Shape Design Sensitivity Analysis for Interface Problem in Axisymmetric Elasticity

Joo Ho Choi*
Department of Aeronautical and Mechanical Engineering, Hankuk Aviation University
Boo Youn Lee
Department of Automotive Engineering, Keimyung University
Jung Suk Han
Department of Prosthodontics, Ewha Womans University Mokdong Hospital

A boundary integral equation method in the shape design sensitivity analysis is developed for the elasticity problems with axisymmetric non-homogeneous bodies. Functionals involving displacements and tractions at the zonal interface are considered. Sensitivity formula in terms of the interface shape variation is then derived by taking derivative of the boundary integral identity. Adjoint problem is defined such that displacement and traction discontinuity is imposed at the interface. Analytic example for a compound cylinder is taken to show the validity of the derived sensitivity formula. In the numerical implementation, solutions at the interface for the primal and adjoint system are used for the sensitivity. While the BEM is a natural tool for the solution, more generalization should be made since it should handle the jump conditions at the interface. Accuracy of the sensitivity is evaluated numerically by the same compound cylinder problem. The endosseous implant-bone interface problem is considered next as a practical application, in which the stress value is of great importance for successful osseointegration at the interface. As a preliminary step, a simple model with tapered cylinder is considered in this paper. Numerical accuracy is shown to be excellent which promises that the method can be used as an efficient and reliable tool in the optimization procedure for the implant design. Though only the axisymmetric problem is considered here, the method can be applied to general elasticity problems having interface.

Key Words : Boundary Integral Equation Method, Interface Problem, Boundary Element Method, Shape Design Sensitivity Analysis, Endosseous Implant

1. Introduction

There are many kinds of multiple zone problems in which the interface shape is an important factor. One of such examples is found mainly in the biomechanics application such as the endosseous implant design in the dentistry or orthopedics in which the proper stress at the interface should be imposed for successful bone remodelling (Rieger et al., 1990, Tadano et al., 1994). Another example is in the thermal mold design in which optimum shape of the interface is to be determined for the mold consisting of two materials with different conductivity to enhance the thermal performance (Rezayat and Jantzen, 1995).

Sensitivity analysis has long been a subject as an efficient and reliable ingredient in optimization study of mechanical components. From numerical point of view, there have been two directions so far. One is discrete method such as finite differences or semi-analytic method, which are to calculate the derivatives of the functional

* Corresponding Author,
Email : jhchoi@mail. hangkong. ac. kr
TEL : +82-2-300-0117 ; **FAX** : +82-2-3158-2191
200-1 Hwajon-dong, Dukyang-ku, Koyang-shi Kyunggi-do, 411-791 Korea. (Manuscript **Received** May 25, 1999 ; **Revised** November 17, 1999)

by finite differences directly or calculate derivatives from the discretized matrices. While these methods are simple to implement, it is known that they are not efficient and reliable since all the process is done in the discretized domain. The other one is continuum method where a sensitivity formula is derived analytically based on the continuum formulation and computation is conducted afterwards using the formula. While this is known better suited for the sensitivity analysis, the process of the derivation is not straightforward, and needs good amount of mathematical knowledge. Depending on the formulation, there have also been two directions in the continuum method. One is domain method in which domain functional is defined, and the sensitivity formula is derived based on the variational equation. For the numerical implementation, finite element method is mostly used (Haug et al., 1986). The other one is boundary method, which is to derive sensitivity for the boundary functionals. Since in most design situations, critical problem usually arises at the boundary, the boundary method is more suited for the design problems. Due to this advantage, the boundary methods have recently been studied in many different literatures, having their own characteristics, e. g., using BEM in the variational formulation by Mota Soares and Choi (1986), the use of Lagrange multiplier by Meric (1987), direct use of the boundary integral equation by Barone and Yang (1988) and Zhang and Mukherjee (1989). Boundary integral method developed by the authors (Choi and Kwak, 1988; Lee and Kwak, 1992) is also one of such directions, which is to utilize a boundary integral identity, a boundary form of Betti's reciprocal formula. This formulation however has advantage over the other boundary methods in that the procedure is more general, and the derived sensitivity formula can be computed by using any kind of solver such as the BEM and FEM, though the BEM is more natural for the implementation.

This paper addresses sensitivity analysis with respect to the interface shape in multi-zone problem. Despite its importance and usefulness, only a few studies are found, which began by the people using domain method, hence, the finite element method. At first they found the resulting formula gives bad accuracy on the interface boundary, and it was overcome afterwards by introducing domain or boundary layer velocity field, which is however inefficient since it causes the computation over the whole or the part of the domain for the sensitivity (Choi and Seong, 1986; Seong and Choi, 1987). While further study not found afterwards, a paper is found very recently in the thermal conduction problem by Dems and Mroz (1998). In the present paper, sensitivity analysis by the boundary integral method is conducted for the axisymmetric elasticity problem as in Lee and Kwak (1992) and Lee (1996), but the formulation is extended to deal with the problem with zone interface. Functionals are considered at the interface, having design parameters as the interface shape. Adjoint system is defined such that a certain jump condition is imposed at the interface, which means more generalization should be made for the analysis code to solve it. A simple compound cylinder problem is considered to show the validity of the present method in analytic as well as numerical way.

