Pattern Formations with Turing and Hopf Oscillating Pattern in a Discrete Reaction-Diffusion System

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Localized structures with fronts connecting a Turing patterns and Hopf oscillations are found in discrete reaction-diffusion system. The Chorite-Iodide-Malonic Acid (CIMA) reaction model is used for a reaction scheme. Localized structures in discrete reaction-diffusion system have more diverse and interesting features than ones in continuous system. Various localized structures can be obtained when a single perturbation is applied with variation of coupling strength of two intermediates. Roles of perturbations are not so simple that perturbations are sources of both Turing patterns and Hopf oscillating domains, and spatial distribution of them is determined by strength of a perturbation applied initially.

Introduction

Pattern formation in a reaction-diffusion system has been subjects of many authors since Turing's seminal work.[1] Diverse and complex patterns in a reaction-diffusion system have been found for many reaction schemes theoretically and experimentally.[2-11] Most of the studies are concerned with a continuous reaction-diffusion system, while discrete reaction-diffusion systems are less studied. Discrete reaction-diffusion systems consist of cells or lattices in which chemical reactions occur, and are coupled with each other by mass transfer through their membranes. Discrete reaction-diffusion systems are more closely related to biological systems than continuous systems and can show very interesting and various patterns in a two-dimensional space in response to a simple perturbation.[11,12] Patterns formed in a reaction-diffusion system usually show only a single kind of spatiotemporal behavior.

A localized structure in continuous system was reported by O. Jensen and co-workers.[2-4] They obtained localized structures with a front connecting a Turing pattern and homogeneous stationary state, or a Turing pattern and temporally oscillating domain using Chorite-Iodide-Malonic Acid (CIMA) reaction model.[13-15] In the 'pinning area', the front is pinned, and the localized structure is maintained. But outside pinning area the front is not stationary, and moves with inconstant speed. The localized structure becomes a Turing pattern or homogeneous stationary state (or temporally oscillating state).

In this work it is reported that there exist more diverse types of localized structure than ones reported formerly. The structures can be formed through simple perturbation in a discrete reaction-diffusion system. The structure consists of Turing patterns and Hopf oscillating spatial subregions and show more diverse and complex characteristics than ones in continuous system. Turing and Hopf oscillating patterns are formed simultaneously in a bounded space in response to a single perturbation. The composite patterns are not transient structure, but maintains their spatial structure unless any serious additional perturbations are applied to them. A geometry of the fronts connecting Turing pattern and Hopf oscillating region is not so simple that Turing patterns are distributed spatially over whole space and spatial composition of two regions show very interesting geometry. The patterns can be observed when the coupling strength for intermediates of the chemical oscillating reaction is set near the Turing bifurcation. The coupling strength can be expressed as the diffusion coefficient and is a measure of how strongly cells are coupled.

The Models and Numerical Calculation

Two-dimensional array of cells will be considered for CIMA model.[13-15] Rate equations in a scaled form are as follows:[13]

$$\frac{dx_{ij}}{dt} = a - x_{ij} - \frac{4x_{ij}y_{ij}}{1+x_{ij}^2} - D_x(4x_{ij} - x_{i+1,j} - x_{i-1,j} - x_{i+j,1} - x_{i-j,i})$$

$$\frac{dy_{ij}}{dt} = b\left(x_{ij} - \frac{4x_{ij}y_{ij}}{1+x_{ij}^2}\right) - D_y(4y_{ij} - y_{i-1,j} - y_{i-1,j} - y_{i,j+1} - y_{i,j-1}) \quad (1)$$

The x_{ij} and y_{ij} are concentrations of two intermediates of the chemical oscillating reaction in a cell located in the ith row and jth column of a two-dimensional array with the length of N. a and b, corresponding to concentrations of two reactants, are kept constant. D_x and D_y are the coupling strength of x and y. It is assumed that mass transfer through the membrane occurs by linear diffusion. The coupling strength will be used for the control parameter. There are four neighborhoods for a cell (two or three neighborhoods for the cells at the boundary since a no-flux boundary conditions are applied). Numerical simulations of this model were carried out with the Gear method.

