Mode I Field Intensity Factors of Infinitely Long Strip in Piezoelectric Media

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We consider the problem of determining the singular stresses and electric fields in a piezoelectric ceramic strip containing a Griffith crack under in-plane normal loading within the framework of linear piezoelectricity. The potential theory method and Fourier transforms are used to reduce the problem to the solution of dual integral equations, which are then expressed to a Fredholm integral equation of the second kind. Numerical values on the field intensity factors are obtained, and the influences of the electric fields for PZT-6B piezoelectric ceramic are discussed.

Key Words : Piezoelectric Medium, Field Intensity Factors, Potential Theory Method

Nomenclature

a	: A half of crack length
c_{11}, c_{13}, c_{33}, c_{44}	: Elastic moduli measured in a constant electric field
d_{11}, d_{33}	: Dielectric permittivities measured at a constant strain
D_x, D_z	: Electric displacement vector
e_{15}, e_{31}, e_{33}	: Piezoelectric constants
E_x, E_z	: Electric field vector
h	: A half of thickness of the strip
K^f	: Field intensity factor
ϕ	: Electric potential
ψ	: Potential function

1. Introduction

Because of its coupling characteristic between electric and mechanical deformation, piezoelectric materials have been widely used in transducers and sensors. Due to the intrinsic brittle property, however, the stress concentration caused by mechanical and/or electric load may induce the initiation and propagation of crack under loading conditions. In order to predict the service lifetime of piezoelectric ceramic components, we must identify the damage and the fracture behavior of the materials. The increasing attention to the study of crack problems in piezoelectric materials in the last decade has led to a lot of significant works being published.

Of particular interest, Pak (1990) obtained the closed form solutions for an unbounded piezoelectric medium under anti-plane loading by employing a complex variable approach. Park and Sun (1995) obtained the closed form solutions for all modes of fracture for an infinite piezoelectric medium containing a center crack subjected to a combined mechanical and electrical loading. Shindo et al. (1996, 1997) obtained the solution for the infinite strip parallel or perpendicular to the crack under anti-plane loading using integral transform method. Recently, Kwon and Lee (2000) obtained the solution of piezoelectric rectangular media with a center crack under anti-plane shear loading.

In contrast to the success of the anti-plane (mode III) crack problem researches mentioned above, there are relatively few papers concerning in-plane (mode I) crack problem due to the

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complexity of analysis. The purpose of the present work is to conduct a systematic study of the in-plane crack problem of a transversely isotropic solid (class 6 mm or hexagonal) by using a potential theory and an integral transform method. In this paper, we consider the crack problem in a piezoelectric ceramic strip under in-plane normal loading. The numerical results for the field intensity factors are shown graphically for PZT-6B piezoelectric ceramic.

2. Potential Theory Method and Governing Equations

In a Cartesian coordinates (x, z), poled with z-axis, the governing equations can be written as follows,

$$c_{11}\frac{\partial^2 u}{\partial x^2}+c_{44}\frac{\partial^2 u}{\partial z^2}+(c_{13}+c_{44})\frac{\partial^2 w}{\partial x\partial z}+(e_{15}+e_{31})\frac{\partial^2 \phi}{\partial x\partial z}=0, \quad (1)$$

$$c_{44}\frac{\partial^2 w}{\partial x^2}+c_{33}\frac{\partial^2 w}{\partial z^2}+(c_{13}+c_{44})\frac{\partial^2 u}{\partial x\partial z}+e_{15}\frac{\partial^2 \phi}{\partial x^2}+e_{33}\frac{\partial^2 \phi}{\partial z^2}=0, \quad (2)$$

$$e_{15}\frac{\partial^2 w}{\partial x^2}+e_{33}\frac{\partial^2 w}{\partial z^2}+(e_{15}+c_{31})\frac{\partial^2 u}{\partial x\partial z}-d_{11}\frac{\partial^2 \phi}{\partial x^2}-d_{33}\frac{\partial^2 \phi}{\partial z^2}=0, \quad (3)$$

where $u(x, z)$, $w(x, z)$, ϕ, $(c_{11}, c_{13}, c_{33}, c_{44})$, (d_{11}, d_{33}) and (e_{15}, e_{31}, e_{33}) are the displacement in x-direction, the displacement in z-direction, the electric potential, the elastic moduli measured in a constant electric field, the dielectric permittivities measured at a constant strain and the piezoelectric constants, respectively.