The motivation of the present study was begun in finding an optimum shape of the post-type dental implant (Rieger et al., 1990; and Lozada et al., 1994). Though there are large number of implant designs nowadays, each claiming unique geometric features and long-term stability, many of those designs lack supporting evidence of their effectiveness. In clinical point of view, it is well known that the implant should be made and used such that permanent osseointegration occur at the implant-bone interface, and this is only possible if the stresses are maintained at the interface with a certain magnitude. If overstressed or understressed, it is known to cause bone resorption or atrophy. The implant shape hence is an important factor that influences dominantly to the stress pattern at the interface, and the sensitivity analysis conducted here can be a good tool for finding true or global optimum design. A tapered cylinder implant, which was treated in Lozada et al. (1994), is chosen as a preliminary step toward more practical study of many screw type implants. Due to its geometry and loading condition, the problem is considered axisymmetric. Once the sensitivity of reasonable accuracy can be successfully obtained, finding optimum design is straightforward, which will be addressed in the next paper.

2. Shape Design Sensitivity of Single Domain

Consider first a single axisymmetric body Ω of (r, z) coordinate as shown in Fig. 1. Stress equilibrium equation under zero body forces is imposed, and the boundary condition for the problem is defined as follows.

$$\left.\begin{array}{l}\frac{1}{r}(r\sigma_{rr})_{,r}+\sigma_{rz,z}-\frac{1}{r}\sigma_{\theta\theta}=0, \text{ when } i=r \\ \frac{1}{r}(r\sigma_{zr})_{,r}+\sigma_{zz,z}=0, \text{ when } i=z\end{array}\right\} \text{ in } \Omega \tag{1}$$

$$u_i=\bar{u}_i \text{ on } \Gamma_u \tag{2}$$
$$p_i=\bar{p}_i \text{ on } \Gamma_p$$

where i, j denote r and z directions respectively, $\sigma_{\theta\theta}$ is hoop stress and p_i is traction vector, i. e., $p_i=\sigma_{ij}n_j$. The detail procedure for deriving sensitivity formula of this problem can be found in Lee and Kwak (1992) or Lee (1996). A boundary integral identity is stated which holds for any two functions that satisfy Eq. (1)

$$\int_\Gamma (u_i p_i^* - p_i u_i^*)\, rds=0 \tag{3}$$

where the two sets of functions (u_i, p_i) and (u_i^*, p_i^*) are called primal and adjoint variables respectively. Consider now a functional involving displacement and traction over the whole boundary as

$$\Psi=\int_\Gamma \Psi(u_i, p_i)\, rds \tag{4}$$

Take material derivative of this functional as well as boundary integral identity with respect to the shape variation which is denoted as velocity vector V_i. Then the analytic formula for the sensitivity of the functional is given after lengthy derivation as

$$\Psi'=\int_\Gamma\{-u_{i,s}\sigma_{ij}^* V_k e_{jk}+p_i u_{i,j}^* V_j+p_i u_i^*(VS_s+\frac{\dot{r}}{r})-\frac{1}{r}u_r\sigma_{\theta\theta}^* V_n\}\, rds+\int_{\Gamma_u}(\Psi_{u_i}-p_i^*)\,\dot{\bar{u}}_i\, rds+\int_{\Gamma_p}(\Psi_{p_i}+u_i^*)\,\dot{\bar{p}}_i\, rds+\int_\Gamma \Psi(VS_s+\frac{\dot{r}}{r})\, rds \tag{5}$$

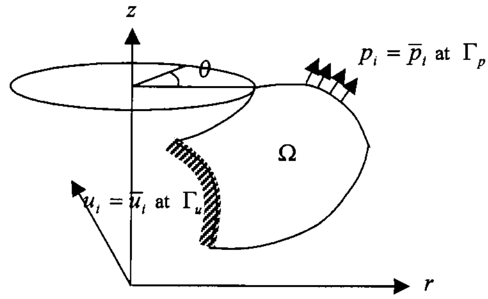

Fig. 1 Single body in axisymmetric elasticity

where V_n, V_s denote normal and tangential components of the shape variation, Ψ_{u_i}, Ψ_{p_i} are the partial derivative of Ψ with respect to u_i, p_i, and

$$\dot{r}=n_r V_n - n_z V_s \tag{6}$$
$$VS_s=V_{k,s}s_k$$
$$e_{jk}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \tag{7}$$

Note here that VS_s is differential of the element ds with respect to the shape variation, which involves only tangential derivatives, hence, is called surface divergence of V_k. The boundary condition for the adjoint variable is defined by

$$u_i^*=-\Psi_{p_i} \text{ on } \Gamma_u \tag{8}$$
$$p_i^*=\Psi_{u_i} \text{ on } \Gamma_p$$

The computational procedure starts from solving the primal problem satisfying the boundary condition (2). Then using the solution, solve for the adjoint problem with the boundary condition (8). Define next the proper velocity vector describing boundary shape variation. Finally the sensitivity is calculated by the formula (5) using both the primal and adjoint solutions at the boundary in terms of velocity vector. As stated previously, the use of boundary element method is natural for the solutions. Then the solution for (u_i, p_i) at the boundary is obtained. However, other forms are also necessary in (5) such as $u_{i,s}$, $u_{i,j}$, σ_{ij}, which are not directly calculable. These can be expressed mostly in terms of u_i, p_i using simple algebraic relation, which does not lose its accuracy. Note however all of them also include

$u_{i,s}$ which should be calculated by a numerical differentiation, and lose accuracy by an order. Nevertheless the resulting accuracy has been proved excellent by many examples especially compared to those by finite elements.

