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Exponentially Weighted Moving Average Control Charts for the Inverse Gaussian Distribution[1]

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Abstract

In this paper, we develop optimal decision interval exponentially weighted moving average(EWMA) scheme for the process mean and the reciprocal measure of dispersion of the inverse Gaussian distribution.

Key Words and Phrases: EWMA control chart, positive skewness, inverse Gaussian distribution, ARL

1. Introduction

EWMA control charts are very effective at detecting persisting special cause. The most common EWMA charts assume that the process measurement being monitored follows the normal distribution. Many industrial problems yield measures with skewed, positive distributions – examples are component reliabilities, times to completion of tasks and insurance claims. Non-normal measures such as these should not be monitored using procedures based on the normal distribution. The inverse Gaussian distribution provides a flexible distribution that can be used to model positive skew quantities, and therefore provides an effective framework for statistical process

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control on processes producing such measures. Positive skewed processes can often be well modelled by the inverse Gaussian distribution. Edgeman(1989) proposed a Shewhart control charting scheme for the inverse Gaussian distribution. Nabar and Bilgi(1994) developed a methodology for cumulative sum(CUSUM) charts for the mean of an inverse Gaussian distribution by using a mask. They approximated the average run lengths(ARLs) by using a method due to Edgeman and Salzberg(1991). Hawkins and Olwell(1997) derived the optimal CUSUM scheme for detecting step changes in the location or the shape parameter of the inverse Gaussian distribution. This paper defines the optimal EWMA control chart schemes of the inverse Gaussian distribution and evaluates its performance in detecting step changes in each of these parameters. For the simulation, an algorithm by Micheal, Schucany & Hass (1976) was used to generate the inverse Gaussian distributed pseudorandom variates.

2. Inverse Gaussian EWMA control limit

2.1 Inverse Gaussian distribution

Suppose that the process characteristic, X, that is being monitored is distributed according to the inverse Gaussian density

$$f(x) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left(\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\right), \quad x>0,\ \mu>0,\ \lambda>0, \tag{1}$$

where the parameter μ is the process mean and is unknown and the parameter λ is a reciprocal measure of dispersion (Johnson & Kotz, 1970) as well as being the shape parameter of the density and may be either known or unknown. In this paper the notation $X \sim \mathrm{IG}(\mu, \lambda)$ will refer to a process characteristic, X, distributed according to (1).

For a random sample $X \sim \mathrm{IG}(\mu, \lambda)$, $n \geq 2$,

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{2}$$

and

$$V = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}}\right) \tag{3}$$

are the maximum likelihood estimates of μ and $\frac{1}{\lambda}$; $\bar{X} \sim \mathrm{IG}(\mu, n\lambda)$, $n\lambda V \sim \chi^2_{n-1}$, the chi-square distribution with $(n-1)df$, and two are stochastically independent (Tweedie, 1957).

Assume that m samples of n items each have been selected from the process when it is in control and that process output is distributed according to the inverse Gaussian probability density function (1). Let $\bar{X}_j$ be the j th sample means and let V_j be the value of (3) for the j th sample. Further, let $\bar{\bar{X}}$ and $\bar{V}$ be defined, respectively, by

$$\bar{\bar{X}} = [\bar{X}_1 + \bar{X}_2 + \cdots + \bar{X}_m]/m$$

and

$$\bar{V} = [V_1 + V_2 + \cdots + V_m]/m.$$

Then $\bar{\bar{X}}$ is a minimum variance unbiased estimator of the process mean, μ, and $\bar{V}$ is a linear unbiased estimator of $\frac{1}{\lambda}$.

2.2 Inverse Gaussian EWMA control limit

The EWMA control chart was introduced by Roberts(1959). The exponentially weighted moving average for the process mean is defined as

$$Z_t = r\bar{X}_t + (1-r)Z_{t-1}, \tag{4}$$

where $0 < r \leq 1$ is a constant and the starting value (required with the first sample at $t = 1$) is $Z_0 = \bar{\bar{X}}$.

To demonstrate that the EWMA Z_t is a weighted average of all previous sample means, we may substitute for Z_{t-1} on the right-hand side of (4) to obtain

$$\begin{aligned} Z_t &= r\bar{X}_t + (1-r)[r\bar{X}_{t-1} + (1-r)Z_{t-2}] \qquad (5) \\ &= r\bar{X}_t + r(1-r)\bar{X}_{t-1} + (1-r)^2 Z_{t-2} \ . \end{aligned}$$

Continuing to substitute recursively for Z_{t-j}, $(j = 2, 3, \cdots, t)$, we obtain

$$Z_t = r\sum_{j=0}^{t-1}(1-r)^j \bar{X}_{t-j} + (1-r)^t Z_0 \ . \tag{6}$$

If the $\bar{X}_j$ are independent random variables with variance $\frac{\mu^3}{n\lambda}$, then the variance of Z_t is

$$\sigma^2{}_{Z_t} = \frac{\mu^3}{n\lambda}\left(\frac{r}{2-r}\right)[1-(1-r)^{2t}]. \tag{7}$$

As t increases, $\sigma^2_{Z_t}$ increases to a limiting value

$$\sigma^2_Z = \frac{\mu^3}{n\lambda}\left(\frac{r}{2-r}\right). \tag{8}$$

Consequently, the upper and lower control limits are

$$\text{UCL} = \bar{\bar{X}} + 3\sqrt{\frac{\mu^3}{n\lambda}\left(\frac{r}{2-r}\right)[1-(1-r)^{2t}]} \tag{9}$$

and

$$\text{LCL} = \max\left\{0,\ \bar{\bar{X}} - 3\sqrt{\frac{\mu^3}{n\lambda}\left(\frac{r}{2-r}\right)[1-(1-r)^{2t}]}\ \right\}. \tag{10}$$

2.3 EWMA control limit for reciprocal measure of dispersion

Similarly, the exponentially weighted moving average for reciprocal measure of dispersion is performed as following that

$$W_t = rV_t + (1-r)W_{t-1}, \tag{11}$$

where $0 < r \leq 1$ is a constant and the starting value (required with the first sample at $t = 1$) is $W_0 = \bar{V}$.

