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Estimations in a Generalized Uniform Distribution

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Abstract

In this paper, we shall derive MLE's, modified MLE, MRE and UMVUE's of the shape and scale parameters in a generalized uniform distribution, and propose several estimators for the right-tail probability in a generalized uniform distribution using the proposed estimators for the shape and scale parameters. And we shall compare exactly MSE of the proposed estimators for the shape and the scale parameters, and compare numerically efficiencies for the several proposed estimators of the right-tail probability in a generalized uniform distribution by Monte Carlo methods.

Key Words and Phrases: generalized uniform, MLE, modified MLE, MRE and UMVUE, Monte Carlo methods.

1. Introduction

A random variable X is said to have a generalized uniform distribution if its density function is of the form

$$f(x:\alpha,\beta)=\frac{\alpha+1}{\beta^{\alpha+1}}x^{\alpha},\quad 0<x<\beta,\ -1<\alpha,\qquad \text{(seeTiwarietal(1996))} \qquad (1.1)$$

where α and $beta$ are referred as the shape and the scale parameters, respectively. It is denoted by $X \sim GUNIF(\alpha,\beta)$.

The mean and variance for the generalized uniform distribution are $((\alpha+1)/(\alpha+2))\beta$ and $((\alpha+1)/((\alpha+2)^2(\alpha+3)))\beta^2$, respectively. The generalized uniform distribution is a uniform distribution over $(0,\beta)$ if $\alpha=0$, and is a standard power-function distribution if $\beta=1$. The density function (1.1) is decreasing of x if $-1<\alpha<0$, and constant if $\alpha=0$ and increasing if $\alpha>0$. Proctor(1987) introduced the four

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parameter generalized uniform distribution which is a counterpart to Burr type XII distribution. And Tiwari, Yang & Zalkikar(1996) studied Bayes Estimation for parameters in the Pareto distribution using the generalized uniform distribution.

In this paper, we shall derive MLE's, modified MLE, minimum risk estimator(MRE) and UMVUE's for the shape and the scale parameters, and propose several estimators for the right-tail probability in the generalized uniform distribution using proposed estimators for the shape and the scale parameters. Also, we shall derive a confidence interval for the shape parameter. We shall compare exactly MSE's for the proposed estimators of the shape and the scale parameters. And proposed estimators for the right-tail probability in the generalized uniform distribution will be numerically compared with each other in the sense of MSE by the Monte Carlo method.

2. Estimations of the Shape and Scale Parameters

Let $X_1, \cdots, X_n$ be a random sample from $X \sim GUNIF(\alpha, \beta)$ and $X_{(1)}, \cdots, X_{(n)}$ be the order statistics of this sample. Then we can obtain the density function of $X_{(i)}$, $i = 1, \cdots, n$, and the joint density function for $X_{(i)}$ and $X_{(j)}$, $1 \leq i < j \leq n$, as follows ;

$$\begin{aligned} f_i(x_i) &= \frac{n!}{(i-1)!(n-i)!} \frac{\alpha+1}{\beta^{n(\alpha+1)}} x_i^{(\alpha+1)i-1} (\beta^{\alpha+1} - x_i^{\alpha+1})^{n-i}, \quad 0 < x_i < \beta, \\ f_{i,j}(x_i, x_j) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \frac{(\alpha+1)^2}{\beta^{n(\alpha+1)}} \\ &\quad \cdot x_i^{(\alpha+1)i-1} (x_j^{\alpha+1} - x_i^{\alpha+1}) x_j^{\alpha} (\beta^{\alpha+1} - x_j^{\alpha+1})^{n-j}, \quad 0 < x_i < x_j < \beta, \end{aligned} \tag{2.1}$$

respectively.

