Bifurcation Modes in the Limit of Zero Thickness of Axially Compressed Circular Cylindrical Shell

Youngjoo Kwon*
(Hongik University)

Bifurcation intability modes of axially compressed circular cylindrical shell are investigated in the limit of zero thickness (i.e., h(thickness) $\rightarrow 0$) analytically, adopting the general stability theory developed by Triantafyllidis and Kwon(1987) and Kwon(1992). The primary state of the shell is obtained in a closed form using the asymptotic technique, and then the straightforward bifurcation analysis is followed according to the general stability theory to obtain the bifurcation modes in the limit of zero thickness in a full analytical manner. Hence, the closed form bifurcation solution is obtained. Finally, the result is compared with the classical one.

Key Words: Strain Energy Density Function, Stören-Rice Hypoelastic Material, Characteristic Equation, Multiple Scales Asymptotic Technique, Bifurcation Modes

1. Introduction

Instability problem in solid structures, e.g., structures with small thickness such as beam, plate, shell etc., has received considerable attention in the literature. This instability phenomenon, physically termed "buckling" and mathematically termed "bifurcation", is still not fully understood. Although the first instabiliy studies in solids go back to Euler (so-called Euler's beam theory), the proper mathematical foundation for the theory of structural instability as a bifurcation problem is a much later achievement and it is essentially due to the works of Koiter (1945) for the elastic case and Hill(1957, 1958) for the more general case of rate independent solids. Even though vast efforts have been made to solve the structural instability problem since the beginning of this century(Lorenz, 1908; Donnell, 1934), the results were not so much satisfactory(Timoshenko, 1961; Hutchinson, 1974). Hence, the structural instability problem of even simple beam is still now under investigation(Pak, 1995).

So far, one of the controversial structural instability problems is the axially compressed circular cylindrical shell buckling. Since the earlier work by Lorenz(1908), many engineers have tried to obtain their theoretical results on the axially compressed cylindrical shell buckling(Donnell, 1934; Von Kārmān et al., 1941; Donnell et al., 1950; Yoshimura, 1955; Hoff et al., 1965; Lee, 1966; Hoff, 1966; Tennyson, 1969), but their results were not so much satisfactory comparing with some experimental works(Evensen, 1964; Almroth et al., 1964; Horton et al., 1965). The discrepancy between the theoretical results and the experimental results was considered to be due to imperfections(unavoidable deviations from the exact shape) (Von Kārmān, 1941; Donnell et al., 1950) and edge conditions(Nachbar et al., 1962; Almroth, 1966). Reviewing their research results show that large number of buckling modes of waveform pattern (e.g., diamond type pattern (Horton et al., 1965) or Yoshimura's triangle type pattern(Yoshimura, 1955)) may occur for circular cylindrical shell thin enough to buckle and not too long to buckle as Euler columns under some axial compressive loading. However,

* Corresponding Author,
E-mail : yjkwon@wow.hongik.ac.kr
TEL : +82-415-860-2484 ; **FAX** : +82-415-866-6598
Department of Mechanical Design and Production Engineering, Hongik University, Chochiwon-eup, Yeonki-gun, Chungnam 339-800, Korea (Manuscript Received June 11, 1999 ; Revised September 13, 1999)