Figure 1 shows a representative example of the patterns

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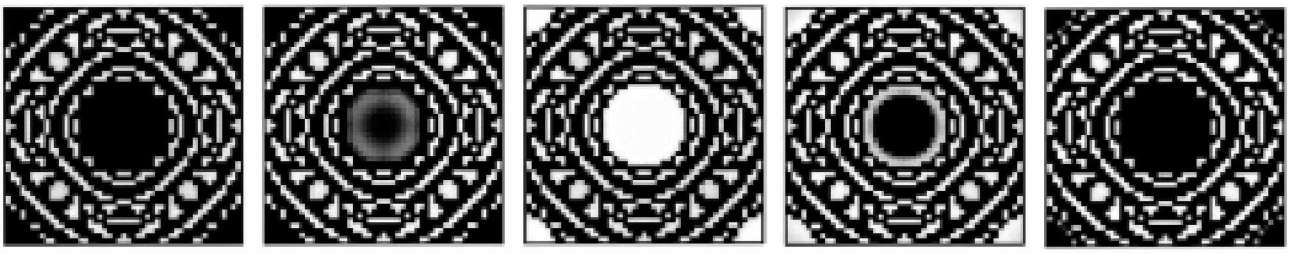


Figure 1. Time evolution of the pattern mixed with stationary and temporal regions (a=30.0, b=1.0, log(D_y/D_x)=1.0, δ_x=log(D_x)=-1.000).

mixed with two spatial regions distinguished by distinct interfaces: stationary mosaic patterns and temporally oscillating region. The composite structure consists of regions of ring-shaped mosaic patterns and oscillating regions in the center and four corners. The composite structure can be formed in response to a single perturbation of x (1.5 times higher value than a stationary state) at the center of 2d-array of cells. The structure is not transient, but permanent if no additional perturbations are applied. One dimensional diagonal cuts of the 2-d patterns are shown in Figure 2. Stationary patterns and temporal oscillating patterns are clearly distinguishable. The mosaic patterns have typical features of complete stationary patterns formed in different conditions. The mosaic patterns consist of multiple motifs and have same qualitative features as the work by Maselko.[12] The temporally oscillating regions at the center and four corners oscillate with the same frequency. However these two oscillating patterns may oscillate with different phase because they are isolated spatially by stationary patterns. They also have different structures in space. Their characteristics will be shown in detail later in Figure 4. Temporal evolution of the composite structures is very complex as depicted in Figure 3. After a cell at the center is perturbed, spatial waves like rings propagate towards the boundaries, and stationary and temporal regions begin to be separated. Temporal regions are formed faster than stationary regions which need more

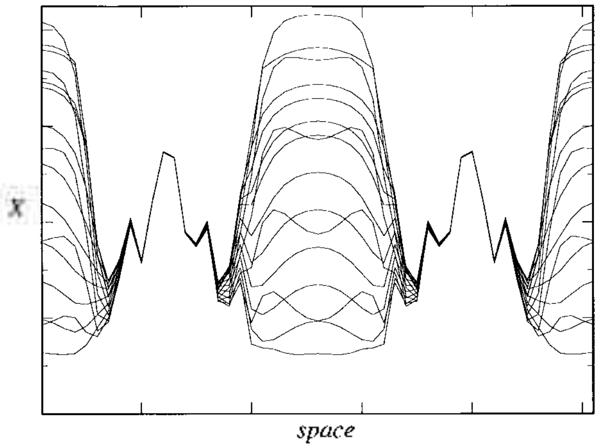


Figure 2. Time evolution of a diagonal cut of the 2d-pattern profile of the pattern (t=500-523, Δt=1, other conditions are the same as ones in Figure 1).