From the density functions in (2.1), we can obtain the mean and the variance for $X_{(i)}$, $i = 1, \cdots, n$, and the covariance between $X_{(i)}$ and $X_{(j)}$, $1 \leq i < j \leq n$, as follows ;

$$\begin{aligned} E(X_{(i)}) &= \frac{\Gamma(n+1)\Gamma(i+\frac{1}{\alpha+1})}{\Gamma(i)\Gamma(n+1+\frac{1}{\alpha+1})}\beta, \\ Var(X_{(i)}) &= \frac{\Gamma(n+1)}{\Gamma(i)} \left[\frac{\Gamma(i+\frac{2}{\alpha+1})}{\Gamma(n+1+\frac{2}{\alpha+1})} - \frac{\Gamma(n+1)\Gamma^2(i+\frac{1}{\alpha+1})}{\Gamma(i)\Gamma^2(n+1+\frac{1}{\alpha+1})} \right] \beta^2, \\ Cov(X_{(i)}, X_{(j)}) &= \frac{\Gamma(n+1)\Gamma(i+\frac{1}{\alpha+1})}{\Gamma(i)} \left[\frac{\Gamma(j+\frac{2}{\alpha+1})}{\Gamma(j+\frac{1}{\alpha+1})\Gamma(n+1+\frac{2}{\alpha+1})} \right. \\ &\quad \left. - \frac{\Gamma(n+1)\Gamma(j+\frac{1}{\alpha+1})}{\Gamma(j)\Gamma^2(n+1+\frac{1}{\alpha+1})} \right] \beta^2 \end{aligned} \tag{2.2}$$

where $\Gamma(a)$ is the gamma function.

Here, we shall consider problems of estimation for the shape and the scale parameters in the generalized uniform distribution. In the generalized uniform distribution, the MLE's for the shape and the scale parameters are given by

$$\widehat{\alpha}_M = \frac{n}{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)} - 1, \quad \widehat{\beta}_M = X_{(n)}. \tag{2.3}$$

Since $-\ln(X/\beta)$ has an exponential distribution with mean $(\alpha+1)^{-1}$, it is well known that $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ has a Gamma distribution with a shape parameter n-1 and a scale parameter 1, and $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ and $X_{(n)}$ are independent(see Johnson et al.(1995)).
Therefore, the MSE's for $\widehat{\alpha}_M$ and $\widehat{\beta}_M$ are

$$\begin{aligned} MSE(\widehat{\alpha}_M) &= \frac{(n^2+2n-6)(\alpha+1)^2}{(n-2)^2(n-3)}, \\ MSE(\widehat{\beta}_M) &= \frac{2\beta^2}{[n(\alpha+1)+1][n(\alpha+1)+2]}. \end{aligned} \tag{2.4}$$

By appling the same method in Lehmann(1983), $(-\sum_{i=1}^{n} \ln X_i,\ X_{(n)})$ are jointly sufficient and complete statistics for (α, β). From Lehmann-Scheffe Theorem, we can obtain the UMVUE's of the shape and the scale parameters in the generalized uniform distribution.

Fact 1. The UMVUE's of the shape parameter α and the scale parameter β in the generalized uniform distribution are

$$\widehat{\alpha}_U = \frac{n-2}{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)} - 1, \quad \widehat{\beta}_U = \left[1 + \frac{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)}{n(n-1)}\right] X_{(n)}. \tag{2.5}$$

Since $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ and $X_{(n)}$ are independent, we can obtain the variances for UMVUE's of α and β as follows ;

$$\begin{aligned} Var(\widehat{\alpha}_U) &= \frac{(\alpha+1)^2}{n-3}, \\ Var(\widehat{\beta}_U) &= \frac{\beta^2}{(n-1)(\alpha+1)[n(\alpha+1)+2]}. \end{aligned} \tag{2.6}$$

Next, we can obtain the minimum risk estimator for the shape parameter α and the modified MLE for the scale parameter β as follows ;

$$\widehat{\alpha}_R = \frac{n-3}{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)} - 1, \quad \widehat{\beta}_{MM} = \left[1 - \frac{\sum_{i=1}^{n} \ln(X_{(n)}/X_i)}{n^2}\right] X_{(n)}. \tag{2.7}$$