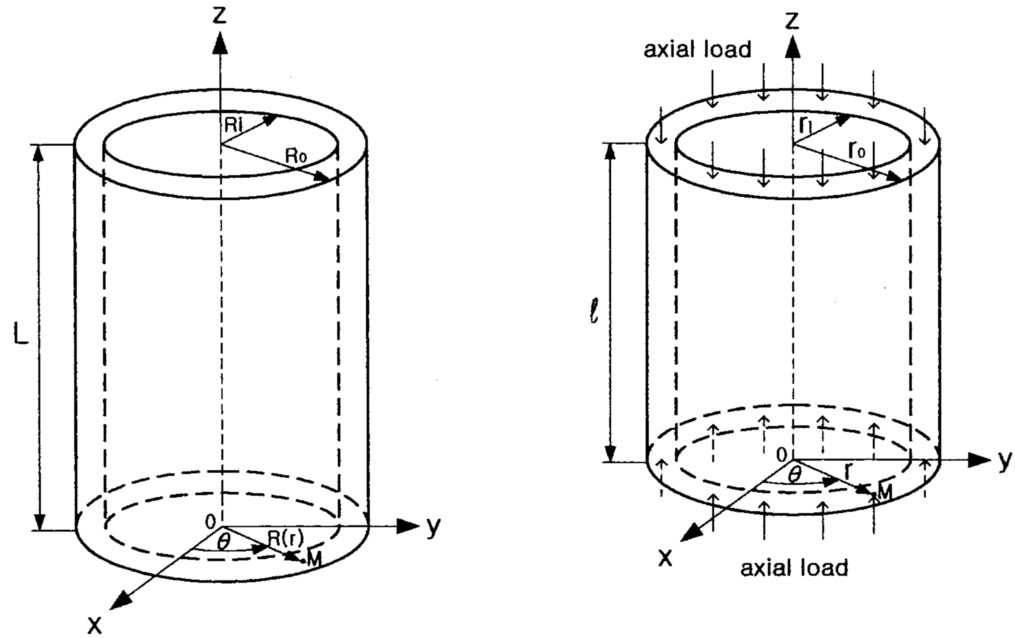

Fig. 1 Schematic diagram of the axially compressed cylindrical shell and sign conventions

clear theoretical explanations on these phenomenon were not given.

Practically the cylindrical shell has been one of the most commonly used elements in modern structures. Its importance and its simple shape have stimulated considerable interest. This interest has led to advances in the theory of thin shells. One of practical applications of thin shells is the application associated with the sheet metal forming problem (Kim et al., 1999). For the purpose of this practical application, some remarkable works to propose a general consistent methodology for the analysis of bifurcation instabilities in solid shells of arbitrary thickness have been done by Triantafyllidis and Kwon (1987) and Kwon (1992). In place of the classical approach, in which a two dimensional nonlinear theory (derived from the three dimensional governing equations of the solid) is linearized about the critical load, the general stability theory by Triantafyllidis and Kwon (1987) and Kwon (1992) starts from the bifurcation equation of the three dimensional solid (which have been obtained by linearization about the critical load of the same three dimensional governing equations for the nonlinear solid equation) and subsequently takes the limits as the structure thickness h tends to zero, following a multiple scales asymptotic technique.

In this study the so-called controversial structural stability problem, i.e., the axially compressed circular cylindrical shell buckling problem, is treated, adopting the aforementioned general stability theory by Triantafyllidis and Kwon (1987) and Kwon (1992). For this analysis, it is assumed that the compressible isotropic hypoelastic cylindrical shell is compressed axially, and then the forthcoming bifurcation instability phenomenon is investigated in the limit of zero thickness. The prebuckling state is solved in an asymptotic manner in the sense that the full analytical primary solution is not required for the asymptotic instability analysis. And the imperfections effect on the buckling is not considered in this analysis.

2. Primary State

In this study we treat the case where the top and bottom surfaces of circular cylidrical shell are lubricated and so they move freely. This is possible through the volume controlled experiment and many waveform bifurcation modes are found in this circumstance (Horton et al., 1965). Thus, the possible prebuckling state is just a cylindrical one as depicted in Fig. 1. Assuming that the material point doesn't rotate during the deformation the material point has the position vectors

$$\boldsymbol{p}=r(\cos\theta\boldsymbol{i}+\sin\theta\boldsymbol{j})+z\boldsymbol{k}, \text{ currently} \quad (1)$$

and

$$\boldsymbol{P}=R(r)(\cos\theta\boldsymbol{i}+\sin\theta\boldsymbol{j})+\frac{z}{\lambda_2}\boldsymbol{k}, \text{ intially} \quad (2)$$

where r and l are the current deformed radius and length and R and L are the initial undeformed radius and length of the cylindrical shell, and $\lambda_2=l/L$ is the axial stretch ratio. Since we assumed that the material point doesn't rotate, the stress state is principal. That is, any shear stress doesn't evolute.