We introduce the following potential function which could transform Eqs. (1)-(3) into the familiar differential equations,

$$u=\frac{\partial\psi}{\partial x},\ w=k_1\frac{\partial\psi}{\partial z},\ \phi=k_2\frac{\partial\psi}{\partial z}, \quad (4)$$

where $\psi(x, z)$ is the potential function introduced, and k_1 and k_2 are unknown constants. Putting Eq. (4) into Eqs. (1)-(3), we have the following equations,

$$c_{11}\frac{\partial^2\psi}{\partial x^2}+[c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2]\frac{\partial^2\psi}{\partial z^2}=0, \quad (5)$$

$$[(c_{13}+c_{44})+c_{44}k_1+e_{15}k_2]\frac{\partial^2\psi}{\partial x^2}+(c_{33}k_1+e_{33}k_2)\frac{\partial^2\psi}{\partial z^2}=0, \quad (6)$$

$$[(e_{15}+e_{31})+e_{15}k_1-d_{11}k_2]\frac{\partial^2\psi}{\partial x^2}+(e_{33}k_1-d_{33}k_2)\frac{\partial^2\psi}{\partial z^2}=0, \quad (7)$$

The terms $\partial^2\psi/\partial x^2$ and $\partial^2\psi/\partial z^2$ are not identically equal to zero. Therefore, a non-trivial solution of Eqs. (5)-(7) is to exist only in the following case,

$$\frac{c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2}{c_{11}}=\frac{c_{33}k_1+e_{33}k_2}{c_{13}+c_{44}+c_{44}k_1+e_{15}k_2}=\frac{e_{33}k_1-d_{33}k_2}{e_{15}+e_{31}+e_{15}k_1-d_{11}k_2}=\lambda \quad (8)$$

Eliminating k_1 and k_2 in Eq. (8), we obtain a cubic algebra equation of λ,

$$A\lambda^3+B\lambda^2+C\lambda+D=0, \quad (9)$$

where

$$A=e_{15}^2+c_{44}d_{11}, \quad (10)$$

$$B=(2c_{13}e_{15}^2-c_{44}e_{31}^2+2c_{13}e_{15}e_{31}-2c_{11}e_{15}e_{33}+c_{13}^2d_{11}+2c_{13}c_{44}d_{11}-c_{11}c_{33}d_{11}-c_{11}c_{44}d_{33})/c_{11}, \quad (11)$$

$$C=[c_{33}(e_{15}+e_{31})^2-2(c_{13}+c_{44})(e_{15}+e_{31})e_{33}+(2c_{44}e_{15}+c_{11}e_{33})e_{33}+c_{33}c_{44}d_{11}-(c_{13}^2+2c_{13}c_{44}-c_{11}c_{33})d_{33}]/c_{11}, \quad (12)$$

$$D=-(e_{33}^2+c_{33}d_{33})c_{44}/c_{11} \quad (13)$$

The three roots of Eq. (9) are denoted by $\lambda_i(i=1, 2, 3)$ and λ_1 is assumed to be a positive real number, λ_2 and λ_3 are to be either positive real numbers or a pair of conjugate complex roots with positive real parts.