3. Shape Design Sensitivity of Multi-Zone in Axisymmetric Elasticity

Consider next domain with two sub-regions which have different material properties denoted by superscript 1 and 2 as shown in Fig. 2. Then the boundary integral identity holds for each sub-region

$$\int_{\Gamma_1} (u_i^1 p_i^{1*} - p_i^1 u_i^{1*})\, rds = 0, \quad \Gamma_1 = \Gamma_a \cup \Gamma_c \qquad (9)$$

$$\int_{\Gamma_2} (u_i^2 p_i^{2*} - p_i^2 u_i^{2*})\, rds = 0, \quad \Gamma_2 = \Gamma_b \cup \Gamma_c$$

At the interface boundary Γ_c, primal variable for zone 1 and 2 should satisfy the following continuity condition

$$\begin{matrix} u_i^1 = u_i^2 = u_i^c \\ p_i^1 = -p_i^2 = p_i^c \end{matrix} \text{ at } \Gamma_c \qquad (10)$$

If the two identities are summed, and the continuity condition is applied, we get

$$\int_{\Gamma_a} (u_i^1 p_i^{1*} - p_i^1 u_i^{1*})\, rds + \int_{\Gamma_b} (u_i^2 p_i^{2*} - p_i^2 u_i^{2*})\, rds + \int_{\Gamma_c} \{u_i^c (p_i^{1*} + p_i^{2*}) - p_i^c (u_i^{1*} - u_i^{2*})\}\, rds = 0 \qquad (11)$$

Functional is defined at the interface Γ_c between adjacent zone as follows, where the design parameter is the interface shape.

$$\Psi = \int_{\Gamma_c} \psi(u_i, p_i)\, rds = \int_{\Gamma_c} \psi(u_i^c, p_i^c)\, rds \qquad (12)$$

Then the material derivative of the functional is

$$\Psi' = \int_{\Gamma_c} (\psi_{u_i} \dot{u}_i^c + \psi_{p_i} \dot{p}_i^c)\, rds + \int_{\Gamma_c} \psi \left(VS_s + \frac{\dot{r}}{r} \right) rds \qquad (13)$$

Take the derivative to each boundary integral identity respectively, sum the resulting two equations, and apply continuity condition (10) afterwards to obtain

$$\int_{\Gamma_a} (\dot{u}_i^1 p_i^{1*} - \dot{p}_i^1 u_i^{1*})\, rds + \int_{\Gamma_b} (\dot{u}_i^2 p_i^{2*} - \dot{p}_i^2 u_i^{2*})\, rds + \int_{\Gamma_c} \{\dot{u}_i^c (p_i^{1*} + p_i^{2*}) - \dot{p}_i^c (u_i^{1*} - u_i^{2*})\}\, rds$$
$$= \int_{\Gamma_c} \left\{ \begin{matrix} -u_{i,s}^c (\sigma_{ij}^{1*} - \sigma_{ij}^{2*}) V_k e_{jk} + p_i^c (u_{i,j}^{1*} - u_{i,j}^{2*}) V_j \\ + p_i^c (u_i^{1*} - u_i^{2*}) \left(VS_s + \frac{\dot{r}}{r} \right) - \frac{1}{r} u_r^c (\sigma_{\theta\theta}^{1*} - \sigma_{\theta\theta}^{2*}) V_n \end{matrix} \right\} rds \qquad (14)$$

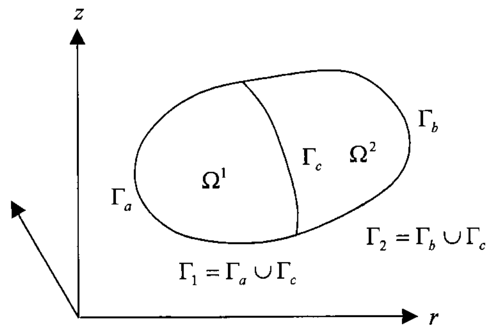

Fig. 2 Two-zone body in axisymmetric elasticity

where the shape variation is limited to the interface boundary. It should be reminded that the tangential and normal directions defined above are for the 1st zone, and those of 2nd zone should be reversed. As a result, the sign of V_n and $\partial/\partial s$ of the 2nd zone has been changed. To relate the implicit derivatives in (13) and (14), define adjoint problem with the interface condition

$$\begin{matrix} u_i^{1*} - u_i^{2*} = -\psi_{p_i} \\ p_i^{1*} + p_i^{2*} = \psi_{u_i} \end{matrix} \text{ at } \Gamma_c \qquad (15)$$

and all the prescribed values at the external boundaries are zero. This means the displacement or the traction should have jump at the interface. Then the desired sensitivity formula is obtained as follows.

$$\Psi' = \int_{\Gamma_c} \left\{ \begin{matrix} -u_{i,s}^c (\sigma_{ij}^{1*} - \sigma_{ij}^{2*}) V_k e_{jk} + p_i^c (u_{i,j}^{1*} - u_{i,j}^{2*}) V_j \\ + p_i^c (u_i^{1*} - u_i^{2*}) \left(VS_s + \frac{\dot{r}}{r} \right) - \frac{1}{r} u_r^c (\sigma_{\theta\theta}^{1*} - \sigma_{\theta\theta}^{2*}) V_n \end{matrix} \right\} rds + \int_{\Gamma_c} \psi \left(VS_s + \frac{\dot{r}}{r} \right) rds \qquad (16)$$

As noted above, the primal as well as the adjoint problem can be solved in principle by any code since the formula is based on continuum formulation. However, since all the computation is done over the boundary variables, the boundary element method is more natural. Use of commercial boundary element S/W such as BEASY is also quite desirable and reduce the unnecessary burden due to its reliability and convenience. However the adjoint problem defined above requires handling jump condition at the interface. While the traction discontinuity is treated in BEASY, imposing displacement discontinuity is not possible. Therefore a special boundary element program has been developed to this end for multi-zone axisymmetric elasticity, which enables such kind of jump conditions.