Now, assuming that the material is a compressible isotropic material, we have the following constitutive equation

$$\tau_i=\lambda_i\frac{\partial W}{\partial\lambda_i}(!) \quad (3)$$

where "(!)" means "no sum", τ_i are principal Kirchhoff stresses, λ_i are principal stretch ratios and $W=W(\mathrm{I}_c, \mathrm{II}_c, \mathrm{III}_c)=W(\lambda_1, \lambda_2, \lambda_3)$ is a strain energy density function with

$$\begin{aligned}\mathrm{I}_c&=\lambda_1^2+\lambda_2^2+\lambda_3^2\\ \mathrm{II}_c&=\lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2\\ \mathrm{III}_c&=\lambda_1^2\lambda_2^2\lambda_3^2\end{aligned} \quad (4)$$

Kinematics

For this special case of axially compressed circular cylindrical shell, the principal stretch ratios are

$$\lambda_1=\frac{r}{R},\quad \lambda_2=\frac{l}{L},\quad \lambda_3=\frac{\partial r}{\partial R} \quad (5)$$

Power-law type material

Here, we will consider a simple material whose uniaxial stress-strain behavior is a piecewise power-law

$$\frac{\varepsilon}{\varepsilon_y}=\frac{\tau}{\tau_y} \text{ for } \tau\leq\tau_y,\ \frac{\varepsilon}{\varepsilon_y}=\left(\frac{\tau}{\tau_y}\right)^m \text{ for } \tau>\tau_y \quad (6)$$

where $\varepsilon_y=\tau_y/E$ is the initial "yield strain", τ_y is the initial "yield stress", and m is the hardening exponent. A strain energy density function W which describes such a material is

$$W(\lambda_1, \lambda_2, \lambda_3)=E\ \varepsilon_y^2\left[\frac{\chi}{\chi+1}\left(\frac{\tau_e}{\tau_y}\right)^{\chi+1}-\frac{1-2\nu}{6}\left(\frac{\tau_e}{\tau_y}\right)^2\right]+\frac{E}{6(1-2\nu)}(\varepsilon_1+\varepsilon_2+\varepsilon_3)^2+C \quad (7)$$

in which the equivalent Kirchhoff stress τ_e is related to the equivalent logarithmic strain ε_e by the relation

$$\left(\frac{\varepsilon_e}{\varepsilon_y}\right)=\left(\frac{\tau_e}{\tau_y}\right)^\chi-\frac{1-2\nu}{3}\left(\frac{\tau_e}{\tau_y}\right) \quad (8)$$

where $\chi=1$ if $\varepsilon_e\leq2(1+\nu)\varepsilon_y/3$ and $\chi=\mathrm{m}$ if $\varepsilon_e>2(1+\nu)\varepsilon_y/3$. Also, E is the Young's modulus, ν is the Poisson's ratio and the constant C which depends on E, ν, m and ε_y is constructed so that it assures the continuity of W at $\varepsilon_e=2(1+\nu)\varepsilon_y/3$.

Here, in the hope that the bifurcation instablity phenomenon will occur in the elastic range and for the simplicity, we treat only the case of m=1. That is, when m=1, we have

$$\tau_i=\frac{E}{1+\nu}\varepsilon_i+\frac{E\nu}{(1-2\nu)(1+\nu)}(\varepsilon_1+\varepsilon_2+\varepsilon_3) \quad (9)$$

where $\varepsilon_i=\ln\lambda_i$. And the current Cauchy stress, when m=1,

$$\sigma_i=-\frac{1}{\lambda_1\lambda_2\lambda_3}\cdot\frac{E}{1+\nu}\left(\ln\frac{1}{\lambda_i}+\frac{\nu}{1-2\nu}\ln\frac{1}{\lambda_1\lambda_2\lambda_3}\right) \quad (10)$$

