Journal of the Korean
Data & Information Science Society
2000, Vol. 11, No. 2, pp. 303 ~ 312

Jackknife Estimation for the Reliability function of Weibull Distribution

In Suk Lee[1] · Jae Soo Kwon[2]

Abstract

We compare the MLE, UMVUE and Jackknife Estimators for the reliability function of the weibull distribution when the sample size is small.

Key Words and Phrases: Weibull distribution, Reliability function, Jackknife estimator

1. INTRODUCTION

Many reliability engineers concern themselves with lifetimes of inanimate objects, such as light bulbs, microprocessors, or drills bits. The Weibull distribution is very often used in this analysis of life data, as there are good theoretical reasons for using it as well as it is the next simplest model to the exponential, and also it is easier to fit a two parameter distribution to a given data set than a one parameter one.

Tanaka(1998) compared four estimators - naive, maximum likelihood (ML), uniformly minimum variance unbiased(UMVU), and invariant optimal estimators - of the reliability function on the Weibull case using the concept of the asymptotic deficiency. That is, he showed that later three estimators have the same asymptotic mean integrated squared error up to the order $o(n^{-2})$.

On the other hand, the Jackknife method is resampling method which was at first introduced by Quenouille(1956) for the purpose of reducing bias. And this

[1]Professor, Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea
[2]Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea

version was latter utilized by Tukey(1958) to develop a general method for obtaining approximate confidence interval. At that time Tukey referred to his method as the "Jackknife". Since then a number of papers have been written on the Tukey method and Quenouille's estimator. They have proven their worth as useful tools for the applied statistician, and the method as well as the estimator are now generally referred to as the Jackknife.

In this paper, we compare the ML, UMVU and Jackknife estimators of the reliability function for the Weibull distribution when the sample size is small.

In section 2, we introduce the Jackknife method and calculate the Jackknife reliability estimator for the ML estimator.

In section 3, we calculate the bias and mean squared error(MSE) of Jackknife reliability estimator by the result of Hurt(1986).

In section 4, we also compare biases and MSEs of ML, UMVU, Jackknife reliability estimator by simulation.

2. JACKKNIFE ESTIMATOR FOR THE ML ESTIMATOR

Suppose $X_1, X_2, \cdots, X_n$ is a sequence of random variables independently and identically distributed according to the Weibull distribution $Wei(\theta, c)$ with a probability density function (p.d.f) of

$$f(x:\theta) = \frac{c}{\theta}\left(\frac{x}{\theta}\right)^{c-1} \exp\left\{-\left(\frac{x}{\theta}\right)^{c}\right\}, (x>0)$$

where c is a known constant and $\theta \in (0, \inf)$ is unknown parameter. Then we well known that the reliability function is given by

$$R_{\theta(t)} = P_\theta\{X_1 \geq t\} = \exp\left\{-\left(\frac{t}{\theta}\right)^{c}\right\}, \quad t>0.$$

Also ML and UMVU estimator for the reliability function on the Weibull case is given by

$$\hat{R}_{ML}(t) = \exp\left\{-\frac{nt^c}{S_n}\right\}, \quad t>0 \tag{2.1}$$

$$\hat{R}_{UMVU}(t) = \left(\frac{S_n - t^c}{S_n}\right)^n - 1, \quad 0<t\leq S_n^{1/c} \tag{2.2}$$

where $\sum_{i=1}^{n} X_i^c = S_n$. By Tanaka(1998), bias of ML estimator is given by

$$E_\theta\left[\hat{R}_{ML}(t)\right] - R_\theta(t) = \frac{1}{2n}\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\} R_\theta(t) + o\left(\frac{1}{n^2}\right) \qquad (2.3)$$

2.1 The Method of Quenouille

Let $\hat{\theta}$ be an estimator defined on the random sample $X_1, X_2, \cdots, X_n$. And we consider the new sample space $(X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ deleted i^{th} subset from original sample. Then we define the estimator $\hat{\theta}^{(i)}$ to be the estimator $\hat{\theta}^{(i)}$ defined on that space which arises when the i^{th} subset has been deleted.