Corresponding to three roots, there are three potential functions $\psi_i(i=1, 2, 3)$ and each of them must satisfy the following equation,

$$\nabla_i^2\psi_i=\frac{\partial^2\psi_i}{\partial x^2}+\frac{\partial^2\psi_i}{\partial z_i^2}=0,\ i=1, 2, 3, \quad (14)$$

where

$$z_i=\frac{z}{\sqrt{\lambda_i}}. \quad (15)$$

3. Problem Statement and Method of Solution

Consider a piezoelectric medium in the form of an infinitely long strip containing a finite crack subjected to the combined in-plane mechanical and electric loads as shown in Fig. 1. A set of cartesian coordinates (x, z) is attached to the center of the crack. The piezoelectric ceramic strip poled with z-axis occupies the region $(-\infty < x < \infty,\ -h \le z \le h)$ and is thick enough in the y-direction to allow a state of plane strain. The crack is situated along the $-a \le x \le a$, $z=0$. Because of the assumed symmetry in geometry and loading, it is sufficient to consider the problem for $0 \le x < \infty$ and $z \ge 0$ only.

How to impose the electrical boundary conditions along the crack surfaces in piezoelectric fracture modelling is controversial. The impermeable boundary condition on the crack surface was widely used in the previous works. This condition shows that the electric displacement intensity factor depends on the electric load, and the energy release rate is always negative only in the presence of electric loading, irrespective of its sign. These contradict the available experimental observations (Park and Sun, 1995; Tobin and Pak, 1993).

In fact, cracks in piezoelectric media will be filled with vacuum or air. This requires that both the normal components of electric displacement and the tangential component of the electric field will be continuous across the crack faces.

Based on this concept, the boundary conditions are written in the forms,

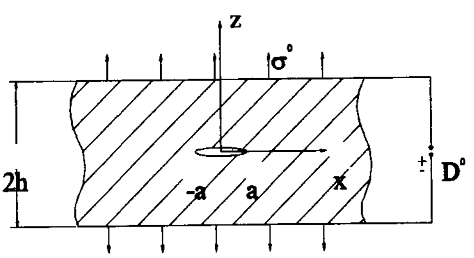

Fig. 1 Infinite piezoelectric strip with a crack subjected to the combined in-plane electric and mechanical loads

$$\sigma_z(x, h) = \sigma_z(x, -h) = \sigma_0, \tag{16}$$

$$D_z(x, h) = D_z(x, -h) = D_0, \tag{17}$$

$$\sigma_{xz}(x, 0) = 0, \tag{18}$$

$$\sigma_{xz}(x, h) = 0, \tag{19}$$

$$E_x(x, 0^+) = E_x(x, 0^-),\ (0 \le x < a), \tag{20a}$$

$$D_z(x, 0^+) = D_z(x, 0^-),\ (0 \le x < a), \tag{20b}$$

$$\phi(x, 0) = 0,\ (0 < x < \infty), \tag{21}$$

$$\sigma_z(x, 0) = 0,\ (0 \le x < a), \tag{22}$$

$$w(x, 0) = 0,\ (a < x < \infty). \tag{23}$$

By applying a Fourier transform to Eq. (14), we can find the displacements, electric fields, stresses and electric displacement components in the forms,

$$\psi_i(x, z_i) = \frac{2}{\pi}\int_0^\infty \frac{1}{s}[A_i(s)\cosh(sz_i) + B_i(s)\sinh(sz_i)]\cos(sx)\,ds, \tag{24}$$

$$u = -\frac{2}{\pi}\int_0^\infty \sum_{i=1}^{3}[A_i(s)\cosh(sz_i) + B_i(s)\sinh(sz_i)]\sin(sx)\,ds, \tag{25}$$

$$w = \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{3}\frac{k_{1i}}{\sqrt{\lambda_i}}[A_i(s)\sinh(sz_i) + B_i(s)\cosh(sz_i)]\cos(sx)\,ds + a_0 z, \tag{26}$$

$$\phi = \frac{2}{\pi}\int_0^\infty \sum_{i=1}^{3}\frac{k_{2i}}{\sqrt{\lambda_i}}[A_i(s)\sinh(sz_i) + B_i(s)\cosh(sz_i)]\cos(sx)\,ds - b_0 z, \tag{27}$$