Equilibrium Equation

The only nontrivial equilibrium equation in the cylindrical coordinates, using the physical components

$$r\frac{\partial\sigma_3}{\partial r}+\sigma_3-\sigma_1=0 \text{ with } \sigma_3=0 \text{ at } r=r_i, r_o \quad (11)$$

where r_i, r_o are current inner, outer radius respectively, or using the above expressions Eq. (5) and Eq. (10), the above Eq. (11) becomes

$$r\frac{\partial}{\partial r}\left[\frac{\partial R}{\partial r}\frac{R}{r}\frac{1}{\lambda_2}\left\{\ln\frac{\partial R}{\partial r}+\frac{\nu}{1-2\nu}\ln\left(\frac{\partial R}{\partial r}\frac{R}{r}\frac{1}{\lambda_2}\right)\right\}\right]+\frac{\partial R}{\partial r}\frac{R}{r}\frac{1}{\lambda_2}\left(\ln\frac{\partial R}{\partial r}\frac{r}{R}\right)=0 \quad (12)$$

This is a nonlinear differential equation with high degree for R(r) whose analytical solution is very difficult to treat. The possible analytical approach is through the asymptotic method.

Nondimensionalization

To obtain a proper dimensionless equation for the asymptotic analysis, we adopt a change of variable as

$$\zeta=r-\left(r_i+\frac{h}{2}\right) \text{ or } r=\zeta+r_i+\frac{h}{2} \text{ and } \partial r=\partial\zeta \quad (13)$$

where h is a constant current thickness. Further, we normalize this coordinate as

$$\zeta^*=\zeta/r_i \quad \text{or} \quad \zeta=r_i\zeta^* \quad (14)$$

then $$r=r_i\left(\zeta^*+1+\frac{\varepsilon}{2}\right) \quad (15)$$

with $$\varepsilon\equiv\frac{h}{r_i}\text{(thickness parameter)} \quad (16)$$

And, we define again,

$$\xi=\zeta^*/\varepsilon \text{ or } \zeta^*=\varepsilon\xi, \quad -\frac{1}{2}\leq\xi\leq\frac{1}{2}$$

then

$$r=r_i\left\{\varepsilon\left(\xi+\frac{1}{2}\right)+1\right\} \text{ and } \partial r=r_i\varepsilon\partial\xi \quad (17)$$

Then, the governing equation Eq. (11) becomes

$$\frac{\partial\sigma_3}{\partial\xi}+\varepsilon\left\{\left(\xi+\frac{1}{2}\right)\frac{\partial\sigma_3}{\partial\xi}+\sigma_3-\sigma_1\right\}=0 \quad (18)$$

This is the desired dimensionless equation for the asymptotic analysis. Together with this equation, we have the following traction free boundary condition

$$\sigma_3\left(\pm\frac{1}{2}\right)=0 \quad (19)$$

and the principal stretch ratios are

$$\lambda_1=\frac{r}{R}=\frac{\varepsilon\left(\xi+\frac{1}{2}\right)+1}{R^*}$$
$$\lambda_2=\frac{l}{L}\equiv\frac{1}{\lambda},\ \lambda\equiv\frac{L}{l}\text{(load parameter)} \quad (20)$$
$$\lambda_3=\frac{\partial r}{\partial R}=\frac{1}{\partial R^*/\varepsilon\partial\xi}$$

where $R^*\equiv R/r_i$. Note that λ is chosen as the load parameter here.

Asymptotic solution

Now the primary solution $R^*=R/r_i$ is a function of ξ and ε, i.e.,

$$R^*=R^*(\xi,\ \varepsilon)$$

and the load parameter λ is a function of ε, i.e.,

$$\lambda=\lambda(\varepsilon)$$

Noting that ε is small, we may have the power series expansions of R^* and λ as follows.

$$R^*(\xi,\ \varepsilon)=\overset{0}{R}+\overset{1}{R}\varepsilon+\overset{2}{R}\varepsilon^2+\overset{3}{R}\varepsilon^3+\cdots \quad (21)$$

where $\overset{0}{R}$, $\overset{1}{R}$, $\overset{2}{R}$, etc. are defined as the lowest order term, the first order term, the second order term, etc., respectively in power series expansion of $R^*(\xi,\ \varepsilon)$ in terms of ε, i.e., $\overset{0}{R}\equiv\lim_{\varepsilon\to0}R^*(\xi,\ \varepsilon)$.