Now let

$$J_i(\hat{\theta}) = n\hat{\theta} - (n-1)\hat{\theta}^{(i)}, \quad i = 1, 2, \cdots, n$$

and

$$\begin{aligned} J(\hat{\theta}) &= \frac{1}{n}\sum_{i=1}^{n} J_i(\hat{\theta}) \qquad (2.4) \\ &= n\hat{\theta} - (n-1)\bar{\hat{\theta}}^{(i)}, \end{aligned}$$

where $\bar{\hat{\theta}}^{(i)} = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}^{(i)}$.

The estimator $J(\hat{\theta})$ is the called the Jackknife and the estimator $J_i(\hat{\theta})$ are called pseudovalue of the Jackknife or pseudo-Jackknife. Both estimators were introduced by Quenouille(1956).

2.2 Calculating The Jackknife Estimator

In this case, we define $\hat{R}_{ML}^{(i)}(t)$ as the ML estimator for the reliability function on the new sample space excepted i^th observation. Also Jackknife estimator is defined by

$$\hat{R}_{JML}(t) = n\hat{R}_{ML}(t) - \frac{n-1}{n}\sum_{i=1}^{n} \hat{R}_{ML}^{(i)} \qquad (2.5)$$

by (2.4). Then we calculated $\hat{R}_{ML}^{(i)}(t) = \hat{R}_{ML}(t; X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n)$ as following

$$\hat{R}_{ML}^{(i)}(t) = \exp\left\{-\frac{(n-1)t^c}{S_n - X_i^c}\right\}, \quad t > 0 \qquad (2.6)$$

by (2.1). Hence we obtain the Jackknife reliability estimator for the ML estimator as following

$$\hat{R}_{JML}(t) = n\exp\left\{-\frac{nt^c}{S_n}\right\} - \frac{n-1}{n}\sum_{i=1}^{n}\exp\left\{-\frac{(n-1)t^c}{S_n - X_i^c}\right\} \tag{2.7}$$

by (2.2).

3. BIAS AND MSE OF ESTIMATORS

Now we introduce the result of Hurt(1986).

Lemma. (Hurt:1986)

(1) If $E|X(t)-\theta|^{q+1} \to 0$ as $\mathrm{t} \to \mathrm{t_0}$, then

$$\begin{aligned} E\,g(X(t),t) &= g(\theta,t) + \sum_{k=0}^{q}\frac{1}{k!}g^{(k)}(\theta,t)E(X(t)-\theta)^k \\ &\quad + o\left(E|X(t)-1|^{q+1}\right). \end{aligned}$$

(2) If $E|X(t)-\theta|^{q+2} \to 0$ as $\mathrm{t} \to \mathrm{t_0}$, then

$$\begin{aligned} \mathrm{Var\,g(X(t),t)} &= \sum_{i=1}^{q}\sum_{j=1}^{q}\frac{1}{i!j!}g^{(i)}(\theta,t)g^{(k)}(\theta,t) \\ &\quad \times \mathrm{Cov[(X(t)-\theta)^i,(X(t)-\theta)^j] + o(E|X(t)-\theta|^{q+2})} \end{aligned}$$

where $j+k <= q+1$.

Using the Lemma, we have the bias of the Jackknife reliability estimator for the ML estimator.

