Historia Mathematica Vol. 13, No. 1, 2000, 125-136

A Completion of Semi-simple MV-algebra

성균관대학교 수학교육과 **박평우**

Abstract

The notion of MV-algebra was introduced by C.C. Chang in 1958 to provide an algebraic proof of the completeness of Lukasiewicz axioms for infinite valued logic. These algebras appear in the literature under different names: Bricks, Wajsberg algebra, CN-algebra, bounded commutative BCK-algebras, etc.

The purpose of this paper is to give a topological lattice completion of semisimple MV-algebras. To this end, we characterize the complete atomic center MV-algebras and semisimple algebras as subalgebras of a cube. Then we define the δ-completion of semisimple MV-algebra and construct the δ-completion. We also study some important properties and extension properties of δ-completion.

0. Introduction

In his classical paper [3], C.C. Chang invented the notion of MV-algebra in order to provide an algebraic proof of the completeness theorem of Lukasiewicz axioms for infinite valued propositional logic.

The Boolean algebra (ring) is the corresponding algebra for the classical two-valued logic. In Boolean algebras(=commutative idempotent unitary ring), ring-operations $+$, $\cdot$, 0 and 1 define lattice-operations(=distributive complemented lattice) so that it forms a Boolean lattice. As in Boolean algebra, MV- algebraic-operations $+$, $\cdot$, $-$, 0 and 1 define lattice-operations so that it forms a bounded lattice (actually, distributive lattice). By using the fact that the set $C(A)$ of all idempotent elements of a MV-algebra A forms a Boolean subalgebra, it is easy to prove that the category of Boolean algebras is a coreflective subcategory of the category of all MV-algebras and their homomorphims.

Unlike Boolean algebra, not all MV-algebras are semi-simple.

In [1, 2], Belluce and [6] Hoo have developed MV-algebras in its algebraic properties and topological properties, in particular, they have characterized the semi-simple MV-algebra in terms of many different notions; In terms of Bold algebra of Fuzzy subset [1], Archimedeanness, quasi-locallyness, and the lattice-completeness and sub-direct product of unit interval MV-algebra [2]. Hoo has shown that A is semi-simple iff the space of maximal ideals of A is dense in the space of prime ideals [6]. In this paper, we first show that if A is a complete MV-algebra and its $C(A)$ is atomic then A is isomorphic to a product of a cube and $\prod A(m)$, where cube means a product of unit interval MV-algebras and $\prod A(m)$ is a product of finite MV-algebras $A(m)$'s. After the proof that any complete atomic one is an atomic center, it follows that if A is complete nonatomic and A has at least one atom, and if $A \cong B \times C$ (Belluce's decomposition: Theorem 9 [2]) then the atomic part $B \cong \prod A(m)$ and atomless part $C \cong I^{\lambda}$ (a cube). It follows immediately that if A is completely atomic then A is the direct product $\prod A(m)$ for some $m \in \Lambda \subset \mathbf{Z}$ (see Theorem 4.2 and the below remark [2]). Secondly we introduce an intrinsic topology on a semi-simple MV-algebra so that it is a topological MV-algebra, we show that every semi-simple MV-algebra A has a completion $\delta(A)$ which is a complete and atomic center (or a compact MV-algebra) and $\delta(A)$ contains A as a dense subset. Futhermore A is a subdirect product of the type $\prod J_{\alpha} \times \prod A(m)$, of MV-algebra where J_{α} is a dense subalgebra of the unit interval MV-algebra I. Finally, we investigate the further properties of the $\delta-$completion, for example, $\delta(A)$ is an extension universal property.

[Preliminaries]

An algebra $A=(A, +, \cdot, -, 0, 1)$ is called an MV-algebra if the following equations are satisfied; for $x, y, z \in A$

(i) $x+y=y+x$,
(ii) $(x+y)+z=x+(y+z)$,
(iii) $x \cdot y=\overline{(\bar{x}+\bar{y})}$,

(iv) $x+0=x$ and $x+1=1$,

(v) $\overline{0}=1$ and $\overline{1}=0$,

(vi) $\overline{(\overline{x}+y)}+y=\overline{(\overline{y}+x)}+x$.