Since $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ has a Gamma distribution with a shape parameter n-1 and a scale parameter 1, and $(\alpha+1)\sum_{i=1}^{n} \ln(X_{(n)}/X_i)$ and $X_{(n)}$ are independent, we can obtain the MSE's for $\widehat{\alpha}_R$ and $\widehat{\beta}_{MM}$ as follows ;

$$\begin{aligned} MSE(\widehat{\alpha}_R) &= \frac{(\alpha+1)^2}{n-2}, \\ MSE(\widehat{\beta}_{MM}) &= \frac{n(\alpha+1)(n+1)+n-1}{n^2(\alpha+1)[n(\alpha+1)+1][n(\alpha+1)+2]}\beta^2. \end{aligned} \tag{2.8}$$

Since certain regular conditions are satisfied, the Frechet-Cramer-Rao lower bound (FCRLB) (see Rohatgi(1976)) for an unbiased estimator of the shape parameter α is

$$(\alpha+1)^2/n. \tag{2.9}$$

From the results (2.4), (2.6), (2.8) and (2.9), we can obtain the followings.

Fact 2. (1) $\widehat{\alpha}_M$, $\widehat{\alpha}_U$ and $\widehat{\alpha}_R$ are consistent estimators for the shape parameter α and $\widehat{\beta}_M$, and $\widehat{\beta}_{MM}$ are consistent estimators for the scale parameters β.

(2) $FCRLB \ < \ MSE(\widehat{\alpha}_R) \ < \ MSE(\widehat{\alpha}_U) \ < \ MSE(\widehat{\alpha}_M)$

(3) $MSE(\widehat{\beta}_M) \ > \ MSE(\widehat{\beta}_U)$, if $\alpha > -\frac{n-3}{n-2}$ and $n \geq 3$
$MSE(\widehat{\beta}_M) \ > \ MSE(\widehat{\beta}_{MM})$, if $\alpha > -\frac{n-1}{n}$
$MSE(\widehat{\beta}_U) \ > \ MSE(\widehat{\beta}_{MM})$, if $\alpha > -1$.

Now, we can obtain the covariance between $\widehat{\alpha}'s$ and $\widehat{\beta}'s$ as follows ;

$$\begin{aligned} \sigma_{ij} \equiv Cov(\widehat{\alpha}_i, \widehat{\beta}_j) &= -a_i b_j \frac{n(\alpha+1)\beta}{(n-2)[n(\alpha+1)+1]}, \\ &i, = 1(M), 2(U), 3(R), \ j = 1(M), 2(U), 3(MM), \end{aligned}$$

where $a_1 = 1, \ a_2 = (n-2)/n, \ a_3 = (n-3)/n, \ b_1 = 0, \ b_2 = 1/(n-1)$ and $b_3 = 1/n$. From the result (2.10) and results of Johnson(1994), we can obtain the followings.

Fact 3. (1) $\widehat{\alpha}'s$ and $\widehat{\beta}_1$ are independent.

(2) correlation coefficients between $\widehat{\alpha}$'s and $\widehat{\beta}$'s are nonpositive.

(3) $\widehat{\alpha}_i$, $i = 1(M), 2(U), 3(R)$ and $\widehat{\beta}_j$, $j = 2(U), 3(MM)$ are asymptotically uncorrelated.