And

$$\lambda(\varepsilon)=\overset{0}{\lambda}+\overset{1}{\lambda}\varepsilon+\overset{2}{\lambda}\varepsilon^2+\overset{3}{\lambda}\varepsilon^3+\cdots \quad (22)$$

where $\overset{0}{\lambda}$, $\overset{1}{\lambda}$, $\overset{2}{\lambda}$, etc. are defined as the lowest order term, the first order term, the second order term, etc., respectively in power series expansion of $\lambda(\varepsilon)$ in terms of ε, i.e., $\overset{0}{\lambda}\equiv\lim_{\varepsilon\to0}\lambda(\varepsilon)$.

Also, in the same manner we have the expansions of λ_i, σ_i, i.e.,

$$\begin{aligned}\lambda_1&=\overset{0}{\lambda}_1+\overset{1}{\lambda}_1\varepsilon+\overset{2}{\lambda}_1\varepsilon^2+\overset{3}{\lambda}_1\varepsilon^3+\cdots\\ \lambda_2&=\overset{0}{\lambda}_2+\overset{1}{\lambda}_2\varepsilon+\overset{2}{\lambda}_2\varepsilon^2+\overset{3}{\lambda}_2\varepsilon^3+\cdots\\ \lambda_3&=\overset{0}{\lambda}_3+\overset{1}{\lambda}_3\varepsilon+\overset{2}{\lambda}_3\varepsilon^2+\overset{3}{\lambda}_3\varepsilon^3+\cdots\end{aligned} \quad (23)$$

and

$$\begin{aligned}\sigma_1&=\overset{0}{\sigma}_1+\overset{1}{\sigma}_1\varepsilon+\overset{2}{\sigma}_1\varepsilon^2+\overset{3}{\sigma}_1\varepsilon^3+\cdots\\ \sigma_2&=\overset{0}{\sigma}_2+\overset{1}{\sigma}_2\varepsilon+\overset{2}{\sigma}_2\varepsilon^2+\overset{3}{\sigma}_2\varepsilon^3+\cdots\\ \sigma_3&=\overset{0}{\sigma}_3+\overset{1}{\sigma}_3\varepsilon+\overset{2}{\sigma}_3\varepsilon^2+\overset{3}{\sigma}_3\varepsilon^3+\cdots\end{aligned} \quad (24)$$

Inserting the above expansions Eqs. (21) ~ (24) into the governing equation Eq. (18) and after quite long algebra, we obtain the asymptotic primary solution as follows.

$$\overset{0}{\sigma}_1=\overset{0}{\sigma}_3=0 \quad (25)$$
$$\overset{0}{\sigma}_2=-E\,(\overset{0}{\lambda})^{\frac{1-\alpha}{1+\alpha}}\ln\overset{0}{\lambda}$$

$$\overset{1}{\sigma}_1=\overset{1}{\sigma}_3=0 \quad (26)$$
$$\overset{1}{\sigma}_2=-E\,(\overset{0}{R})^2\overset{0}{\lambda}\left(1+\frac{1-\alpha}{1+\alpha}\ln\overset{0}{\lambda}\right)\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}$$

$$\overset{2}{\sigma}_1=\overset{2}{\sigma}_3=0 \quad (27)$$
$$\overset{2}{\sigma}_2=-E\,(\overset{0}{R})^2\overset{0}{\lambda}\left[\frac{1-3\alpha}{2(1+\alpha)}\left(\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}\right)^2+\frac{\overset{2}{\lambda}}{\overset{0}{\lambda}}+\frac{1-\alpha}{1+\alpha}\left\{-\frac{\alpha}{1+\alpha}\left(\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}\right)^2+\frac{\overset{2}{\lambda}}{\overset{0}{\lambda}}\right\}\ln\overset{0}{\lambda}\right]$$