Theorem. The bias of the Jackknife reliability estimator for the ML estimator is following as

$$E_\theta\left[\hat{R}_{JML}(t)\right] - R_\theta(t) = o\left(\frac{1}{n}\right). \tag{3.1}$$

Proof : In order to use the Lemma, Let

$$Z_i = \frac{S_i}{(n-1)\theta^c}, \qquad \eta = \eta(\theta,t) = \left(\frac{t}{\theta}\right)^c,$$

and

$$h(z) = \exp\left(-\frac{\eta}{z}\right), \quad \text{for} \quad z > 0.$$

Then we have

$$\begin{aligned}
h(z)|_{z\to 1} &= R_\theta(t), \\
h'(z)|_{z\to 1} &= -R_\theta(t)\log R_\theta(t), \\
h''(z)|_{z\to 1} &= R_\theta(t)\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\},
\end{aligned}$$

$$\begin{aligned}
E[Z_i - 1] &= 0, \\
E[(Z_i - 1)^2] &= \frac{1}{(n-1)}, \\
E[(Z_i - 1)^3] &= \frac{2}{(n-1)^2}.
\end{aligned}$$

Hence the approximate expectation of the estimator $\hat{R}^{(i)}_{ML}(t)$ is given by

$$E_\theta\left[\hat{R}^{(i)}_{ML}(t)\right] = R_\theta(t) + \frac{1}{2(n-1)}\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\}R_\theta(t) + o\left(\frac{1}{n^2}\right) \quad (3.2)$$

up to the order $o(n^{-2})$ by lemma. Therefore we have the conclusion (3.1) by (3.2) and

$$E\left[\hat{R}_{JML}(t)\right] = nE\left[\hat{R}_{ML}(t)\right] - (n-1)E\left[\hat{R}^{(i)}_{ML}(t)\right].$$

Next, in order to calculate the MSE of estimators, by lemma we first calculate the variance of estimators as following

$$\begin{aligned}
\mathrm{Var}_\theta\left[\hat{\mathrm{R}}_{\mathrm{ML}}(\mathrm{t})\right] &= \frac{1}{n}R_\theta(t)^2\log^2 R_\theta(t) + \frac{2}{n^2}R_\theta(t)^2\log R_\theta(t) \\
&\quad \times\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\} + o\left(\frac{1}{n^2}\right).
\end{aligned} \quad (3.3)$$

Since

$$\begin{aligned}
\mathrm{Var}_\theta\left[\hat{\mathrm{R}}^{(\mathrm{i})}_{\mathrm{ML}}(\mathrm{t})\right] &= \frac{1}{(n-1)}R_\theta(t)^2\log^2 R_\theta(t) + \frac{2}{(n-1)^2}R_\theta(t)^2 \\
&\quad \times\log R_\theta(t)\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\} + o\left(\frac{1}{n^2}\right), \quad (3.4)
\end{aligned}$$

we have the variance of Jackknife reliability estimator as following

$$\begin{aligned}\mathrm{Var}_\theta\left[\hat{\mathrm{R}}_{\mathrm{JML}}(\mathrm{t})\right] &= n^2\mathrm{Var}\left[\hat{\mathrm{R}}_{\mathrm{ML}}(\mathrm{t})\right] - \frac{(\mathrm{n}-1)^2}{\mathrm{n}}\mathrm{Var}\left[\mathrm{R}_{\mathrm{ML}}^{(1)}(\mathrm{t})\right] \\ &= nR_\theta^2(t)\log^2 R_\theta(t) + 2R_\theta(t)^2\log R_\theta(t) \\ &\quad \times\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\} + \frac{(n-1)}{n}R_\theta^2(t)\log^2 R_\theta(t) \\ &\quad +\frac{2}{n}R_\theta^2(t)\log R_\theta(t)\left\{\log^2 R_\theta(t) + 2\log R_\theta(t)\right\} + o(1).\end{aligned} \tag{3.5}$$

Hence we can calculate the MSE of Jackknife reliability estimator as following

$$MSE\left[\hat{R}_{JML}(t)\right] = \mathrm{Var}\left[\hat{\mathrm{R}}_{\mathrm{JML}}(\mathrm{t})\right] + \mathrm{bias}^2\left[\hat{\mathrm{R}}_{\mathrm{JML}}(\mathrm{t})\right]$$

by (3.1) and (3.5).