(4) $0 = |\sigma_{R\ M}| < |\sigma_{R\ MM}| < |\sigma_{R\ U}| < |\sigma_{U\ MM}| < |\sigma_{U\ U}| < |\sigma_{M\ MM}| < |\sigma_{M\ U}|$

Now, we shall consider a confidence interval of the shape parameter α in the generalized uniform distribution. Since $2(\alpha+1)\sum_{i=1}^{n}\ln(X_{(n)}/X_i)$ is a pivotal quantity, which has a chi-square distribution with a degree of freedom 2(n-1), we can obtain the $(1-\gamma)$ 100% confidence interval for the shape parameter α in the generalized uniform distribution as follows ;

$$\left(\frac{\chi^2_{2(n-1),1-\gamma/2}}{2\sum_{i=1}^{n}\ln(X_{(n)}/X_i)} - 1,\ \frac{\chi^2_{2(n-1),\gamma/2}}{2\sum_{i=1}^{n}\ln(X_{(n)}/X_i)} - 1\right), \tag{2.10}$$

where $\gamma = \int_0^{\chi^2_{2(n-1),\gamma}} \chi^2_{2(n-1)}(t)dt$, $\chi^2_{2(n-1)}(t)$ is the density function of a chi-square distribution with a degree of freedom 2(n-1).

3. Estimations of the Right-Tail Probability

Here we shall consider the estimations for the right-tail probability in the generalized uniform distribution. The distribution function of the generalized uniform distribution is

$$F(t) = \left(\frac{t}{\beta}\right)^{\alpha+1}, \quad 0 < t < \beta, \tag{3.1}$$

and so the right-tail probability of the generalized uniform distribution is

$$R(t) = 1 - \left(\frac{t}{\beta}\right)^{\alpha+1}, \quad 0 < t < \beta. \tag{3.2}$$

Using several estimators for the shape and the scale parameters proposed in the previous section 2, we can propose the following estimators for the right-tail probability in the generalized uniform distribution

$$\widehat{R}_{ij}(t) = 1 - \left(\frac{t}{\widehat{\beta}_j}\right)^{\widehat{\alpha}_i+1}, \quad i = 1(M), 2(U), 3(R) \ \ j = 1(M), 2(U), 3(MM). \tag{3.3}$$

To compare the performances of the proposed estimators$\widehat{R}_{ij}$ for the right-tail probability in the generalized uniform distribution, the Monte Carlo simulations were carried out for the generalized uniform distribution. Table 1 through Table 3 show the simulated MSE's for the proposed right-tail probability estimators in the generalized uniform distribution for the sample size n=10(5)40, the shape parameter $\alpha = 1/2, 0, -1/2$, the scale parameter $\beta = 1$, and the right-tail probability $R(t) = 0.1$.

From Table 1, when the sample sizes are small and $R(t) = 0.1$, $\widehat{R}_{R\ MM}$-estimator based on the MRE of the shape parameter and the modified MLE of the scale parameter is more efficient than any other proposed estimators. When the sample sizes are large and $R(t) = 0.1$, $\widehat{R}_{R\ U}$-estimator based on the MRE of the shape parameter and the UMVUE of the scale parameter is more efficient than any other proposed estimators. When sample sizes are large$n \geq 20$, $\widehat{R}_{R\ M}$-estimator based on the MRE of the shape parameter and the MLE of the scale parameter is worse in a sense of MSE than any other proposed estimators. From (4) in Fact 3 and the preceding simulated results, we can obtain ;

Fact 4. When sample sizes are large($n \geq 30$) and $R(t) = 0.1$, the simulated MSE's of$\widehat{R}_{Rj}$ based on the MRE of the shape parameter and $j = 1$(MLE), $j = 2$(UMVUE) and $j = 3$(modified MLE) of the scale parameter have reverse ordering of covariances between the MRE of the shape parameter and $j = 1$(MLE), $j = 2$(UMVUE) & $j = 3$(modified MLE) of the scale parameter.