If we define that for $x, y \in A$ $x \vee y = x+\overline{x}y$ and $x \wedge y = x(\overline{x}+y)$ then $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.

For all basic terminologies of MV-algebra, we refer to [1, 2, 3]. For an MV- algebra A, $C(A)$ is called the *center* of A, its element is called a *center element* of A. For $a \in A$, $\downarrow a$ denotes the subset $\{z \in A \mid x \leq a\}$ and dually $\uparrow a$ denotes $\{x \in A \mid a \leq x\}$.

1. Atomic MV-algebras

Let A be an MV-algebra. For $x, y \in A$ '*y covers x*' means that $x < y$ and there is no element between x and y. If a covers 0 then a is called an *atom* of A. In this section we show first that if A is an atomic complete MV-algebra, then the center $C(A)$ must be a power-set Boolean subalgebra, namely, $C(A)$ is an atomic complete Boolean algebra.

We first prove the following theorem.

Theorem 1.1 For an MV-algebra A, the following are equivalent; for $x, y \in A$

(i) y covers x,

(ii) $\overline{x}y$ covers 0,

(iii) 1 covers $x+\overline{y}$.

Proof. (i) $\Rightarrow$ (ii). Assume that $0 < u < \overline{x}y$ for some $u \in A$. Then we have $x \leq x+u \leq x+\overline{x}y = x \vee y = y$. We claim that $x < x+u < y$ which is absurd. Indeed, if $x=x+u$ then $u=0$, because $0 \leq \overline{x}$ and $u \leq \overline{x}y \leq \overline{x}$ imply $u=0$ by Theorem 1.14 [3], which is a contradiction. Thus $x < x+u$. Now if $x+u=y$ then $\overline{x}(x+u) = \overline{x}y$, i.e., $\overline{x} \wedge u = \overline{x}y$. On the other hand, we have $u < \overline{x}y \leq \overline{x}$, i.e., $\overline{x} \wedge u = u$. Thus $\overline{x}y = u$ which is also a contradiction. So we have $x+u < y$. Hence we have $x < x+u < y$ which is absurd to (i). Thus (ii) holds.

(ii) $\Rightarrow$ (i). Soppose that $x < z < y$ for some $z \in A$. Then we have $0 \leq \bar{x}z \leq \bar{x}y$. If $0 = \bar{x}z$ then $d(x, z) = x\bar{z} = 0$, since $x < z$. Thus $x = z$. Similarly, if $\bar{x}z = \bar{x}y$ then $x + \bar{z} = x + \bar{y}$. Since $\bar{z}$ and $\bar{y}$ are both bounded by $\bar{x}$, we have $\bar{z} = \bar{y}$, i.e., $z = y$. Hence $0 < \bar{x}z < \bar{x}y$ which is a contradiction to (ii).

(ii) $\Leftrightarrow$ (iii). (ii) and (iii) are simply dual each other. The proof is complete. □

The following corollary is immediate.

Corollary [2]. If an MV-algebra has no atoms then it densely ordered.

The following lemma is immediate from Theorem 5 [2].

Lemma 1.2. If A is a complete MV-algebra then so is $C(A)$, i.e., $C(A)$ is a complete Boolean subalgebra of A.

Lemma 1.3. Let A be a complete MV-algebra. For $S \subset A$ with $S \neq \varnothing$, if $c = \sup S^{\perp}$, then $c \in C(A)$, where $S^{\perp} = \{x \in A \mid x \wedge s = 0 \text{ for all } s \in S\}$.