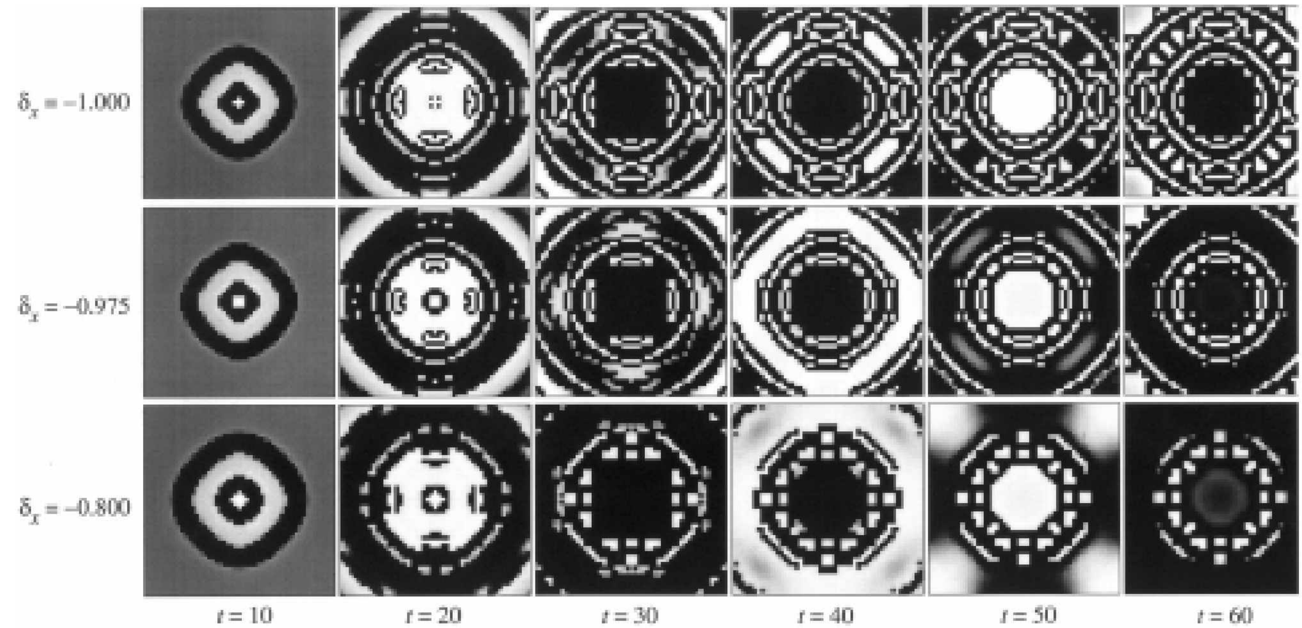
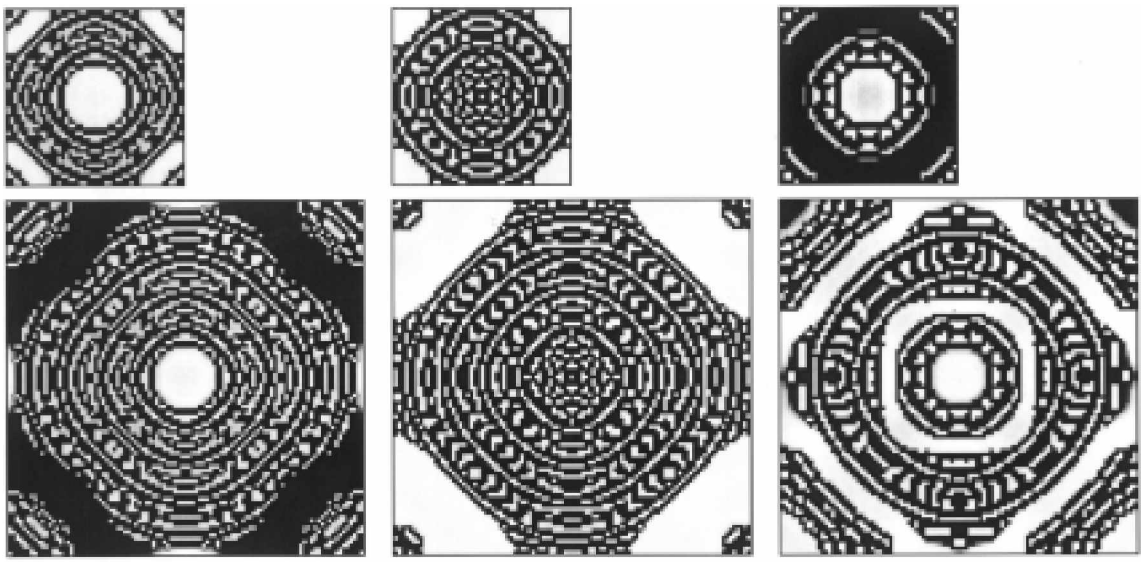