$$E_x = \frac{2}{\pi}\int_0^\infty s\sum_{i=1}^{3}\frac{k_{2i}}{\sqrt{\lambda_i}}[A_i(s)\sinh(sz_i) + B_i(s)\cosh(sz_i)]\sin(sx)\,ds, \tag{28}$$

$$\sigma_z = \frac{2}{\pi}\int_0^\infty s\sum_{i=1}^{3}\alpha_i[A_i(s)\cosh(sz_i) + B_i(s)\sinh(sz_i)]\cos(sx)\,ds + c_0, \tag{29}$$

$$\sigma_{xz} = -\frac{2}{\pi}\int_0^\infty s\sum_{j=1}^{3}\beta_i[A_i(s)\sinh(sz_i) + B_i(s)\cosh(sz_i)]\sin(sx)\,ds, \tag{30}$$

$$D_z = \frac{2}{\pi}\int_0^\infty s\sum_{i=1}^{3}\gamma_i[A_i(s)\cosh(sz_i) + B_i(s)\sinh(sz_i)]\cos(sx)\,ds + d_0, \tag{31}$$

where

$$\alpha_i = \frac{c_{33}k_{1i} + e_{33}k_{2i}}{\lambda_i} - c_{13}, \tag{32}$$

$$\beta_i = \frac{c_{44}(1+k_{1i}) + e_{15}k_{2i}}{\sqrt{\lambda_i}}, \tag{33}$$

$$\gamma_i=\frac{e_{33}k_{1i}-d_{33}k_{2i}}{\lambda_i}-e_{31}, \tag{34}$$

and $A_i(s)$, $B_i(s)$ $(i=1, 2)$ are the unknowns to be solved, and a_0, b_0, $c_0=c_{33}a_0-e_{33}b_0$ and $d_0=e_{33}a_0+d_{33}b_0$ are unknown constants, which will be determined from the far field loading conditions.

By applying the far field loading conditions (16)-(17), the relations between $A_i(s)$ and $B_i(s)$ as well as the constants a_0, b_0, c_0 and d_0 are evaluated as follows,

$$a_0=\frac{d_{33}\sigma_0+e_{33}D_0}{c_{33}d_{33}+e_{33}^2},\ b_0=\frac{-e_{33}\sigma_0+c_{33}D_0}{c_{33}d_{33}+e_{33}^2}, \tag{35}$$

$$c_0=\sigma_0,\ d_0=D_0,$$

$$\alpha_1\cosh(sh_1)A_1(s)+\alpha_2\cosh(sh_2)A_2(s)+\alpha_3\cosh(sh_3)A_3(s)=-\alpha_1\sinh(sh_1)B_1(s)-\alpha_2\sinh(sh_2)B_2(s)-\alpha_3\sinh(sh_3)B_3(s), \tag{36}$$

$$\gamma_1\cosh(sh_1)A_1(s)+\gamma_2\cosh(sh_2)A_2(s)+\gamma_3\cosh(sh_3)A_3(s)=-\gamma_1\sinh(sh_1)B_1(s)-\gamma_2\sinh(sh_2)B_2(s)-\gamma_3\sinh(sh_3)B_3(s), \tag{37}$$

where

$$h_i=h/\sqrt{\lambda_i},\ i=1, 2, 3. \tag{38}$$

Also from Eqs. (18)-(21), we obtain,

$$\beta_1B_1(s)+\beta_2B_2(s)+\beta_3B_3(s)=0, \tag{39}$$

$$\beta_1\sinh(sh_1)A_1(s)+\beta_2\sinh(sh_2)A_2(s)+\beta_3\sinh(sh_3)A_3(s)=-\beta_1\cosh(sh_1)B_1(s)-\beta_2\cosh(sh_2)B_2(s)-\beta_3\cosh(sh_3)\beta_3(s), \tag{40}$$

$$\frac{k_{21}}{\sqrt{\lambda_1}}B_1(s)+\frac{k_{22}}{\sqrt{\lambda_2}}B_2(s)+\frac{k_{23}}{\lambda_3}B_3(s)=0. \tag{41}$$