Proof. For any $s \in S$, $s \wedge c = s \wedge \sup S^{\perp} = \sup(s \wedge S^{\perp}) = 0$. Thus $c \in S^{\perp}$. Since $S^{\perp}$ is always an ideal of A, $2c \in S^{\perp}$, and hence $2c = c$. □

Proposition 1.4. If A is a complete MV-algebra and a_0 is an atom of A, then there exists a unique atom c_0 of $C(A)$ such that $a_0 \leq c_0$.

Proof. We have either

(i) $\{a_0\}^{\perp} = \{0\}$ or
(ii) $\{a_0\}^{\perp} \neq \{0\}$

For case (i), if $y \in A$ with $y \neq 0$ then $a_0 \wedge y = a_0$ because $a_0 \wedge y = a_0$ or $a_0 \wedge y = 0$ since a_0 is an atom of A. If $y \wedge a_0 = 0$ then $y \in \{a_0\}^{\perp}$. Hence $y \in \uparrow a_0$. Thus $\uparrow a_0 = A - \{0\}$. Now let $c_0 = \inf\{c \in C(x) \mid a_0 \leq c\}$. Then c_0 is an atom of $C(A)$.

For case (ii), firstly we note that $\{a_0\} \subset \{a_0\}^{\perp\perp}$. Let $d = \{a_0\}^{\perp\perp}$. Then $d_0 \in C(A) \cap \uparrow a_0$.

Clearly $d_0 \in \{a_0\}^{\perp\perp}$. So we have $\{a_0\}^{\perp\perp} = \downarrow d_0$. Now let $e_0 = \inf(C(A) \cap \uparrow a_0)$. Evidently $e_0 \in C(A) \cap \uparrow a_0$. We claim that e_0 is an atom of $C(A)$ and $a_0 \leq c_0$. Indeed, if there exists $e_0 \in C(A)$ such that $0 < e_0 < c_0 < d_0$, then $e_0 \in \{a_0\}^{\perp\perp} = \downarrow d_0$ is an ideal of A. Hence since a_0 is an atom of A we have either $e_0 \wedge a_0 = a_0$ or $e_0 \wedge a_0 = e_0$ or $e_0 \wedge a_0 = 0$ which implies $a_0 \leq e_0$ or $e_0 \leq a_0$ or $e_0 \in \{a_0\}^{\perp}$, respectively. Thus we have $e_0 = c_0$ or $e_0 = a_0$ or $e_0 \in \{a_0\}^{\perp}$, respectively. But none of which is possible, because $e_0 < c_0$, $e_0 = a_0$ implies $e_0 = c_0$, and if $e_0 \in \{a_0\}^{\perp}$ then $e_0 = 0$.

For the uniqueness of such atoms, if c_0 and c_1 are two atoms of $C(A)$ and $a_0 \leq c_0$, $a_0 \leq c_1$ then $a_0 \leq c_1 \wedge c_2 = 0$ and hence $a_0 = 0$, a contradiction. □

The following corollary is obvious from lemma 1 and the above theorem:

Corollary 1.5. If A is a complete atomic MV-algebra, then $C(A)$ is a power set Boolean algebra.

For a decomposition of an MV-algebra A, the center $C(A) \neq 2$ plays a very important role, as in lattice theory.

If $C(A)$ is atomic and $C(A) \neq 2$ for an MV-algebra A, A is said to be *atomic center*. If $C(A) = \{0, 1\} = 2$, then we say that A is *irreducible*.

2. Decompositions of complete atomic center MV-algebras

It is well known that for an ideal P of an MV-algebra A, P is a prime ideal iff for $x, y \in A$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is also known that the quotient MV-algebra A/P is linearly ordered for any prime ideal P of A [4].

The following two lemmas are obvious:

Lemma 2.1. If P is a prime ideal of A and $a \in C(A)$ then either $a \in P$ or $\bar{a} \in P$. Moreover, $a \in P$ iff $\bar{a} \in P$.

Lemma 2.2. If A is a complete MV-algebra, and if $x \wedge y = 0$ and $c = \sup\{a_0\}^{\perp}$ for $0 \neq x \neq 1$, $0 \neq y \neq 1$ then $c \in C(A)$ and $0 \neq c \neq 1$.

