $Var(\xi) = \sigma^2$,that is, $\xi \sim N(0, \sigma^2)$. Thus, we will use the convention $\tilde{X} \sim FN(\tilde{\mu}, \sigma^2)$ if $\tilde{X} = \tilde{\mu} \oplus \hat{\xi}$ and $\xi \sim N(0, \sigma^2)$.

The f.r.v.'s $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ is called a fuzzy random sample from $\tilde{X}$ if $j(\tilde{X}_1), j(\tilde{X}_2), \ldots, j(\tilde{X}_n)$ is a random sample from $j(\tilde{X})$. For the fuzzy random sample $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ from $\tilde{X}$, we define the fuzzy sample mean $\overline{\tilde{X}}$ and the sample variance $S^2_{\tilde{X}}$ by

$$\overline{\tilde{X}} = \frac{1}{n} \oplus_{i=1}^{n} \tilde{X}_i, \tag{3.1}$$

$$S^2_{\tilde{X}} = \frac{1}{n-1} \sum_{i=1}^{n} d^2(\tilde{X}_i, \overline{\tilde{X}}). \tag{3.2}$$

If $\tilde{X} \sim FN(\tilde{\mu}, \sigma^2)$, then it follows that $\overline{\tilde{X}} = \tilde{\mu} \oplus \hat{\overline{\xi}}$ and

$$\begin{aligned} S^2_{\tilde{X}} &= \frac{1}{n-1} \sum_{i=1}^{n} d^2(\tilde{\mu} \oplus \hat{\xi}_i, \tilde{\mu} \oplus \hat{\overline{\xi}}) \\ &= \frac{1}{n-1} \sum_{i=1}^{n} |\xi_i - \overline{\xi}|^2 = S^2_\xi, \end{aligned} \tag{3.3}$$

where $\overline{\xi}$ and S^2_ξ are the sample mean and the sample variance from ξ, respectively .

Theorem 3.2 If $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ is a fuzzy random sample from $\tilde{X} \sim FN(\tilde{\mu}, \sigma^2)$, then $\overline{\tilde{X}}$ and $S^2_{\tilde{X}}$ are unbiased estimators of $\tilde{\mu}$ and σ^2, respectively.

Now we wish to test the hypothesis $H_0 : \tilde{\mu} = \tilde{\mu}_0$, $H_1 : \tilde{\mu} \neq \tilde{\mu}_0$. Since $\overline{\tilde{X}}$ is an unbiased estimator of $\tilde{\mu}$, it is natural that the test statistic should be taken as

$$Z = \frac{d(\overline{\tilde{X}}, \tilde{\mu}_0)}{\sigma/\sqrt{n}},$$

and that the critical region be given by $Z \geq c$ for some c. Since

$$d(\overline{\tilde{X}}, \tilde{\mu}) = d(\tilde{\mu} \oplus \hat{\bar{\xi}}, \tilde{\mu}) = |\bar{\xi}|$$

and $\bar{\xi} \sim N(0, \frac{\sigma^2}{n})$, we have

$$P(\frac{d(\overline{\tilde{X}}, \tilde{\mu})}{\sigma/\sqrt{n}} \geq z_{\frac{\alpha}{2}}) = P(\frac{|\bar{\xi}|}{\sigma/\sqrt{n}} \geq z_{\frac{\alpha}{2}}) = \alpha.$$

This implies that under $H_0, P(Z \geq z_{\frac{\alpha}{2}}) = \alpha$. and the critical region at a significance level α is $Z \geq z_{\frac{\alpha}{2}}$. Therefore the test for the fuzzy mean can be formulated as follows;

Theorem 3.3 Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ be a fuzzy random sample from $FN(\tilde{\mu}, \sigma^2)$. Then the test statistics for the null hypothesis $H_0 : \tilde{\mu} = \tilde{\mu}_0$ is

$$Z = \frac{d(\overline{\tilde{X}}, \tilde{\mu}_0)}{\sigma/\sqrt{n}},$$

and the critical region at a significance level α is $Z \geq z_{\frac{\alpha}{2}}$.