etc., where $\alpha \equiv \frac{\nu}{1-\nu}$

And

$$\overset{0}{R} = (\overset{0}{\lambda})^{-\frac{\alpha}{1+\alpha}}$$

$$\overset{1}{R} = \left(\xi + \frac{1}{2} - \frac{\alpha}{1+\alpha}\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}\right)\overset{0}{R}$$

$$\overset{2}{R} = -\frac{\alpha}{1+\alpha}\left\{\left(\xi+\frac{1}{2}\right)\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}} - \frac{1+2\alpha}{2(1+\alpha)}\left(\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}\right)^2 + \frac{\overset{2}{\lambda}}{\overset{0}{\lambda}}\right)\overset{0}{R} \quad (28)$$

$$\overset{3}{R} = \frac{\alpha}{1+\alpha}\left\{\frac{1+2\alpha}{2(1+\alpha)}\left(\frac{\overset{1}{\lambda}}{\overset{0}{\lambda}}\right)^2 - \frac{\overset{2}{\lambda}}{\overset{0}{\lambda}}\right\}\overset{0}{R}\xi + C_3$$

where C_3 is the unknown constant to be determined.

3. Constitutive Law

Next, attention is focused on the choice of a proper constitutive law, for a compressible elastic-plastic solid the stress increments are related to the strain increments by a constitutive equation of the form

$$\dot{\tau}^{ij} = L^{ijkl}\dot{E}_{kl} \quad (29)$$

The instaneous incremental moduli L^{ijkl} have the symmetries $L^{ijkl} = L^{klij} = L^{lkij} = L^{klji}$ and have two branches, the one corresponding to plastic loading, and the other to elastic unloading. There are several constitutive models. Here, specially, we are interested in the compressible Stören-Rice hypoelastic material. The incremental moduli for this finite strain J_2-deformation theory is, especially when m=1 (no hardening, elastic case),

$$L^{ijkl} = \frac{E}{1+\nu}\left\{\frac{1}{2}(g^{ik}g^{jl}+g^{jk}g^{il}) + \frac{\nu}{1-2\nu}g^{ij}g^{kl}\right\} - \frac{1}{2}(g^{ik}\tau^{jl}+g^{jl}\tau^{ik}+g^{jk}\tau^{il}+g^{il}\tau^{jk}) \quad (30)$$

Now, for this special cylindrical system, we have

$$L^{1111} = \left\{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} - 2\tau_1\right\}\frac{1}{r^4}$$

$$L^{2222} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} - 2\tau_2$$

$$L^{3333} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} - 2\tau_3$$

$$L^{1122} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{r^2} \quad (31)$$

$$L^{1133} = \frac{E\nu}{(1+\nu)(1-2\nu)}\frac{1}{r^2}$$

$$L^{2233} = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$L^{1212} = \frac{1}{2}\left\{\frac{E}{1+\nu} - (\tau_1+\tau_2)\right\}\frac{1}{r^2}$$

$$L^{1313} = \frac{1}{2}\left\{\frac{E}{1+\nu} - (\tau_3+\tau_1)\right\}\frac{1}{r^2}$$

$$L^{2323} = \frac{1}{2}\left\{\frac{E}{1+\nu} - (\tau_2+\tau_3)\right\}$$

and others are zero, note that $L^{ijkl} = L^{klij} = L^{jikl} = L^{ijlk}$. Or the physical components defined as

$$L^{<ijkl>} = L^{ijkl}\sqrt{g_{ii}g_{jj}g_{kk}g_{ll}} \text{ (!) ((!) means "no sum")} \quad (32)$$

are, noting $g_{11} = r^2$, $g_{22} = g_{33} = 1$

$$L^{<1111>} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} - 2\tau_1$$

$$L^{<2222>} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} - 2\tau_2$$

$$L^{<3333>} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} - 2\tau_3$$

$$L^{<1122>} = L^{<1133>} = L^{<2233>} = \cdots = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (33)$$