4. Analytic Example in Axisymmetric Elasticity

Consider a compound cylinder as shown in Fig. 3 where the radii of the inner and outer cylinder are a and b respectively, and uniform tension with magnitude $\bar{\sigma}$ is applied in the radial direction at the external radius of outer cylinder. Material properties for the inner and outer cylinder are μ^i, ν^i and μ^o, ν^o respectively. For the axial direction, plane stress condition is assumed. The purpose is to get sensitivity formula for displacement and stresses at the interface with respect to the radius a. Analytic solution for the displacement and stresses of this problem can be found in any literature as follows (Ugural and Fenster, 1975). Let the displacement and stress in the radial direction at the interface be u^c and σ^c. Then the continuity condition for the traction at the interface is given by

$$p_r^i = \sigma^c,\ p_r^o = -\sigma^c \text{ at } r = a \tag{17}$$

Analytic solution for the displacement and stress at the interface are

$$u^c = \frac{a}{2\mu^i c_4^i}\sigma^c \tag{18}$$

and

$$\sigma^c = \frac{(1+c_4^o)\,b^2}{(\beta + c_4^o)\,b^2 + (1-\beta)\,a^2}\bar{\sigma} \tag{19}$$

where $\beta = \mu^o c_4^o / \mu^i c_4^i$ and $C_4 = \frac{1+\nu}{1-\nu}$. Sensitivities of u^c and σ^c with respect to a are then obtained by taking derivative by a as

$$\frac{d\sigma^c}{da} = -\frac{2(1-\beta)\,a}{(\beta + c_4^o)\,b^2 + (1-\beta)\,a^2}\sigma^c \tag{20}$$

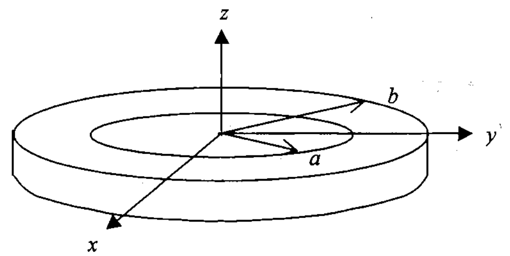

Fig. 3 Compound cylinder problem

$$\frac{du^c}{da} = \frac{1}{2\mu^i c_4^i}\frac{(\beta + c_4^o)\,b^2 - (1-\beta)\,a^2}{(\beta + c_4^o)\,b^2 + (1-\beta)\,a^2}\sigma^c \tag{21}$$

Consider now a displacement and stress functional defined at the interface as

$$\Psi_u = \int_{\Gamma_c} u^c r ds \Big/ \int_{\Gamma_c} r ds = u^c \tag{22}$$

$$\Psi_\sigma = \int_{\Gamma_c} \sigma^c r ds \Big/ \int_{\Gamma_c} r ds = \sigma^c \tag{23}$$

where the length in z-direction is assumed unit. If the sensitivity formula derived previously is used, adjoint problem for the displacement functional is defined as follows.

$$\left.\begin{aligned} & u_k^{i*} - u_k^{o*} = -\psi_{p_k} = 0 \\ & p_k^{i*} + p_k^{o*} = \psi_{u_k} = 1 \text{ when } k = r \text{ only} \end{aligned}\right\} \text{at } \Gamma_c \tag{24}$$

which means that traction discontinuity with unit magnitude should be imposed at the interface. The solution for this problem can also be obtained analytically, which is for the inner region at the interface,

$$\begin{aligned} & \sigma_r^{i*} = \sigma^{ic*},\ \sigma_\theta^{i*} = \sigma^{ic*} \\ & u_r^{i*} = \frac{\sigma^{ic*}}{2\mu^i c_4^i}a,\ u_{r,r}^{i*} = \frac{\sigma^{ic*}}{2\mu^i c_4^i} \end{aligned} \tag{25}$$

and for the outer region at the interface

$$\begin{aligned} & \sigma_r^{o*} = \sigma^{oc*},\ \sigma_\theta^{o*} = -\frac{b^2+a^2}{b^2-a^2}\sigma^{oc*} \\ & u_r^{o*} = -\frac{a}{2\mu^o c_4^o}\frac{c_4^o b^2 + a^2}{b^2 - a^2}\sigma^{oc*}, \\ & u_{r,r}^{o*} = \frac{1}{2\mu^o c_4^o}\frac{c_4^o b^2 - a^2}{b^2 - a^2}\sigma^{oc*} \end{aligned} \tag{26}$$

Applying condition at the interface, solution for adjoint problem is obtained as follows.

$$\sigma^{oc*} = \frac{-\beta(b^2 - a^2)}{(\beta + c_4^o)\,b^2 + (1-\beta)\,a^2} \tag{27}$$

$$\sigma^{ic*}=1+\sigma^{oc*}=\frac{(c_4^o b^2+a^2)}{(\beta+c_4^o)\,b^2+(1-\beta)\,a^2}$$