If σ^2 is unknown, we replace σ^2 by its unbiased estimator $S^2_{\tilde{X}}$. Since $S_{\tilde{X}} = S_\xi$ and

$$\frac{\bar{\xi}}{S_\xi/\sqrt{n}} \sim t(n-1),$$

we have the following;

Theorem 3.4 Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ be a fuzzy random sample from $FN(\tilde{\mu}, \sigma^2)$. If σ^2 is unknown, then the test statistics for the null hypothesis $H_0 : \tilde{\mu} = \tilde{\mu}_0$ is

$$T = \frac{d(\overline{\tilde{X}}, \tilde{\mu}_0)}{S_{\tilde{X}}/\sqrt{n}},$$

and the critical region at a significance level α is $T \geq t_{\frac{\alpha}{2}}(n-1)$.

Turning to the test of hypotheses for a variance, if we apply the usual test about variance, together with the fact (3.3), we can conclude the followings.

Theorem 3.5 Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ be a fuzzy random sample from $FN(\tilde{\mu}, \sigma^2)$. Then the test statistics for the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ is

$$\chi^2 = \frac{(n-1)S_{\tilde{X}}^2}{\sigma_0^2}.$$

(1) If $H_1 : \sigma^2 > \sigma_0^2$, the critical region at a significance level α is $\chi^2 \geq \chi_\alpha^2(n-1)$.

(2) If $H_1 : \sigma^2 < \sigma_0^2$, the critical region at a significance level α is $\chi^2 \leq \chi_{1-\alpha}^2(n-1)$.

(3) If $H_1 : \sigma^2 \neq \sigma_0^2$, the critical region at a significance level α is $\chi^2 \geq \chi_{\frac{\alpha}{2}}^2(n-1)$ or $\chi^2 \leq \chi_{1-\frac{\alpha}{2}}^2(n-1)$.

Example 1 A triangular fuzzy number $\tilde{u}$ denoted by $< l, m, r >$ is a fuzzy set $\tilde{u} : R \to [0,1]$ defined by

$$\tilde{u}(x) = \begin{cases} \frac{x-l}{m-l}, & \text{if } l < x < m \\ 1, & \text{if } x = m \\ \frac{r-x}{r-m}, & \text{if } m < x < r \\ 0, & \text{otherwise,} \end{cases}$$

where $l < m < r$. If $m = r$, we understood

$$\tilde{u}(x) = \begin{cases} \frac{x-l}{m-l}, & \text{if } l < x < m \\ 1, & \text{if } x = m \\ 0, & \text{otherwise,} \end{cases}$$

and if $m = l$,

$$\tilde{u}(x) = \begin{cases} 1, & \text{if } x = m \\ \frac{r-x}{r-m}, & \text{if } m < x < r \\ 0, & \text{otherwise.} \end{cases}$$

We note that if $\tilde{\mu} = E\tilde{X}$ is triangular, $\tilde{X}(\omega)$ is also triangular for all $\omega \in \Omega$. To test the hypothesis $H_0 : \tilde{\mu} =< 2.5, 3.0, 3.5 >$, suppose that we have a fuzzy random sample observation $\tilde{x}_1, \ldots, \tilde{x}_{10}$ from $FN(\tilde{\mu}, \sigma^2)$ as follows;

$$\begin{aligned}
&\tilde{x}_1 =< 2.896, 2.968, 3.135 >, \ \tilde{x}_2 =< 1.903, 2.435, 2.931 >,\\
&\tilde{x}_3 =< 1.874, 2.459, 3.357 >, \ \tilde{x}_4 =< 3.136, 3.609, 3.918 >,\\
&\tilde{x}_5 =< 2.524, 2.845, 3.225 >, \ \tilde{x}_6 =< 2.596, 3.193, 4.263 >,\\
&\tilde{x}_7 =< 2.803, 3.281, 3.786 >, \ \tilde{x}_8 =< 3.097, 3.435, 4.052 >,\\
&\tilde{x}_9 =< 2.908, 3.283, 3.604 >, \ \tilde{x}_{10} =< 1.793, 2.152, 3.039 > .
\end{aligned}$$

First we have that $\overline{\tilde{x}} =< 2.553, 2.966, 3.531 >$. We note that if $\tilde{u} =< l_1, m_1, r_1 >$ and $\tilde{v} =< l_2, m_2, r_2 >$,

$$d(\tilde{u}, \tilde{v}) = \max(|l_1 - l_2|, |m_1 - m_2|, |r_1 - r_2|).$$

Thus, we obtain

$$\begin{aligned}
&d(\tilde{x}_1, \overline{\tilde{x}}) = 0.396, \ d(\tilde{x}_2, \overline{\tilde{x}}) = 0.65, \ d(\tilde{x}_3, \overline{\tilde{x}}) = 0.679, \ d(\tilde{x}_4, \overline{\tilde{x}}) = 0.643,\\
&d(\tilde{x}_5, \overline{\tilde{x}}) = 0.306, \ d(\tilde{x}_6, \overline{\tilde{x}}) = 0.732, \ d(\tilde{x}_7, \overline{\tilde{x}}) = 0.315, \ d(\tilde{x}_8, \overline{\tilde{x}}) = 0.544,\\
&d(\tilde{x}_9, \overline{\tilde{x}}) = 0.355, \ d(\tilde{x}_{10}, \overline{\tilde{x}}) = 0.81.
\end{aligned}$$