We may substitute for W_{t-1} on the right-hand side of (11) to obtain

$$W_t = r\sum_{j=0}^{t-1}(1-r)^j V_{t-j} + (1-r)^t W_0. \tag{12}$$

Since $n\lambda V \sim \chi^2_{n-1}$, variance of V is $\frac{2(n-1)}{(n\lambda)^2}$. Then the variance of W_t is

$$\sigma^2_{W_t} = \frac{2(n-1)}{(n\lambda)^2}\left(\frac{r}{2-r}\right)[1-(1-r)^{2t}]. \tag{13}$$

As t increases, $\sigma^2_{W_t}$, increases to a limiting value

$$\sigma^2_W = \frac{2(n-1)}{(n\lambda)^2}\left(\frac{r}{2-r}\right). \tag{14}$$

Consequently, the upper and lower control limits for reciprocal measure of dispersion are

$$\text{UCL} = \bar{V} + 3\sqrt{\frac{2(n-1)}{(n\lambda)^2}\left(\frac{r}{2-r}\right)[1-(1-r)^{2t}]} \tag{15}$$

and

$$\text{LCL} = \max\left\{0,\ \bar{V} - 3\sqrt{\frac{2(n-1)}{(n\lambda)^2}\left(\frac{r}{2-r}\right)[1-(1-r)^{2t}]}\ \right\}. \tag{16}$$

3. Simulation study and Conclusion

If the X_n is inverse Gaussian with parameters μ and λ and we define the rescaling

$$Y_n = \frac{\lambda X_n}{\mu^2} \tag{17}$$

then Y_n follows distribution $\text{IG}(\phi, \phi^2)$ where

$$\phi = \frac{\lambda}{\mu}. \tag{18}$$

This is a single-parameter distribution, reducing the tabulation problem to one of double rather than triple entries, and making the production of useful ARL tables a reasonable objective. The extent to which an inverse Gaussian density differs from a normal density essentially depends on its shape. The inverse Gaussian density is unimodal and its shape depends only on the value of ϕ (see Figure 1 of Johnson & and Kotz(1970)). As ϕ tends to infinity, with μ fixed, the inverse Gaussian density tends to a unit normal distribution (Johnson & Kotz, 1970).

To investigate the advantage of EWMA control chart in using inverse Gaussian distribution rather than normal distribution when, in fact, the measured process characteristic $X \sim \text{IG}(\mu, \lambda)$, 5000 samples each of size $n = 4, 6, 8$ and 10 were generated from inverse Gaussian density (1) with $\phi = 0.25, 0.50, 1, 2, 4, 8, 16, 32, 100, 1000$ and 5000. Results of the simulation are recorded in Table 1. Examination of Table 1 reveals that, for small values of ϕ, the EWMA control chart gives progressively better (yet still poor) results as the sample size, n, increase. But, for large values of ϕ, ARLs are not under influence of the sample size.

Table 2 presented here can be used in a similar way to construct EWMA charts with specified ARLs at $\phi = \phi_0$ and at $\phi = \phi_1$, some deviation from target. It shows that the EWMA control chart is very effective against small process shifts for the small r.

Table 1. ARL values for EWMA control chart

Φ	r							
	0.2				0.5			
	n				n			
	4	6	8	10	4	6	8	10
0.25	146.148	167.417	179.935	194.661	68.982	69.541	76.448	83.380
0.5	182.309	213.260	233.589	250.912	73.011	84.963	93.613	106.946
1	228.983	274.116	297.343	319.036	98.253	119.964	132.200	150.156
2	297.084	325.257	362.076	368.551	133.298	153.181	175.393	200.978
4	355.480	389.166	400.788	411.832	174.427	212.996	233.343	251.959
8	399.247	419.672	434.571	435.117	228.696	261.196	275.373	265.227
16	432.718	443.057	438.571	447.376	281.200	291.980	312.944	318.556
32	444.912	445.447	454.471	454.039	339.086	323.250	346.776	331.070
100	457.460	464.625	459.983	464.008	346.154	356.275	350.182	353.365
1000	528.151	536.472	534.302	548.574	359.768	361.691	362.089	356.327
50000	543.402	536.338	537.679	541.341	363.469	362.475	372.350	362.976

Table 2. ARL values of Schemes (optimal in detecting $\phi_0 = 1.0$, $n = 10$)

Φ	r				
	1	0.8	0.5	0.2	0.1
1.0	96.210	106.814	144.775	319.982	484.207
1.1	52.034	55.320	50.181	56.094	33.104
1.2	30.094	25.348	22.686	9.049	-
1.3	18.715	14.125	9.665	0.070	-
1.4	10.456	8.259	5.092	-	-
1.5	7.383	4.969	1.908	-	-
1.6	4.434	3.143	0.249	-	-
1.7	2.822	1.930	0.249	-	-
1.8	2.048	1.234	0.050	-	-
1.9	1.374	0.705	0.015	-	-
2.0	1.074	0.406	-	-	-
2.5	0.148	0.012	-	-	-
3.0	0.010	-	-	-	-

The dash denotes an ARL that is less than 0.001.

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