Note in this case that integration path s is in z direction. Therefore $\dot{r}=V_n=\delta a$, $V_s=0$, and derivatives with respect to z are all zero. Then the sensitivity for the displacement functional is given by

$$\Psi_u'=\frac{1}{\int_{\Gamma_c} ads}\left[\begin{array}{l}\int_{\Gamma_c}\left\{\sigma^c(u_{r,r}^{i*}-u_{r,r}^{o*})\,\delta a-\frac{1}{a}u^c(\sigma_\theta^{i*}-\sigma_\theta^{o*})\,\delta a\right\}ads\\ +\int_{\Gamma_c}(u^c-\Psi)\,\delta ads\end{array}\right]$$

$$=\sigma^c(u_{r,r}^{i*}-u_{r,r}^{o*})\,\delta a-\frac{1}{a}u^c(\sigma_\theta^{i*}-\sigma_\theta^{o*})\,\delta a \quad (28)$$

If (25-27) are applied, we get the identical result to (21). Similarly for the stress functional, the adjoint condition is given by

$$\begin{array}{l}u_k^{i*}-u_k^{o*}=-\psi_{p_k}=-1 \text{ when } k=r \text{ only}\\ p_k^{i*}+p_k^{o*}=\psi_{u_k}=0\end{array}$$

at Γ_c (29)

and the solution is

$$\sigma^{oc*}=-\frac{2\mu^o c_4^o}{a}\frac{(b^2-a^2)}{(\beta+c_4^o)\,b^2+(1-\beta)\,a^2} \quad (30)$$

$$\sigma^{ic*}=\sigma^{oc*}$$

The sensitivity for the stress functional is given by

$$\Psi_p'=\frac{1}{\int_{\Gamma_c} ads}\left[\begin{array}{l}\int_{\Gamma_c}\left\{\begin{array}{l}\sigma^c(u_{r,r}^{i*}-u_{r,r}^{o*})\,\delta a\\ +\sigma^c(u_r^{i*}-u_r^{o*})\frac{\delta a}{a}\\ -\frac{1}{a}u^c(\sigma_\theta^{i*}-\sigma_\theta^{o*})\,\delta a\end{array}\right\}ads\\ +\int_{\Gamma_c}(\sigma^c-\Psi)\,\delta ads\end{array}\right]$$

$$=\sigma^c(u_{r,r}^{i*}-u_{r,r}^{o*})\,\delta a+\sigma^c(u_r^{i*}-u_r^{o*})\frac{\delta a}{a}$$

$$-\frac{1}{a}u^c(\sigma_\theta^{i*}-\sigma_\theta^{o*})\,\delta a \quad (31)$$

which will lead to the same expression as (20).

5. Numerical Example of Design Sensitivity Analysis

This section considers more general case of compound cylinder by numerical way. Consider a compound cylinder having outer radius 2 and height 1. The interface between the inner and outer cylinder is expressed in this case as straight line with abscissa a and the angle α from the vertical axis as follows. (see Fig. 4)

$$r=a+z\tan\alpha \quad (32)$$

Shear modulus and Poisson ratio are given as 0. 3847, 0.3 for the inner, 0.5771 and 0.3 for the outer cylinder. Displacement in the z direction is fixed at the bottom line of the cylinder, and unit traction is exerted at the external end. Design variables in this problem are a and α of the interface. Then the velocity vector and its tangential derivative is for a

$$\begin{array}{l}V_r=\frac{\partial r}{\partial a}=1,\ V_z=0\\ V_{r,s}=0,\ V_{z,s}=0\end{array} \quad (33)$$

and for α

$$\begin{array}{l}V_r=z\sec^2\alpha,\ V_z=0\\ V_{r,s}=\sec\alpha,\ V_{z,s}=0\end{array} \quad (34)$$

Functionals are defined at the part of the interface Γ_ε as the square averages of displacement and traction, i. e.,

$$\begin{array}{l}\Psi_u=\int_{\Gamma\varepsilon}(u_r^2+u_z^2)\,m_p r ds/\int_{\Gamma\varepsilon}m_p r ds\\ \Psi_p=\int_{\Gamma\varepsilon}(p_r^2+p_z^2)\,m_p r ds/\int_{\Gamma\varepsilon}m_p r ds\end{array} \quad (35)$$

Then the adjoint condition can be defined for Ψ_u

$$\begin{array}{ll}u_i^{1*}-u_i^{2*}=0 & \text{at } \Gamma_c\\ p_i^{1*}+p_i^{2*}=\begin{cases}\psi_{u_i}=2u_i m_p, & \text{at } \Gamma_\varepsilon\subset\Gamma_c\\ 0, & \text{elsewhere of } \Gamma_c\end{cases}\end{array} \quad (36)$$

and those for Ψ_p can be defined similarly. Note here that a mollifier function m_p is introduced to alleviate the discontinuity occurring at the end points of Γ_ε. Note further that in the sensitivity formula (16), following term should be added due to the presence of denominator in (35).

$$-\Psi\int_{\Gamma\varepsilon}(VS_s+\frac{\dot{r}}{r})\,m_p r ds \quad (37)$$

Boundary element model is made with 5 elements at every external boundary segment, and 8 elements at the interface. Four numbers of functionals are considered with each having 2 elements for Γ_ε. Three cases are considered for the mollifier function. First and second are quadratic and constant function as