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Table 1. The simulated MSE's of proposed right-tail probability estimators for a generalized uniform distribution with $\alpha = 1/2$ and $\beta = 1$.

n	10	15	20	25	30	35	40
$\widehat{R}_{M\ M}$	0.002641	0.002114	0.001787	0.001506	0.001310	0.001071	0.000912
$\widehat{R}_{M\ U}$	0.005129	0.002708	0.001699	0.001271	0.000992	0.000788	0.000672
$\widehat{R}_{M\ MM}$	0.004472	0.002517	0.001629	0.001232	0.000975	0.000780	0.000672
$\widehat{R}_{U\ M}$	0.002966	0.002382	0.002015	0.001681	0.001456	0.001191	0.001012
$\widehat{R}_{U\ U}$	0.002371	0.001686	0.001243	0.000996	0.000835	0.000682	0.000595
$\widehat{R}_{U\ MM}$	0.002609	0.001763	0.001267	0.001011	0.000839	0.000682	0.000594
$\widehat{R}_{R\ M}$	0.003325	0.002599	0.002174	0.001799	0.001550	0.001268	0.001075
$\widehat{R}_{R\ U}$	0.001913	0.001483	0.001143	0.000937	0.000799	0.000654	0.000574
$\widehat{R}_{R\ MM}$	0.001842	0.001455	0.001140	0.000932	0.000800	0.000658	0.000578

where simulations were repeated 5000 times when $R(t) = 0.1$.

Table 2. The simulated MSE's of proposed right-tail probability estimators for a generalized uniform distribution with $\alpha = -1/2$ and $\beta = 1$.

n	10	15	20	25	30	35	40
$\widehat{R}_{M\ M}$	0.002789	0.002157	0.001827	0.001606	0.001306	0.001109	0.000924
$\widehat{R}_{M\ U}$	0.005078	0.002557	0.001746	0.001300	0.001001	0.000775	0.000654
$\widehat{R}_{M\ MM}$	0.004504	0.002388	0.001687	0.001274	0.000983	0.000770	0.000654
$\widehat{R}_{U\ M}$	0.003035	0.002426	0.002042	0.001786	0.001447	0.001233	0.001025
$\widehat{R}_{U\ U}$	0.002454	0.001609	0.001309	0.001073	0.000842	0.000684	0.000585
$\widehat{R}_{U\ MM}$	0.002651	0.001673	0.001324	0.001079	0.000847	0.000681	0.000583
$\widehat{R}_{R\ M}$	0.003361	0.002643	0.002196	0.001905	0.001539	0.001311	0.001089
$\widehat{R}_{R\ U}$	0.001987	0.001420	0.001205	0.001021	0.000806	0.000659	0.000566
$\widehat{R}_{R\ MM}$	0.001939	0.001401	0.001208	0.001025	0.000807	0.000666	0.000570

Table 3. The simulated MSE's of proposed right-tail probability estimators for a generalized uniform distribution with $\alpha = 0$ and $\beta = 1$.

n	10	15	20	25	30	35	40
$\widehat{R}_{M\ M}$	0.002687	0.002126	0.001845	0.001517	0.001266	0.001093	0.000935
$\widehat{R}_{M\ U}$	0.005275	0.002729	0.001735	0.001237	0.001032	0.000805	0.000688
$\widehat{R}_{M\ MM}$	0.004595	0.002552	0.001679	0.001209	0.001021	0.000796	0.000682
$\widehat{R}_{U\ M}$	0.002960	0.002380	0.002067	0.001699	0.001405	0.001212	0.001034
$\widehat{R}_{U\ U}$	0.002674	0.001773	0.001313	0.001001	0.000866	0.000700	0.000610
$\widehat{R}_{U\ MM}$	0.002406	0.001710	0.001302	0.000994	0.000866	0.000698	0.000610
$\widehat{R}_{R\ M}$	0.003299	0.002590	0.002223	0.001820	0.001496	0.001288	0.001096
$\widehat{R}_{R\ U}$	0.001944	0.001488	0.001194	0.000936	0.000819	0.000672	0.000591
$\widehat{R}_{R\ MM}$	0.001842	0.001474	0.001202	0.000940	0.000826	0.000674	0.000593