Therefore, $s^2_{\tilde{x}} = 0.36301, d(\overline{\tilde{x}}, \tilde{\mu}_0) = 0.531$ and the test statistic is

$$t = \frac{d(\overline{\tilde{x}}, \tilde{\mu}_0)}{s_{\tilde{x}}/\sqrt{10}} = 2.78701.$$

Since $t_{0.025}(9) = 2.26216$ and $t_{0.005}(9) = 3.24984$, the null hypothesis H_0 is rejected at at a significance level $\alpha = 0.05$ but H_0 is not rejected at a significance level $\alpha = 0.01$.

Similarly, for the hypothesis $H_0 : \sigma = 0.4, H_1 : \sigma > 0.4$, the test statistic is

$$\chi^2 = \frac{(n-1)S^2_{\tilde{x}}}{0.4^2} = 20.41905.$$

Since $\chi^2_{0.05}(9) = 16.91896$ and $\chi^2_{0.01}(9) = 21.66605$, the null hypothesis H_0 is rejected at at a significance level $\alpha = 0.05$ but H_0 is not rejected at a significance level $\alpha = 0.01$.

Now we would like to test the null hypothesis $H_0 : \tilde{\mu}_1 = \tilde{\mu}_2$ for two independent fuzzy Gaussian distribution $FN(\tilde{\mu}_1, \sigma_1^2)$ and $FN(\tilde{\mu}_2, \sigma_2^2)$. Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ and $\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_m$ be two fuzzy random samples of sizes n and m from $FN(\tilde{\mu}_1, \sigma_1^2)$ and $FN(\tilde{\mu}_2, \sigma_2^2)$, respectively. If we write

$$\begin{aligned}
\tilde{X}_i &= \tilde{\mu}_1 \oplus \xi_i, i = 1, \ldots, n,\\
\tilde{Y}_j &= \tilde{\mu}_2 \oplus \zeta_j, j = 1, \ldots, m,
\end{aligned}$$

then under H_0,

$$d(\overline{\tilde{X}}, \overline{\tilde{Y}}) = d(\tilde{\mu}_1 \oplus \hat{\bar{\xi}}, \tilde{\mu}_2 \oplus \hat{\bar{\zeta}})$$
$$= d(\hat{\bar{\xi}}, \hat{\bar{\zeta}}) = |\bar{\xi} - \bar{\zeta}|.$$

Since $\bar{\xi} - \bar{\zeta} \sim N(0, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m})$, we have

$$P(\frac{d(\overline{\tilde{X}}, \overline{\tilde{Y}})}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \geq z_{\frac{\alpha}{2}}) = P(\frac{|\bar{\xi} - \bar{\zeta}|}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \geq z_{\frac{\alpha}{2}}) = \alpha.$$

This leads to the following result.

Theorem 3.6 Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ and $\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_m$ be two fuzzy random samples of sizes n and m from two independent fuzzy Gaussian distribution $FN(\tilde{\mu}_1, \sigma_1^2)$ and $FN(\tilde{\mu}_2, \sigma_2^2)$, respectively. Then the test statistics for the null hypothesis $H_0 : \tilde{\mu}_1 = \tilde{\mu}_2$ is

$$Z = \frac{d(\overline{\tilde{X}}, \overline{\tilde{Y}})}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}},$$

and the critical region at a significance level α is $Z \geq z_{\frac{\alpha}{2}}$.