In the following lemma we prove first that the ideal generated by $\bar{a}$ is prime in a complete MV-algebra A but it will turn out that it is actually a maximal ideal in the latter.

Lemma 2.3. If A is a complete MV-algebra, and $C(A)\neq 2$ and if a is an atom of $C(A)$, then $\downarrow \bar{a}$ is a prime ideal of A. Furthermore, the ideal $\downarrow a$ is linearly ordered.

Proof. Clearly $\downarrow \bar{a}$ is a proper ideal of A. Assume that $\downarrow \bar{a}$ is not prime. Namely, there exists two elements x and y in A such that $x\wedge y\in \downarrow \bar{a}$ but $x\notin \downarrow \bar{a}$ and $y\notin \downarrow \bar{a}$. Then $ax\wedge ay=0$ by Theorem 3.1 [2]. Note that $ax\neq 0$ since $ax=0$ implies $\bar{a}\vee x=\bar{a}$, and hence $x\leq \bar{a}$. Similarly $ay\neq 0$. Further $ax\neq 1$ since $\bar{a}+\bar{x}=0$ implies $\bar{a}=0$ and similarly $ay\neq 1$. Further $ax\neq 1$ since $\bar{a}+\bar{x}=0$ implies $\bar{a}=0$ and similarly $ay\neq 1$. By Lemma 2.2, $c=\sup\{ay\}^{\perp}\in C(A)$ and $0\neq c\neq 1$. Since $ax\in\{ay\}^{\perp}$, we have $0<ax\leq c$; since $\bar{a}\in\{ay\}^{\perp}$, we have $\bar{a}\leq c$, i.e., $\bar{c}\leq a$. Note that $\bar{c}\neq a$ because if so, $ax\leq c=\bar{a}$ and hence $ax=a^2x\leq a\bar{a}=0$. It follows that $\bar{c}\in C(A)$ and $0<\bar{c}<a$, which is a contradiction to the fact that a is an atom of $C(A)$.

The 2nd part of the lemma follows from the first isomorphism Theorem; Let $f: A\to I_a=\downarrow a$ by $f(x)=ax$. (See Theorem 7,8 [2]). Then the kernel of f is $I_a^{\perp}=\downarrow \bar{a}$. Thus the quotient of A modulo $\downarrow \bar{a}$ is isomorphic to $\downarrow a$. □

Remark. For a prime ideal P of A, there exists at most one atom a of $C(A)$ such that $\bar{a}\in P$. For, if P contains two such atoms a_1 and a_2 of $C(A)$ ($a_1\neq a_2$), then $\bar{a_1}\vee \bar{a_2}=\bar{a_1}+\bar{a_2}=1\in P$.

Proposition 2.4. If B is a complete subalgebra of the unit interval MV-algebra I ($=[0, 1]$), then B is either I itself or a finite MV-algebra $A(m)$ for some $m\in \mathbf{Z}^+$.

Proof. If B has an atom b, say its order is m, then evidently B is isomorphic to $A(m)$. Now assume that B does not have atoms. Then B must be I. For, suppose $I-B\neq\varnothing$. Then for any $x\in I-B$, let $b_0=\sup\{\downarrow x\cap B\}$ and $b_1=\inf\{\uparrow x\cap B\}$ then b_1 covers b_0 in B since $b_0<x<b_1$ and $x\notin B$. By Theorem 1.1, B has an atom. This is a contradiction. □

Lemma 2.5. Let A be a complete atomic center MV-algebra. Let a be an atom of $C(A)$. Then the MV-algebra $I_a = < \downarrow a, +, \cdot, \sim, 0, a >$ is isomorphic to either $A(m)$ for some $m \in \mathbb{Z}$ or the unit interval MV-algebra.