[Note] By (2.3) and (3.3), also we can calculate the MSE of ML estimator as following

$$MSE\left[\hat{R}_{ML}(t)\right] = \mathrm{Var}\left[\hat{\mathrm{R}}_{\mathrm{ML}}(\mathrm{t})\right] + \mathrm{bias}^2\left[\hat{\mathrm{R}}_{\mathrm{ML}}(\mathrm{t})\right].$$

4. SIMULATION AND CONCLUSION

Design parameters in the simulation experiments include sample size N, shape parameter C, scale parameter θ and time T. We consider the following combinations;

$$\begin{aligned} N &= 8, 14, 20 \\ \theta &= 5.0, 5.2, 5.4, 5.6, 5.8 \\ C &= 2.0, 2.4, 2.8, \\ T &= 2.0, 2.5, 3.0 \end{aligned}$$

For each combination of N, θ, C and T, we generate Weibull random variables by using IMSL and run 3000 simulations. We estimate each estimators and calculated their bias and MSE and the simulation results are given by table 1 and 2.

From the above results, we conclude that

(1) On the Weibull case, we can conclude that the Jackknife reliability estimator is more efficient than the ML estimator, regardless parameters, because it has considerable reduction of bias without increasing mean squared error when the sample size is small.

(2) Comparing to the uniformly minimum variance unbiased estimator, mean squared error of Jackknife reliability estimator is not smaller but it has little difference when the sample size N is 20.

References

1. Tanaka, H. (1998). On reliability estimation in the Weibull case, *J. Japan Statist, Soc.* Vol. 28, No. 2 193-203

2. Hurt, J. (1986). Asymptotic expansions for moments of stochastic processes and their applications, Statistics & Decisions, 4, 251-271.

3. Tanaka, H. and Akahira, M. (1997). The comparison of estimators of a reliability function in the Weibull case, (In Japanese), Proc. Symp., Res. Inst. Math. Sci., Kyoto Univ., 1007, 1-10.