Figure 3. Time evolutions of the patterns with different coupling strength (other conditions are the same as ones in Figure 1).



$\delta_x = -1.075$(SC) $\delta_x = -1.050$(C) $\delta_x = -0.900$(SRC)

Figure 4. Comparison of temporal regions appeared in the patterns in systems with different mesh size (upper low - 51 × 51 mesh size; lower row - 101 × 101; other conditions are the same as ones in Figure 1).

time to form complete patterns. The patterns with different coupling strengths evolve temporally in similar fashions at initial steps of evolution, but as mosaic patterns begin to be formed, stationary and temporal regions are separated and their composition intrinsic to applied conditions are determined.

Important difference of the composite structure from the localized structure reported earlier[2-4] is spatial arrangement of two patterns composing of the structure. The localized structure is composed of hexagonal stationary pattern centering around the position at which perturbation is applied and oscillatory subregion around the stationary subregion. But the composite structure of the pattern has more diverse and interesting features than the localized structure. The pattern centering around perturbed point can be not only stationary but also oscillatory, and two patterns are not localized in a simple way of stationary pattern centered at perturbed point and oscillatory pattern around the centered pattern. The way of localization of two patterns in the composite structure is more various than one of the localized structure as described above.

By changing coupling strength of X and Y with constant ratio of them, patterns with various spatial composition of stationary and temporal regions can be obtained. When the ratio of D_X and D_Y is set to 10, the patterns appear in conditions of $-1.100 \leq \delta_X (= \log(D_X)) \leq -0.775$. If δ_X is smaller than -1.100, only Turing patterns are obtained, while if δ_X is larger than -0.775, homogeneously oscillating patterns appear. Among results the patterns with δ_X = -1.075, -1.050, and -0.900 are presented in Figure 4. The patterns are obtained using mesh size 51 × 51 and 101 × 101 repectively. The oscillating regions in the patterns can be classified by their geometric features. Three elements can be defined for the classification of temporal region in the patterns. They correspond to the circular spot at the center(S), the ring-shaped region(R), and regions at the four corners(C). The oscillating patterns have one or more elements with co-existing stationary mosaic patterns which isolate it (them) spatially; they are labeled as S, R, C, and combinations of them in Figure 4. The R subregion is new type of component of the composite pinning structure. The R regions can appear repeatedly under certain condition (as shown in the third pattern in Figure 4). The S and R regions have geometric features distinct from the C regions. They have an intrinsic length and shape that does not depend on the geometric features of the system. The C regions interact with boundaries and fit their size and shape to boundaries of the system. The differences between the types can be seen clearly in the patterns of a larger mesh size (101 × 101) in Figure 4. The S and R regions preserve not only a geometric size and shape, but the positions in which they appear. But the C regions appears at four corners and fit their size to boundaries. The C regions may or may not be formed depending on the geometric size of the system, but S and R regions always appear if the geometric size of the system is larger than the regions.