Proof. Since A is complete, so is I_a. Thus I_a is a complete semi-simple (actually, simple) linearly ordered MV-algebra. It follows that I_a is Archimedean and hence I_a is locally finite (Theorem 31, 32 [1]).

We have $A/I_{\bar{a}}$ is locally finite, since $A/I_{\bar{a}} \cong I_a$. It follows that I_a is embedded into I. (see the remark on page 2 [2]). By Proposition 2.4. I_a is isomorphic to either $A(m)$ for some $m \in \mathbb{Z}^+$ or I. □

Remark. In the above proof, since $A/I_{\bar{a}}$ is locally finite we have $I_{\bar{a}} = \downarrow \bar{a}$ is actually a maximal ideal of A by (Theorem 4.7 [3]).

Proposition 2.6. Let A be a complete atomic center MV-algebra and $\{a_\alpha | \alpha \in \Gamma\}$ be the set of all atoms of $C(A)$. Then A is isomorphic to $\prod\{I_\alpha \mid \alpha \in \Gamma\}$ where $I_\alpha = I_{a_\alpha}$ for each $\alpha \in \Gamma$.

Proof. Define $\varphi: A \to \prod I_\alpha$ by $\varphi(x) = \langle a_\alpha x \rangle_{\alpha \in \Gamma}$ for each $x \in A$, and define $\psi: \prod I_\alpha \to A$ by $\psi(\langle x_\alpha \rangle) = \sup\{x_\alpha \mid \alpha \in \Gamma\}$ for each element $\langle x_\alpha \rangle$ of $\prod I_\alpha$. Then clearly φ and ψ are both MV-homomorphisms. By Theorem 5 [2] it is easy to see that $\psi \circ \varphi = \mathrm{id}_A$ and $\varphi \circ \psi = \mathrm{id}_{\prod I_\alpha}$. □

In summary, the following theorem has completely characterized complete atomic center MV-algebras.

Theorem 2.7. If A has a complete atomic center MV-algebra, then A is isomorphic to a direct product of a cube and $\prod\{A(m) \mid m \in \mathbb{Z}^+\}$, where cube means a product of I's.

In [2], it is shown that if A is a complete nonatomic MV-algebra and if A has at least one atom, then $A \cong B \times C$ where B is complete atomic, C is complete atomless MV-algebra. Furthermore if A is atomic, then $A \cong B$ and C is disappeared as follows:

Corollary 2.8. If A is a complete atomic MV-algebra with $C(A)\neq 2$, then $A\cong\prod A(m)$.

3. The δ-completion of semi-simple MV-algebra

By a topological MV-algebra, we mean a pair (A, τ) where A is an MV-algebra and τ is a Hausdorff topology on A such that all operations $+$, $\cdot$ and $-$ are continuous.

Clearly every topological MV-algebra (A, τ) is also a topological distributive lattice and $C(A)$ is a closed subset of A.

The following lemma is well-known [5]:

Lemma 3.1. If (A, τ) is a compact topological MV-algebra, then

(i) A is a complete lattice
(ii) $C(A)$ is a compact Boolean algebra, i.e., it is a power set Boolean algebra.

By Theorem 2.7, we then have the following lemma:

Lemma 3.2. If (A, τ) is a compact MV-algebra with $C(A)\neq 2$, then $A\cong I^{\lambda}\times\prod\{A(m)\mid m\in\Lambda\subset \mathbf{Z}^{+}\}$ where the cube I^{λ} is the connected atomless part of A for some cardinal λ, and $\prod\{A(m)\}$ is the totally disconnected atomic part of A for some subset Λ of $\mathbf{Z}^{+}$.

Now we turn to characterize semi-simple algebras as subalgebras of a cube.

First of all, we note that the unit interval algebra $(I, \oplus, \odot, ^{-}, 0, 1)$ is a topological MV-algebra under the ordinal topology. For $x, y\in I$, $x\oplus y=\min\{1, x+y\}=\frac{1}{2}\{1+x+y-|1-x-y|\}$ and $\bar{x}=1-x$ are continuous and hence $x\odot y$ is continuous, where $+$, $-$ are the real operations of I.