4. H. L. Gray and W. R. Schucany (1972). The Generalized Jackknife Statistic, Marcel Dekker, Inc. New York.

5. Quenouille, M. (1956). Notes on bias in estimation. *Biometrika*, 43, 353-360.

6. Miller. R. G (1974). The Jackknife-a review, *Biometrika,* vol 61, 1-15.

7. Tukey, G. R. (1958). Bias and confidence in not quite large samples, *Ann. Math. Statist.*, 29, 614.

8. Lawrence M. Leemis (1995). *Reliability Probabilistic Models and Statistical Methods,* Prentice-Hall, Inc. New Jersy.

Table 1. Bias of the estimators

T	C	θ	$N=8$ Bias of MLE	$N=8$ Bias of Jackknife	$N=14$ Bias of MLE	$N=14$ Bias of Jackknife	$N=20$ Bias of MLE	$N=20$ Bias of Jackknife
2.0	2.0	5.0	-1.7351E-02	1.8277E-03	-7.6657E-03	9.7828E-04	-4.5964E-03	9.6927E-04
		5.2	-1.6404E-02	1.8229E-03	-7.2351E-03	9.3541E-04	-4.3392E-03	9.1408E-04
		5.4	-1.5516E-02	1.8057E-03	-6.8334E-03	8.9377E-04	-4.0991E-03	8.6267E-04
		5.6	-1.4686E-02	1.7791E-03	-6.4590E-03	8.5358E-04	-3.8752E-03	8.1495E-04
		5.8	-1.3911E-02	1.7456E-03	-6.1104E-03	8.1509E-04	-3.6666E-03	7.7071E-04
	2.4	5.0	-1.3168E-02	1.7059E-03	-5.7772E-03	7.7725E-04	-3.4671E-03	7.2836E-04
		5.2	-1.2207E-02	1.6443E-03	-5.3477E-03	7.2730E-04	-3.2098E-03	6.7402E-04
		5.4	-1.1331E-02	1.5781E-03	-4.9575E-03	6.8053E-04	-2.9760E-03	6.2466E-04
		5.6	-1.0533E-02	1.5099E-03	-4.6027E-03	6.3708E-04	-2.7633E-03	5.7988E-04
		5.8	-9.8049E-03	1.4414E-03	-4.2800E-03	5.9680E-04	-2.5698E-03	5.3915E-04
	2.8	5.0	-9.7186E-03	1.4329E-03	-4.2418E-03	5.9200E-04	-2.5468E-03	5.3435E-04
		5.2	-8.8372E-03	1.3413E-03	-3.8522E-03	5.4222E-04	-2.3132E-03	4.8526E-04
		5.4	-8.0528E-03	1.2529E-03	-3.5064E-03	4.9727E-04	-2.1057E-03	4.4165E-04
		5.6	-7.3541E-03	1.1687E-03	-3.1991E-03	4.5656E-04	-1.9213E-03	4.0294E-04
		5.8	-6.7305E-03	1.0894E-03	-2.9254E-03	4.1988E-04	-1.7570E-03	3.6844E-04
2.5	2.0	5.0	-2.2950E-02	1.4949E-03	-1.0254E-02	1.1876E-03	-6.1352E-03	1.3099E-03
		5.2	-2.1974E-02	1.6076E-03	-9.7969E-03	1.1578E-03	-5.8644E-03	1.2479E-03
		5.4	-2.1021E-02	1.6917E-03	-9.3528E-03	1.1256E-03	-5.6010E-03	1.1889E-03