Table 1 Comparison of sensitivity by m_p variation in compound cylinder

		$m_p=1-\eta^2$			$m_p=1$			$m_p=1$ Equivalent		
		Finite Diff.	Present Method	Ratio (%)	Finite Diff.	Present Method	Ratio (%)	Finite Diff.	Present Method	Ratio (%)
$\frac{\partial \Psi_u}{\partial a}$	1	.57E+0	.56E+0	99	.85E+0	.91E+0	108	.85E+0	.84E+0	99
	2	.56E+0	.56E+0	99	.85E+0	.98E+0	116	.85E+0	.84E+0	99
	3	.56E+0	.56E+0	99	.84E+0	.97E+0	116	.84E+0	.83E+0	99
	4	.55E+0	.55E+0	100	.82E+0	.89E+0	108	.82E+0	.81E+0	99
$\frac{\partial \Psi_p}{\partial a}$	1	.72E−1	.78E−1	108	.11E+0	.11E+0	102	.11E+0	.11E+0	101
	2	.72E−1	.77E−1	106	.11E+0	.11E+0	106	.11E+0	.11E+0	99
	3	.73E−1	.78E−1	108	.11E+0	.11E+0	105	.11E+0	.11E+0	103
	4	.77E−1	.59E−1	78	.11E+0	.12E+0	104	.11E+0	.11E+0	96
$\frac{\partial \Psi_u}{\partial \alpha}$	1	.19E−2	.19E−2	99	.29E−2	.32E−2	111	.29E−2	.29E−2	99
	2	.37E−2	.37E−2	100	.55E−2	.64E−2	116	.55E−2	.54E−2	99
	3	.53E−2	.53E−2	100	.80E−2	.93E−2	116	.80E−2	.79E−2	99
	4	.67E−2	.67E−2	101	.10E−1	.11E−1	107	.10E−1	.98E−2	98
$\frac{\partial \Psi_p}{\partial \alpha}$	1	.12E−2	.10E−2	81	.20E−2	.21E−2	103	.20E−2	.20E−2	98
	2	.40E−3	.44E−3	108	.61E−3	.68E−3	112	.61E−3	.65E−3	106
	3	.22E−4	.10E−3	459	.37E−4	−.82E−4	###	.37E−4	.15E−3	409
	4	.58E−3	.28E−5	1	.11E−2	.99E−3	93	.11E−2	.94E−3	89

$$m_p=\begin{cases}1-\eta^2, & \text{for the 1st case,}\\ 1, & \text{for the 2nd case,}\end{cases}\quad \eta=(-1,\ 1)\subset\Gamma_\varepsilon \tag{38}$$

While the 1st one is continuous at the end points of Γ_ε, 2nd one brings about jumps. To remove this problem, last one is to use equivalent value solving an additional equation from the 2nd case by weighted residual concept as

$$\int_{\Gamma_c} f_{eq} Wrds=\int_{\Gamma_\varepsilon} fWrds \tag{39}$$

where f and f_{eq} denote original and equivalent function value defined at Γ_ε and Γ_c, and W is a weighting function. In this case, the shape function used for boundary element method is employed. As the analytic solution is not available, the computed sensitivity is compared to the finite differences with 1% design variable increment. Table 1 is the compared result of m_p when the design variables are $a=1$, $\alpha=0$. Upper two row blocks are sensitivity w. r. t. the translation of a while the lower two are for the rotation of angle α. Number in each block denotes the functional id increasing from the bottom to the top. From the table, use of $m_p=1$ with equivalent value is found best. Therefore using $m_p=1$ and the equivalent values, the sensitivities are computed when, $\alpha=0°$, 10° and 20°, which is shown in the Table 2. From the result, we can observe stable accuracy even with the different geometry.

6. Design Sensitivity Analysis of Endosseous Implant Shape

The endosseous implant-bone interface problem is considered next as a practical application, in which the stress value is of great importance for successful osseointegration at the interface. A tapered cylinder implant, which is relatively simple in geometry, is chosen as a preliminary step toward more practical study of many screw type implants. Due to its geometry and loading condition, the problem is considered axisymmetric. The model consists of three zones, which are implant, cancellous bone and cortical bone. The problem is to find the height h and the taper angle α such that the stress value at the interface between the cancellous bone and the implant is maintained at a certain specified level. The lower end of the implant has semi-circular shape. The radius and

Table 2 Sensitivity by angle α increase in compound cylinder

		$\alpha=0°$			$\alpha=10°$			$\alpha=20°$		
		Finite Diff.	Present Method	Ratio (%)	Finite Diff.	Present Method	Ratio (%)	Finite Diff.	Present Method	Ratio (%)
$\frac{\partial \Psi_u}{\partial a}$	1	.85E+0	.84E+0	99	.89E+0	.89E−1	100	.94E+0	.48E−1	101
	2	.85E+0	.84E+0	99	.92E+0	.92E−1	100	.10E+1	.51E−1	101
	3	.84E+0	.83E+0	99	.95E+0	.95E−1	100	.11E+1	.55E−1	101
	4	.82E+0	.81E+0	99	.97E+0	.97E−1	100	.12E+1	.58E−1	101
$\frac{\partial \Psi_p}{\partial a}$	1	.11E+0	.11E+0	101	.11E+0	.11E−1	98	.11E+0	.53E−2	102
	2	.11E+0	.11E+0	99	.12E+0	.11E−1	98	.11E+0	.53E−2	94
	3	.11E+0	.11E+0	103	.13E+0	.13E−1	103	.13E+0	.70E−2	105
	4	.11E+0	.11E+0	96	.13E+0	.13E−1	98	.16E+0	.81E−2	99
$\frac{\partial \Psi_u}{\partial \alpha}$	1	.29E−2	.29E−2	99	.33E−1	.33E−2	100	.79E−1	.40E−2	101
	2	.55E−2	.54E−2	99	.64E−1	.64E−2	100	.16E+0	.79E−2	101
	3	.80E−2	.79E−2	99	.96E−1	.96E−2	100	.25E+0	.12E−1	101
	4	.10E−1	.98E−2	98	.13E+0	.13E−1	100	.35E+0	.17E−1	101
$\frac{\partial \Psi_p}{\partial \alpha}$	1	.20E−2	.20E−2	98	−.27E−1	−.28E−2	101	−.14E+0	−.67E−2	99
	2	.61E−3	.65E−3	106	−.43E−1	−.42E−2	99	−.17E+0	−.86E−2	99
	3	.37E−4	.15E−3	409	−.46E−1	−.44E−2	97	−.17E+0	−.84E−2	98
	4	.11E−2	.94E−3	89	−.39E−1	−.41E−2	105	−.16E+0	−.85E−2	103