Let A be a semi-simple MV-algebra and let $H=\hom(A, I)$ be the set of all homomorphisms of A to the I.

Clearly, the cube $I^H = \prod\{I_f \mid f \in H\}$ has the compact topology τ, its product topology, for which (I^H, τ) is a compact MV-algebra.

Let e: $A \to I^H$ be the evaluation map: $e(x) = \langle f(x) \rangle$ for each $x \in A$. Since A is semi-simple, e is injective and hence A is embedded into I^H. Then $A \cong e(A) \subset I^H$. Since $(e(A), \tau_{e(A)})$ is a topological MV-subalgebra of I^H under the relative topology $\tau_{e(A)}$ of τ.

It us known [5] that the closure of subalgebra B of a topological universal algebra A is again a subalgebra.

Setting $\delta(A) = \Gamma(e(A))$ where Γ is the closure operation of I^H, we call $\delta(A)$ the δ *-completion* of A.

Evidently, $\delta(A)$ is a compact Hausdorff MV-algebra under its relative topology and hence $\delta(A)$ is a complete atomic center MV-algebra.

Again by Theorem 2.7, we have that $\delta(A)$ has the following type: $\delta(A) \cong I^{H_o} \times \prod\{A(m) \mid m \in \wedge \subset \mathrm{Z}^+\}$, where $H_o \subset H$.

Let A be a semi-simple MV-algebra. Then A is embedded into I^H. Since $A \cong (e(A), \tau_{e(A)})$, A has the topology τ_A so that $(A, \tau_A) \cong (e(A), \tau_{e(A)})$ is isomorphic algebraically and topologically. τ_A is called the *intrinsic topology* of A.

Then we have the following Theorem:

Theorem 3.3. Any semi-simple MV-algebra A is densely embedded into $I^{H_o} \times \prod\{A(m) \mid m \in \wedge \mathrm{Z}^+\}$ under its intrinsic topology, where $H_o \subset H$, a subset $\wedge$ of Z^+. And $\| H_o \| + \| \wedge \| = \| H \|$ where $H = \mathrm{hom}(A, I)$. Furthermore, A is a subdirect product of $\prod_{f \in H_o} J_f \times \prod A(m)$ where J_f is a dense subalgebra of I for each $f \in H_o$.

Proof. The first part of the theorem already has been shown in the above. For the second part, let $\delta(A)$ be the δ-completion of A and $A \xrightarrow{e} e(A) \subset \delta(A) \subset I^H$. For each $f \in H$ and the f^{th} projection p_f of I^H onto I, setting $p_f(e(A)) = J_f$ for each $f \in H_0$, it is easy to see that J_f is a dense subalgebra of I. Note that for the atomic part $\prod A(m)$, A has a exactly same copy of subalgebra as $\prod A(m)$ because $p_m(e(A)) = A(m)$ for each $m \in \wedge$, p_m is the m^{th} projection of I^H onto $A(m)$. □

Examples. We give several typical examples of dense subalgebras of I.

1. I itself.
2. The subalgebra of all rationals in I.
3. The subalgebra of all algebraic numbers in I.
4. The subalgebra of dyadic numbers in I.
5. The subalgebra of all numbers of type $r+s\sqrt{2}$ in I for all rationals r and s.

Here we study some important properties of the δ-completion; among those, a useful property in an extension property. From this property one can easily show that the category of a complete and atomic center MV-algebras is an epireflective subcategory of the category of all semi-simple MV-algebras.

Lemma 3.4. Let A, B be two semi-simple MV-algebras and τ_A, τ_B are their intrinsic topologies, respectively. If $\phi : A \to B$ is an MV-homomorphism then the $\varphi : (A, \tau_A) \to (B, \tau_B)$ is continuous.