		5.6	-2.0099E-02	1.7519E-03	-8.9254E-03	1.0916E-03	-5.3471E-03	1.1326E-03
		5.8	-1.9210E-02	1.7923E-03	-8.5161E-03	1.0569E-03	-5.1035E-03	1.0791E-03
	2.4	5.0	-1.9458E-02	1.7826E-03	-8.6299E-03	1.0668E-03	-5.1712E-03	1.0940E-03
		5.2	-1.8276E-02	1.8179E-03	-8.0876E-03	1.0183E-03	-4.8481E-03	1.0236E-03
		5.4	-1.7160E-02	1.8279E-03	-7.5787E-03	9.6975E-04	-4.5444E-03	9.5807E-04
		5.6	-1.6113E-02	1.8186E-03	-7.1032E-03	9.2189E-04	-4.2604E-03	8.9716E-04
		5.8	-1.5134E-02	1.7947E-03	-6.6606E-03	8.7541E-04	-3.9958E-03	8.4067E-04
	2.8	5.0	-1.6051E-02	1.8175E-03	-7.0751E-03	9.1896E-04	-4.2436E-03	8.9351E-04
		5.2	-1.4783E-02	1.7828E-03	-6.5026E-03	8.5835E-04	-3.9013E-03	8.2060E-04
		5.4	-1.3621E-02	1.7310E-03	-5.9803E-03	8.0042E-04	-3.5887E-03	7.5421E-04
		5.6	-1.2559E-02	1.6682E-03	-5.5050E-03	7.4573E-04	-3.3041E-03	6.9390E-04
		5.8	-1.1591E-02	1.5987E-03	-5.0732E-03	6.9453E-04	-3.0453E-03	6.3935E-04
3.0	2.0	5.0	-2.6919E-02	5.9275E-04	-1.2167E-02	1.2532E-03	-7.2509E-03	1.5889E-03
		5.2	-2.6186E-02	8.4149E-04	-1.1805E-02	1.2513E-03	-7.0429E-03	1.5317E-03
		5.4	-2.5405E-02	1.0523E-03	-1.1424E-02	1.2425E-03	-6.8222E-03	1.4746E-03
		5.6	-2.4595E-02	1.2292E-03	-1.1035E-02	1.2283E-03	-6.5946E-03	1.4184E-03
		5.8	-2.3773E-02	1.3755E-03	-1.0643E-02	1.2097E-03	-6.3645E-03	1.3634E-03
	2.4	5.0	-2.4849E-02	1.1776E-03	-1.1157E-02	1.2332E-03	-6.6658E-03	1.4356E-03
		5.2	-2.3754E-02	1.3786E-03	-1.0634E-02	1.2092E-03	-6.3591E-03	1.3621E-03
		5.4	-2.2649E-02	1.5328E-03	-1.0113E-02	1.1788E-03	-6.0515E-03	1.2906E-03
		5.6	-2.1556E-02	1.6474E-03	-9.6016E-03	1.1440E-03	-5.7486E-03	1.2219E-03
		5.8	-2.0489E-02	1.7288E-03	-9.1059E-03	1.1063E-03	-5.4544E-03	1.1563E-03
	2.8	5.0	-2.2405E-02	1.5614E-03	-9.9986E-03	1.1714E-03	-5.9838E-03	1.2751E-03
		5.2	-2.1022E-02	1.6916E-03	-9.3533E-03	1.1256E-03	-5.6014E-03	1.1889E-03
		5.4	-1.9682E-02	1.7729E-03	-8.7332E-03	1.0756E-03	-5.2327E-03	1.1075E-03
		5.6	-1.8401E-02	1.8154E-03	-8.1449E-03	1.0236E-03	-4.8823E-03	1.0310E-03
		5.8	-1.7189E-02	1.8279E-03	-7.5918E-03	9.7101E-04	-4.5523E-03	9.5972E-04