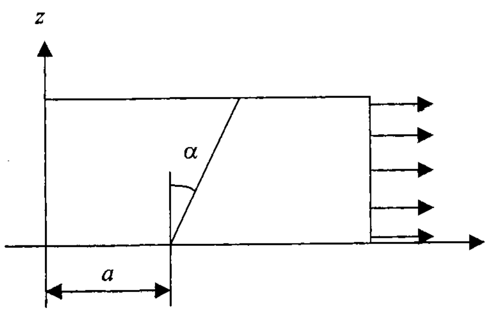

Fig. 4 Compound cylinder problem for numerical implementation

height of cortical bone are 6.5 and $10mm$. The thickness at the bottom and top of the cortical bone is 1.5, and at the right side is $2mm$. Implant radius is $2mm$. At the top of the implant, uniform traction is applied such that magnitude of the total force is $100N$. Shear moduli for the implant, cortical and cancellous bone are 117.e3, 20.e3, 2.e3N/mm^2 and Poisson ratios are 0.3 all together. Interface to be varied is the tapered straight line between the implant and cancellous bone. Though the reasonable measure for the stress evaluation is von Mises stress, this paper treats only traction values as a preliminary study. Therefore same functionals as in the previous

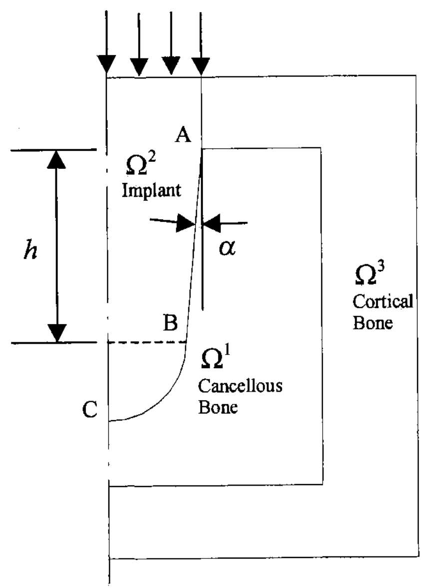

Fig. 5 Tapered cylinder implant problem

example are considered. Having 2 elements per a functional, 6 functionals are defined at the interface from point A to B of Fig. 5. Deformed shape of the whole domain and square of the traction distribution along the interface are shown in Fig. 6, where severe concentration of the traction is found at point A. The sensitivity values are

Table 3 Sensitivity result of the implant problem

		Finite Diff.	Present Method	Ratio (%)			Finite Diff.	Present Method	Ratio (%)
$\frac{\partial \Psi_u}{\partial h}$	1	−.10E−7	−.10E−7	101	$\frac{\partial \Psi_u}{\partial a}$	1	.47E−8	.46E−8	98
	2	1.13E−7	−.13E−7	100		2	.46E−8	.45E−8	98
	3	−.15E−7	−.15E−7	100		3	.45E−8	.44E−8	98
	4	−.17E−7	−.17E−7	100		4	.42E−8	.41E−8	98
	5	−.18E−7	−.18E−7	101		5	.39E−8	.38E−8	98
	6	−.19E−7	−.21E−7	107		6	.36E−8	.38E−8	104
$\frac{\partial \Psi_p}{\partial h}$	1	−.29E+1	−.30E+1	104	$\frac{\partial \Psi_p}{\partial a}$	1	.93E+0	.82E+0	88
	2	−.34E+0	−.33E+0	98		2	.17E+0	.17E+0	101
	3	−.24E+0	−.24E+0	101		3	.14E+0	.14E+0	100
	4	−.22E+0	−.23E+0	100		4	.12E+0	.12E+0	99
	5	−.25E+0	−.24E+0	95		5	.11E+0	.11E+0	100
	6	−.28E+0	−.17E+0	62		6	.89E−1	.89E−1	101