The patterns in which temporal and stationary subregions coexist are very sensitive to initial perturbations. But after the patterns are settled down, additional perturbations do not change the patterns seriously. Effects of additional perturbations on the patterns are limited to only local area near perturbed cells such. By changing the strength of the perturbation, geometrical features of the patterns and the composition of temporal and stationary regions can be varied completely. Figure 5 shows various patterns formed with the increasing strength of perturbation. The relation of perturbation strength and change of the patterns is not so simple. It is worthwhile to note that the tolerance of the numerical method can be a factor to influence formation of the patterns when it is not sufficiently small. Tolerance must be relatively small (10^{-8}~10^{-9}) to obtain patterns indepen-

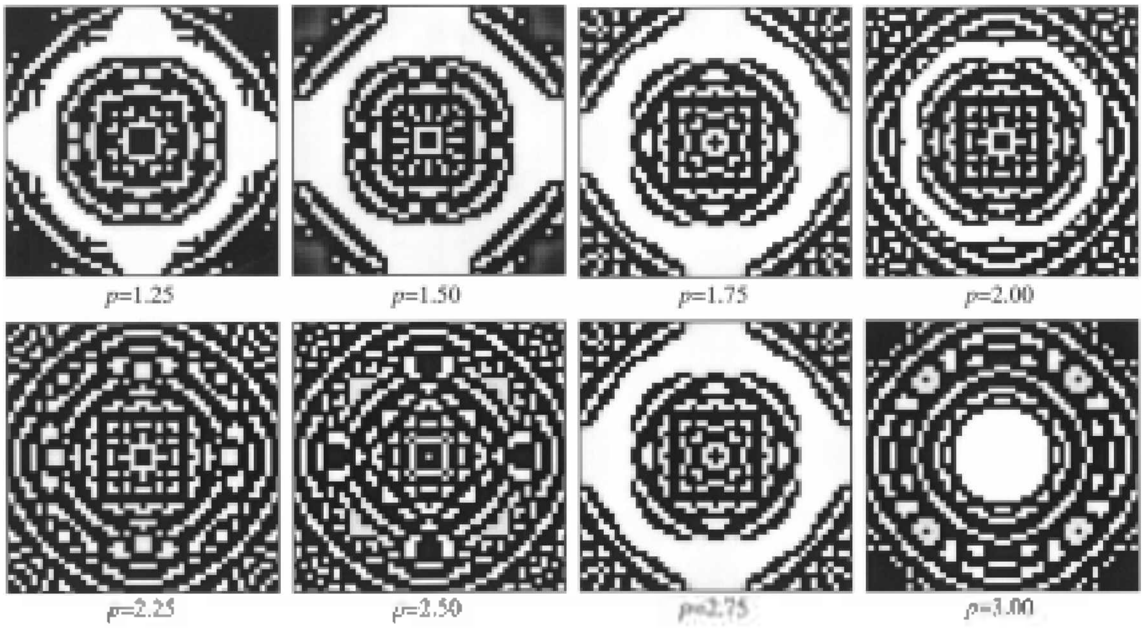


Figure 5. Pattern changes in the variation of perturbation strength (The cell (26,26) is perturbed with x value to $p \times x_{ss}$. x_{ss} is a stationary value of x. other conditions are the same as ones in Figure 1).

dent of the tolerance.

Discussion

It is shown in this paper that the localized structure can exist in diverse types in discrete reaction-diffusion system. The type of the patterns can be expected in other reaction models. such as autocatalytic model and Brusselator. The patterns are very sensitive to the perturbation strength applied to the system. and their geometric features change as the strength of perturbation varies. Roles of perturbation or initial conditions are very important to pattern formations in the system and have to be studied in more detail. Even though compositions of temporal and stationary regions change. there seems to be some rules for organization of the patterns. Many stationary regions appearing in different conditions have qualitative consistency. Mosaic patterns in spationary regions consist of multiple motifs and look like chaotic. But they have simple symmetry and ways of formation. Mosaic patterns may show some ways to form very complex structures which are constructed by cooperative behaviors of many components constituting a total system. With two distinct temporal features. the patterns have more abundant aspects to describe phenomena occuring in reaction-diffusion system and reaction-diffusion system get more broad applications to pattern formation phenomena. These kinds of pattern formations may give rise to more diverse and flexible pattern formations in nature. especially in a biological system even though their definite examples have not yet been found.

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