If $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and σ^2 is unknown, its unbiased estimator is

$$S_p^2 = \frac{(n-1)S_{\tilde{X}}^2 + (m-1)S_{\tilde{Y}}^2}{n+m-2}.$$

Then under H_0,

$$\frac{d(\overline{\tilde{X}}, \overline{\tilde{Y}})}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} = \frac{|\bar{\xi} - \bar{\zeta}|}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}}.$$

Since

$$\frac{\bar{\xi} - \bar{\zeta}}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t(n+m-2),$$

we can conclude the following:

Theorem 3.7 Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ and $\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_m$ be two fuzzy random samples of sizes n and m from two independent fuzzy Gaussian distribution $FN(\tilde{\mu}_1, \sigma^2)$ and

$FN(\tilde{\mu}_2, \sigma^2)$, respectively. If σ^2 is unknown, then the test statistics for the null hypothesis $H_0 : \tilde{\mu}_1 = \tilde{\mu}_2$ is

$$T = \frac{d(\bar{\tilde{X}}, \bar{\tilde{Y}})}{S_p\sqrt{\frac{1}{n} + \frac{1}{m}}},$$

and the critical region at a significance level α is $T \geq t_{\frac{\alpha}{2}}(n + m - 2)$.

Example 2 For a given $n = 10$ observations $\tilde{x}_1, \ldots, \tilde{x}_{10}$ from $FN(\tilde{\mu}_1, \sigma^2)$ in example 1, $\bar{\tilde{x}} =< 2.553, 2.966, 3.531 >$ and $s_{\tilde{x}}^2 = 0.36301$. Now suppose that we have $m = 8$ observations $\tilde{y}_1, \ldots, \tilde{y}_8$ from $FN(\tilde{\mu}_2, \sigma^2)$ as follows;

$$\begin{aligned}
&y_1 =< 3.136, 3.667, 4.501 >,\ y_2 =< 2.432, 3.163, 3.929 >,\\
&y_3 =< 1.902, 2.998, 3.997 >,\ y_4 =< 3.127, 3.557, 4.516 >,\\
&y_5 =< 2.515, 2.961, 3.879 >,\ y_6 =< 2.874, 3.485, 4.215 >,\\
&y_7 =< 2.481, 2, 824, 3.627 >,\ y_8 =< 2.533, 2.953, 3.936 > .
\end{aligned}$$

Then $\bar{\tilde{y}} =< 2.625, 3.210, 4.075 >$ and $s_{\tilde{y}}^2 = 0.21051$. Thus, $s_p^2 = 0.29629$ and the test statistic is

$$t = \frac{d(\bar{\tilde{x}}, \bar{\tilde{y}})}{s_p\sqrt{\frac{1}{10} + \frac{1}{8}}} = 2.10693.$$

Since $t_{0.025}(16) = 2.11991$, the null hypothesis $H_0 : \tilde{\mu}_1 = \tilde{\mu}_2$ is not rejected at at a significance level $\alpha = 0.05$.

To test the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$, we can apply the usual test for the equality of two variances for normal populations thanks to (3.3).

Theorem 3.8 Let $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n$ and $\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_m$ be two fuzzy random samples of sizes n and m from two independent fuzzy Gaussian distribution $FN(\tilde{\mu}_1, \sigma^2)$ and $FN(\tilde{\mu}_2, \sigma^2)$, respectively. Then the test statistics for the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ is

$$F = \frac{S_{\tilde{X}}^2}{S_{\tilde{Y}}^2}.$$

If $H_1 : \sigma_1^2 > \sigma_2^2$, the critical region at a significance level α is $F \geq F_\alpha(n-1, m-1)$.

If $H_1 : \sigma_1^2 < \sigma_2^2$, the critical region at a significance level α is $F \leq F_{1-\alpha}(n-1, m-1)$.

If $H_1 : \sigma_1^2 \neq \sigma_2^2$, the critical region at a significance level α is $F \geq F_{\frac{\alpha}{2}}(n-1, m-1))$ or $F \leq F_{1-\frac{\alpha}{2}}(n-1, m-1))$.

Example 3 For example 2, we shall test the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ against $H_1 : \sigma_1^2 \neq \sigma_2^2$. At an $\alpha = 0.05$ significance level, H_0 is rejected if

$$\frac{s_{\tilde{x}}^2}{s_{\tilde{y}}^2} \geq F_{0.025}(9,7) = 4.82322$$

or

$$\frac{s_{\tilde{x}}^2}{s_{\tilde{y}}^2} \leq F_{0.975}(9,7) = \frac{1}{F_{0.025}(7,9)} = 0.23826.$$

Using the data in Example 2, we obtain $\frac{s_{\tilde{x}}^2}{s_{\tilde{y}}^2} = 1.72441$. Therefore, we do not reject H_0.

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