It is convenient to use the following definition,

$$M(s)\equiv\frac{k_{11}}{\sqrt{\lambda_1}}B_1(s)+\frac{k_{12}}{\sqrt{\lambda_2}}B_2(s)+\frac{k_{13}}{\sqrt{\lambda_3}}B_3(s) \tag{42}$$

From Eqs. (39), (41) and (42), we can find $B_i(s)$ in the forms,

$$B_1(s)=N_1M(s),\ B_2(s)=N_2M(s),\ B_3(s)=N_3M(s), \tag{43}$$

where

$$N_1=(-k_{22}-k_{13}k_{22}+k_{23}+k_{12}k_{23})\sqrt{\lambda_1}/\Delta, \tag{44}$$

$$N_2=(k_{21}+k_{13}k_{21}-k_{23}-k_{11}k_{23})\sqrt{\lambda_2}/\Delta, \tag{45}$$

$$N_3=(-k_{21}-k_{12}k_{21}+k_{22}+k_{11}k_{22})\sqrt{\lambda_3}/\Delta, \tag{46}$$

$$\Delta=k_{12}k_{21}-k_{13}k_{21}-k_{11}k_{22}+k_{13}k_{22}+k_{11}k_{23}-k_{12}k_{23}. \tag{47}$$

Therefore the unknowns $A_i(s)$ $(i=1, 2, 3)$ can be determined in terms of $M(s)$ from Eqs. (36)-(37) and (40).

Using the mixed boundary conditions, Eqs. (22) and (23), we obtain the following dual integral equations,

$$\int_0^\infty sf_1(s)M(s)\cos(sx)\,ds=-\frac{\pi}{2}c_0,\ (0\le x<a), \tag{48}$$

$$\int_0^\infty M(s)\cos(sx)\,ds=0,\ (a<x<\infty), \tag{49}$$

where

$$f_1(s)=-\frac{\alpha_1A_1(s)+\alpha_2A_2(s)+\alpha_3A_3(s)}{(\alpha_1N_1+\alpha_2N_2+\alpha_3N_3)M(s)}. \tag{50}$$

To solve the dual integral equations, we define $M(s)$ in the form,

$$M(s)=\int_0^a\xi\mho(\xi)J_0(s\xi)\,d\xi, \tag{51}$$

where $J_0(s\xi)$ is the zero-order Bessel function of the first kind.

Inserting Eq. (51) into Eqs. (48) and (49), we can find that the auxiliary function $\mho(\xi)$ is given by the following Fredholm integral equation of the second kind,

$$\Omega(\Xi)+\int_0^1L(\Xi, H)\Omega(H)\,dH=\sqrt{\Xi}, \tag{52}$$

where

$$L(\Xi, H)=\sqrt{\Xi H}\int_0^\infty S[f_1(S/a)-1]J_0(SH)J_0(S\Xi)\,dS, \tag{53}$$

$$\begin{cases} S=sa,\ \Xi=\dfrac{\xi}{a},\ H=\dfrac{\eta}{a}, \\ \Omega(\Xi)=-\dfrac{2}{\pi c_0}\sqrt{\Xi}\mho(\xi), \\ \Omega(H)=-\dfrac{2}{\pi c_0}\sqrt{H}\mho(\eta). \end{cases} \tag{54}$$

4. Field Intensity Factors

The mode I stress intensity factor, K^σ is defined and determined in the form,

$$K^\sigma \equiv \lim_{x\to a^+}\sqrt{2\pi(x-a)}\;\sigma_z(x,0)=c_0\sqrt{\pi a}\,\Omega(1). \tag{55}$$