Proof. Since (B, τ_B) is embedded into I^G, where $G=\hom(B, I)$ τ_B is the initial topology with respective to the source $G_B = \{g|_B | g \in G\}$. And τ_A is also the initial topology with respect to $H_A = \{f|_A | f \in H\}$. Since $g|_B \circ \varphi \in H_A$ for each $g \in H$, and hence $g|_B \circ \varphi$ is continuous. Thus φ is continuous. □

Lemma 3.5. If A is a complete atomic center MV-algebra, then (A, τ_A) is a compact MV-algebra.

Proof. Since $A \cong I^{H_0} \times \prod A(m)$ by Theorem 3.3, A is closely-embedded into I^H. Therefore (A, τ_A) is compact. □

The following lemma is easy to prove:

Lemma 3.6. If $f: A \to \prod\{A_i \mid i \in I\}$ is a map and if $p_i f$ is a homomorphism for all projection $p_i: \prod A_i \to A_i$ for all $i \in I$ then f is a homomorphism.

Now we prove the extension properties of the δ-completion.

Theorem 3.7. Let A be a semi-simple MV-algebra and $e: A \hookrightarrow \delta(A)$ be the embedding. For each complete atomic center MV-algebra B and a homomorphism $f: A \to B$, there exists a homomorphism $F: \delta(A) \to B$ such that $F \circ e = f$.

Proof. Let $H = \hom(A, I)$, $G = \hom(B, I)$ and let $e: A \to e(A) \subset I^H = \prod\{I_f \mid f \in H\}$ and $e': B \to e'(B) \subset I^G = \{I_g | g \in G\}$ be the evaluation maps of A and B, respectively. Note that e and e' are point-separating, i.e., they are both injective.

For a homomorphism $f: A \to B$, we have the situation:

$$\begin{array}{l} A \hookrightarrow e(A) \subset \prod\{ I_h \mid h \in H\} \\ f\downarrow \\ B \hookrightarrow e'(B) \subset \prod\{ I_g \mid g \in G\}. \end{array}$$

We define a map $\bar{f}: \prod I_h \to \prod I_g$ as follows: for each $x \in \prod I_h$, $\langle \bar{f}(x) \rangle_g = g \cdot f(x)$, i.e., $q_g(\bar{f}(x)) = p_{gf}(x)$ where p_h and q_g are the h^{th} and the g^{th} projection of $\prod I_h$ and $\prod I_g$ onto I_h and I_g for each $h \in H$ and $g \in G$, respectively. First, we show that $\bar{f}$ is a homomorphism. Indeed $q_g \bar{f} = p_{fg}$ is a homomorphism for all $g \in G$. By Lemma 3.6, $\bar{f}$ is a homomorphism.

Secondly, we show that $\bar{f}(e(A)) \subset e'(B)$. For each $a \in A$, $q_g(\bar{f}(e(a)) = p_{gf}(e(a)) = g \circ f(a) = q_g(e'(f(a))$ for all $g \in G$. Thus $\bar{f}(e(a)) = e'(f(a))$.

Thirdly, we prove that $\bar{f}(\delta(A)) \subset e'(B)$. Since B is a complete atomic center, (B, τ_B) is compact, and hence $e'(B)$ is closed (or compact) in $\prod I_g$. Further, $\bar{f}$ is continuous, so we have that $\bar{f}(\delta(A)) = \bar{f}(\Gamma(e(A)) \subset \Gamma'(\bar{f}(e(A)) \subset \Gamma'(e'(B)) = e'(B)$, where Γ and Γ' are the closure operation of the product spaces $\prod I_h$ and $\prod I_g$, respectively.

Finally, setting $F = e'^{-1}(\bar{f} \mid_{\delta(A)})$, we show that $F: \delta(A) \rightarrow B$ is a required extension of f. Indeed $F \circ e(a) = e'^{-1}(\bar{f} \mid \delta(A))(e(a)) = e'^{-1}(e'(f(a))) = f(a)$ for all $a \in A$. The proof is complete. □

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