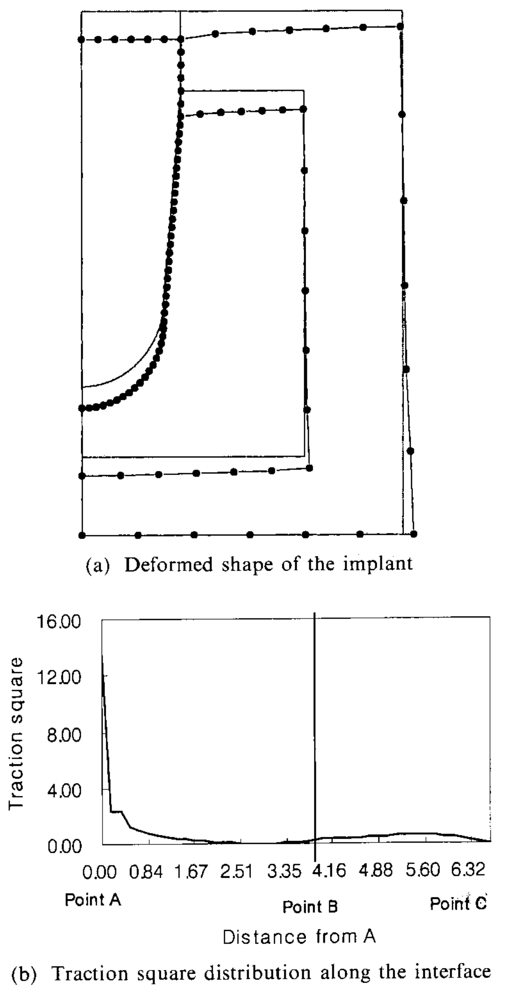

Fig. 6 Boundary element analysis result of the implant problem

compared with the finite differences, and listed in Table 3. Since this problem deals with 3 zones, accuracy of the 1st functional which includes point A is more concerned. The resulting values 104% and 88% shows favorable accuracy. Moreover even for the regions with lower magnitude, the accuracy does not degrade much which promises that the method can be used as an efficient and reliable tool in the optimization procedure for the implant design.

7. Conclusion

Boundary integral method is applied to the interface problem of axisymmetric elasticity. Sensitivity formulae for displacement and traction functionals are derived in terms of the interface shape variation. The adjoint system is then defined such that a jump condition is imposed across the interface, which means more generalization should be made for the analysis code to enable the jump condition. A special boundary element program has been developed to this end for multi-zone axisymmetric elasticity, which enables such kind of jump conditions. A simple compound cylinder problem is considered to show the validity of the present method in analytic as well as numerical way. Three cases of mollifier function are studied and the use of equivalent values for the unit step function defined over the boundary is found best for the sensitivity. As a more practical example, implant design problem with a tapered cylinder shape is considered. Accuracy of the sensitivity at the implant bone interface is shown to be excellent even at the point with 3 zones in common, which promises that the method can be used as an efficient and reliable tool in the optimization procedure for the implant design. Study for more complex implant model and optimization such as screw type will be reported in near future.

Acknowledgements

This study was supported by the Korea Science and Engineering Foundation (KOSEF) under Grant Number 981-0711-087-2, which is gratefully acknowledged.

References

Barone, M. R. and Yang, R. J., 1988, "Boundary Integral Equations for Recovery of Design Sensitivity in Shape Optimization," *AIAA Journal*, Vol. 26, pp. 589~594.

Choi, J. H. and Kwak, B. M., 1988, "Boundary Integral Equation Method for Shape Optimization of Elastic Structures," *International Journal for Numerical Methods in Engineering*, Vol. 26, pp. 1579~1595.

Choi, K. K. and Seong, H. G., 1986, "A Domain Method for Shape Design Sensitivity Analysis of Built-Up Structures," *Computer Methods in Applied Mechanics and Engineering*, Vol. 57, pp. 1~15.

Dems, K. and Mroz, Z., 1998, "Sensitivity Analysis and Optimal Design of External Boundaries and Interfaces for Heat Conduction Systems," *Journal of Thermal Stresses*, Vol. 21, pp. 461~488.

Haug, E. J., Choi, K. K. and Komkov, K., 1986, "Design Sensitivity Analysis of Structural Systems," Academic Press,.

Lee, B. Y., 1996, "Consideration of Body Forces in Axisymmetric Design Sensitivity Analysis Using the BEM," *Computers & Structures*, Vol. 61, pp. 587~596.

Lee, B. Y. and Kwak, B. M., 1992, "Axisymmetric Thermoelastic Shape Sensitivity Analysis and Its Application to Turbine Disc Design," *International Journal for Numerical Methods in Engineering*, Vol. 33, pp. 2073~2089.

Lozada, J. L., Abbate, M. F., Pizzarello, F. A. and James, R. A., 1994, "Comparative 3-Dimensional Analysis of Two Finite Element Endosseous Implant Designs," *Journal of Oral Implantology*, Vol. 20, pp. 315~321.

Meric, R. A., 1987, "Boundary Elements in Shape Design Sensitivity Analysis of Thermoelastic Solids," in C. A. Mota Soares (ed.), *Computer Aided Optimal Design : Structural and Mechanical Systems*, Springer-Verlag, pp. 643~652.

Mota Soares, C. A. and Choi, K. K., 1986, "Boundary Elements in Shape Optimal Design of Structures," in J. A. Bennett and M. E. Botkin (ed.), *The Optimum Shape : Automated Structural Design*, Plenum, NewYork, pp. 199~231.

Rezayat, M., Jantzen, B., 1995, "Effects of Inserts on the Injection Molding Process," *Polymer Engineering and Science*, Vol. 35, pp. 247~251.

Rieger, M. R., Adams, W. K., Kinzel, S. L., 1990, "A Finite Element Survey of Eleven Endosseous Implants," *Journal of Prosthetic Dentistry*, Vol. 63, pp. 457~465.

Seong H. G. and Choi, K. K., 1987, "Boundary Layer Approach to Shape Design Sensitivity Analysis," *Mechanics of Structures and Machines*, Vol. 15, pp. 241~263.

Tadano, S., Ukai, T., Ochiai, H., Sasaki, T., 1994, "Three-Dimensional Stress Analysis in Femur Before and After Total Hip Replacement," *Transactions of the Japan Society of Mechanical Engineers, Part A*, Vol. 60, pp. 278~284.

Ugural, A. C. and Fenster, S. K., 1975, "Advanced Strength and Applied Elasticity," Elsevier.

Zhang, Q., Mukherjee, S., 1989, "Design Sensitivity Coefficients for Linear Elasticity Problems by Boundary Element Methods," *IUTAM/IACM Symposium on Discretization Methods in Structural Mechanics*, Austria.