Extending the traditional concept of stress intensity factor to other field variables (Pak, 1990) and considering four possible far field conditions, we have

$$K^f=C_f\sigma_0\sqrt{\pi a}\,\Omega(1),\ (f=\sigma,\ \varepsilon,\ E,\ D), \tag{56}$$

where

$$C_\sigma=1, \tag{57}$$

$$C_\epsilon=\frac{\frac{k_{11}}{\lambda_1}\mathrm{N}_1+\frac{k_{12}}{\lambda_2}\mathrm{N}_2+\frac{k_{13}}{\lambda_3}\mathrm{N}_3}{\alpha_1\mathrm{N}_1+\alpha_2\mathrm{N}_2+\alpha_3\mathrm{N}_3}, \tag{58}$$

$$C_E=-\frac{\frac{k_{21}}{\lambda_1}\mathrm{N}_1+\frac{k_{22}}{\lambda_2}\mathrm{N}_2+\frac{k_{23}}{\lambda_3}\mathrm{N}_3}{\alpha_1\mathrm{N}_1+\alpha_2\mathrm{N}_2+\alpha_3\mathrm{N}_3}, \tag{59}$$

$$C_D=\frac{\gamma_1\mathrm{N}_1+\gamma_2\mathrm{N}_2+\gamma_3\mathrm{N}_3}{\alpha_1\mathrm{N}_1+\alpha_2\mathrm{N}_2+\alpha_3\mathrm{N}_3}, \tag{60}$$

and K^ϵ, K^E and K^D are strain intensity, electric field intensity and electric displacement intensity factor, respectively.

From Eq. (56), it is noted that the uniform electric load has no influence on the field singularities under constant stress loading, and the electric displacement intensity factor depends on the material constant C_D and the applied mechan ical load σ_0 but not on the applied electrical load These are well agreed with the results of Gao anc Fan (1999).

In case of $h\to\infty$, the field intensity factors fo an infinite piezoelectric ceramic can be obtainec in the form,

$$K^f=C_f c_0\sqrt{\pi a},\ (f=\sigma,\ \varepsilon,\ E,\ D), \tag{61}$$

These are also agreed with the results of Ga(and Fan (1999).

To examine the effect of electromechanica interactions on the field intensity factors, equa tion (52) is computed numerically by Gaussiar quadrature formula. We consider PZT-6I ceramic which material properties are as follows

Elastic constants (10^{10} N/m^2):

$$c_{11}=16.8,\ c_{33}=16.3,\ c_{44}=2.71,\ c_{13}=6.0,$$

Piezoelectric constants (C/m^2):

$$e_{15}=4.6,\ e_{31}=-0.9,\ e_{33}=7.1,$$

Dielectric permittivity (10^{-10} F/m):

$$d_{11}=36,\ d_{33}=34.$$

Figure 2 displays the variation of the normal ized field intensity factors, $K^f/C_f c_0\sqrt{\pi a}$, agains the a/h values. They increase with the increase o the a/h ratio.

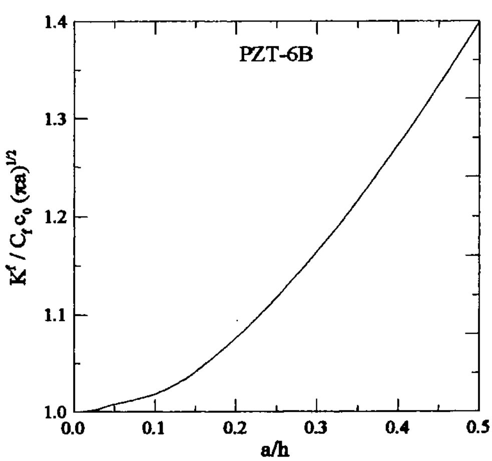

Fig. 2 Normalized field intensity factors $K^f/C_f c_0\sqrt{\pi a}$ vs. a/h

5. Conclusions

The electroelastic crack problem in a transver sely isotropic piezoelectric ceramic strip under in-plane normal loading was analyzed by the intro duction of potential functions and the integra transform approach. The traditional concept o linear elastic fracture mechanics is extended t(include the piezoelectric effects and the results ar expressed in terms of the field intensity factors Especially, the electric loadings have no influenc on the field singularities under constant stres loading.

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