Table 2. Mean Squared Error of estimators

T	C	θ	$N=8$ MSE of MLE	$N=8$ MSE of Jackknife	$N=8$ MSE of UMVUE	$N=14$ MSE of MLE	$N=14$ MSE of Jackknife	$N=14$ MSE of UMVUE
2.0	2.0	5.0	3.7848E-03	3.3268E-03	2.9900E-03	1.4542E-03	1.3021E-03	1.2835E-03
		5.2	3.3572E-03	2.9372E-03	2.6304E-03	1.2798E-03	1.1416E-03	1.1254E-03
		5.4	2.9840E-03	2.6001E-03	2.3206E-03	1.1295E-03	1.0041E-03	9.8993E-04
		5.6	2.6580E-03	2.3080E-03	2.0531E-03	9.9960E-04	8.8591E-04	8.7344E-04
		5.8	2.3728E-03	2.0540E-03	1.8216E-03	8.8709E-04	7.8401E-04	7.7302E-04
	2.4	5.0	2.1163E-03	1.8270E-03	1.6153E-03	7.8682E-04	6.9360E-04	6.8391E-04
		5.2	1.8087E-03	1.5564E-03	1.3706E-03	6.6777E-04	5.8673E-04	5.7858E-04
		5.4	1.5511E-03	1.3311E-03	1.1678E-03	5.6909E-04	4.9860E-04	4.9170E-04
		5.6	1.3348E-03	1.1429E-03	9.9916E-04	4.8699E-04	4.2557E-04	4.1971E-04
		5.8	1.1525E-03	9.8499E-04	8.5831E-04	4.1840E-04	3.6480E-04	3.5979E-04
	2.8	5.0	1.1319E-03	9.6713E-04	8.4241E-04	4.1066E-04	3.5796E-04	3.5304E-04
		5.2	9.3214E-04	7.9479E-04	6.8950E-04	3.3617E-04	2.9224E-04	2.8825E-04
		5.4	7.7140E-04	6.5665E-04	5.6757E-04	2.7676E-04	2.4003E-04	2.3677E-04
		5.6	6.4149E-04	5.4533E-04	4.6977E-04	2.2910E-04	1.9829E-04	1.9560E-04
		5.8	5.3599E-04	4.5515E-04	3.9090E-04	1.9065E-04	1.6472E-04	1.6250E-04
2.5	2.0	5.0	7.0548E-03	6.4373E-03	5.9035E-03	2.8613E-03	2.6315E-03	2.5938E-03
		5.2	6.3761E-03	5.7715E-03	5.2744E-03	2.5584E-03	2.3403E-03	2.3067E-03
		5.4	5.7638E-03	5.1803E-03	4.7181E-03	2.2902E-03	2.0846E-03	2.0547E-03
		5.6	5.2131E-03	4.6559E-03	4.2266E-03	2.0528E-03	1.8602E-03	1.8335E-03
		5.8	4.7186E-03	4.1908E-03	3.7923E-03	1.8428E-03	1.6631E-03	1.6393E-03
	2.4	5.0	4.8529E-03	4.3166E-03	3.9096E-03	1.8995E-03	1.7162E-03	1.6917E-03
		5.2	4.2328E-03	3.7391E-03	3.3720E-03	1.6393E-03	1.4734E-03	1.4524E-03
		5.4	3.6960E-03	3.2456E-03	2.9149E-03	1.4178E-03	1.2685E-03	1.2505E-03
		5.6	3.2319E-03	2.8238E-03	2.5260E-03	1.2292E-03	1.0952E-03	1.0797E-03
		5.8	2.8310E-03	2.4627E-03	2.1947E-03	1.0683E-03	9.4839E-04	9.3503E-04
	2.8	5.0	3.2055E-03	2.7999E-03	2.5040E-03	1.2185E-03	1.0854E-03	1.0701E-03
		5.2	2.6949E-03	2.3410E-03	2.0833E-03	1.0142E-03	8.9920E-04	8.8655E-04
		5.4	2.2708E-03	1.9636E-03	1.7393E-03	8.4711E-04	7.4791E-04	7.3744E-04
		5.6	1.9184E-03	1.6527E-03	1.4575E-03	7.1008E-04	6.2465E-04	6.1595E-04
		5.8	1.6254E-03	1.3959E-03	1.2260E-03	5.9743E-04	5.2387E-04	5.1661E-04
3.0	2.0	5.0	1.0653E-02	1.0180E-02	9.4863E-03	4.5731E-03	4.3329E-03	4.3329E-03
		5.2	9.8364E-03	9.2956E-03	8.6321E-03	4.1673E-03	3.9207E-03	3.9207E-03
		5.4	9.0655E-03	8.4808E-03	7.8496E-03	3.7941E-03	3.5467E-03	3.5467E-03
		5.6	8.3443E-03	7.7349E-03	7.1366E-03	3.4531E-03	3.2089E-03	3.2089E-03
		5.8	7.6742E-03	7.0551E-03	6.4894E-03	3.1428E-03	2.9048E-03	2.9048E-03
	2.4	5.0	8.5630E-03	7.9595E-03	7.3510E-03	3.5558E-03	3.3102E-03	3.3102E-03
		5.2	7.6591E-03	7.0399E-03	6.4750E-03	3.1359E-03	2.8981E-03	2.8981E-03
		5.4	6.8393E-03	6.2246E-03	5.7022E-03	2.7645E-03	2.5381E-03	2.5381E-03
		5.6	6.1015E-03	5.5052E-03	5.0236E-03	2.4375E-03	2.2248E-03	2.2248E-03
		5.8	5.4413E-03	4.8724E-03	4.4292E-03	2.1507E-03	1.9525E-03	1.9525E-03
	2.8	5.0	6.6688E-03	6.0572E-03	5.5440E-03	2.6883E-03	2.4648E-03	2.4648E-03
		5.2	5.7645E-03	5.1810E-03	4.7187E-03	2.2905E-03	2.0849E-03	2.0849E-03
		5.4	4.9771E-03	4.4333E-03	4.0185E-03	1.9522E-03	1.7656E-03	1.7656E-03
		5.6	4.2960E-03	3.7975E-03	3.4262E-03	1.6656E-03	1.4979E-03	1.4979E-03
		5.8	3.7093E-03	3.2578E-03	2.9261E-03	1.4233E-03	1.2735E-03	1.2735E-03

Table 2. (continued)

T	C	θ	N = 20 MSE of MLE	MSE of Jackknife	MSE of UMVUE
2.0	2.0	5.0	9.0351E-04	8.4153E-04	8.3451E-04
		5.2	7.9383E-04	7.3760E-04	7.3145E-04
		5.4	6.9952E-04	6.4856E-04	6.4317E-04
		5.6	6.1822E-04	5.7207E-04	5.6732E-04
		5.8	5.4796E-04	5.0615E-04	5.0196E-04
	2.4	5.0	4.8546E-04	4.4769E-04	4.4399E-04
		5.2	4.1141E-04	3.7862E-04	3.7550E-04
		5.4	3.5017E-04	3.2167E-04	3.1903E-04
		5.6	2.9931E-04	2.7450E-04	2.7225E-04
		5.8	2.5689E-04	2.3526E-04	2.3334E-04
	2.8	5.0	2.5211E-04	2.3084E-04	2.2895E-04
		5.2	2.0613E-04	1.8842E-04	1.8689E-04
		5.4	1.6952E-04	1.5473E-04	1.5347E-04
		5.6	1.4020E-04	1.2780E-04	1.2677E-04
		5.8	1.1658E-04	1.0615E-04	1.0529E-04
2.5	2.0	5.0	1.7989E-03	1.7045E-03	1.6904E-03
		5.2	1.6046E-03	1.5152E-03	1.5026E-03
		5.4	1.4333E-03	1.3491E-03	1.3378E-03
		5.6	1.2822E-03	1.2034E-03	1.1933E-03
		5.8	1.1489E-03	1.0755E-03	1.0665E-03
	2.4	5.0	1.1849E-03	1.1100E-03	1.1007E-03
		5.2	1.0202E-03	9.5260E-04	9.4465E-04
		5.4	8.8059E-04	8.1978E-04	8.1294E-04
		5.6	7.6203E-04	7.0753E-04	7.0165E-04
		5.8	6.6123E-04	6.1250E-04	6.0742E-04
	2.8	5.0	7.5534E-04	7.0122E-04	6.9539E-04
		5.2	6.2738E-04	5.8067E-04	5.7585E-04
		5.4	5.2302E-04	4.8281E-04	4.7882E-04
		5.6	4.3771E-04	4.0312E-04	3.9979E-04
		5.8	3.6775E-04	3.3799E-04	3.3522E-04
3.0	2.0	5.0	2.9131E-03	2.8139E-03	2.7916E-03
		5.2	2.6464E-03	2.5446E-03	2.5241E-03
		5.4	2.4026E-03	2.3006E-03	2.2818E-03
		5.6	2.1810E-03	2.0804E-03	2.0633E-03
		5.8	1.9802E-03	1.8824E-03	1.8668E-03
	2.4	5.0	2.2476E-03	2.1464E-03	2.1288E-03
		5.2	1.9758E-03	1.8780E-03	1.8625E-03
		5.4	1.7367E-03	1.6438E-03	1.6301E-03
		5.6	1.5273E-03	1.4401E-03	1.4281E-03
		5.8	1.3444E-03	1.2633E-03	1.2528E-03
	2.8	5.0	1.6878E-03	1.5961E-03	1.5829E-03
		5.2	1.4335E-03	1.3493E-03	1.3380E-03
		5.4	1.2183E-03	1.1420E-03	1.1325E-03
		5.6	1.0368E-03	9.6844E-04	9.6035E-04
		5.8	8.8402E-04	8.2304E-04	8.1617E-04