$$L^{<1212>} = \frac{1}{2}\left\{\frac{E}{1+\nu} - (\tau_1+\tau_2)\right\}$$

$$L^{<1313>} = \frac{1}{2}\left\{\frac{E}{1+\nu} - (\tau_3+\tau_1)\right\}$$

$$L^{<2323>} = \frac{1}{2}\left\{\frac{E}{1+\nu} - (\tau_2+\tau_3)\right\}$$

and others are zero. And so, the power series expansion of the primary state can be formed as

$$M^{ijkl} \equiv L^{ijkl} + \tau^{ik}g^{jl} \equiv \overset{0}{M}{}^{ijkl} + \overset{1}{M}{}^{ijkl}\varepsilon + \overset{2}{M}{}^{ijkl}\varepsilon^2 + \cdots \quad (34)$$

with same definitions as in Eqs. (21) ~ (22)

4. Bifurcation Analysis

Using the updated Lagrangian formulation, we have the general asymptotic bifurcation solution for the asymptotic expansion of the bifurcation mode and see Kwon (1992) for more details, i.e.,

$$\Delta u_l = \overset{0}{u}_l + \overset{1}{u}_l\varepsilon + \overset{2}{u}_l\varepsilon^2 + \cdots \quad (35)$$

Or

$$v_\theta \equiv \Delta u_1 = \overset{0}{v}_\theta + \overset{1}{v}_\theta\varepsilon + \overset{2}{v}_\theta\varepsilon^2 + \cdots$$

$$v_z \equiv \Delta u_2 = \overset{0}{v}_z + \overset{1}{v}_z\varepsilon + \overset{2}{v}_z\varepsilon^2 + \cdots$$

$$v_r \equiv \Delta u_3 = \overset{0}{v}_r + \overset{1}{v}_r\varepsilon + \overset{2}{v}_r\varepsilon^2 + \cdots$$

where we use the same definition as in Eq. (21). For the constant current thickness, i.e. $h=h_o$ or $g(\theta^\alpha)=1$ (Kwon, 1992), the solutions are

The lowest order solution

$$(\overset{0}{G}{}^{\alpha j\beta l}\,\overset{0}{u}_{l,\beta})_{,\underset{\alpha}{0}}=0 \tag{36}$$

with

$$\overset{0}{G}{}^{\alpha j\beta l}=\overset{0}{M}{}^{\alpha j\beta l}-\overset{0}{M}{}^{\alpha j3m}(\overset{0}{M}{}^{\beta n3m})^{-1}\overset{0}{M}{}^{\beta n\beta l}$$

where the notation $(\,)_{,\underset{\alpha}{0}}$ is defined as $T^{\alpha j}{}_{,\underset{\alpha}{0}}\equiv\partial T^{\alpha j}/\partial\theta^\alpha+\overset{0}{\Gamma}{}^{\alpha}_{m\alpha}T^{mj}+\overset{0}{\Gamma}{}^{j}_{m\alpha}T^{m\alpha}$ and $\Gamma^i_{jk}\equiv\overset{0}{\Gamma}{}^i_{jk}+\overset{1}{\Gamma}{}^i_{jk}\varepsilon+\overset{2}{\Gamma}{}^i_{jk}\varepsilon^2+\cdots$ is Christoffel symbols (Kwon, 1992) with same definitions as in Eq. (21).

The first order solution

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\Big\{(\overset{0}{G}{}^{\alpha j\beta l}\overset{1}{u}_{l,\beta}+\overset{0}{G}{}^{\alpha j\beta l}\overset{0}{u}_{l,\beta})_{,\underset{\alpha}{0}}+\overset{1}{\Gamma}{}^{\alpha}_{m\alpha}\overset{0}{G}{}^{mj\beta l}\overset{0}{u}_{l,\beta}+\overset{1}{\Gamma}{}^{j}_{m\alpha}\overset{0}{G}{}^{\alpha m\beta l}\overset{0}{u}_{l,\beta}\Big\}\partial\xi=0 \tag{37}$$

with

$$\overset{1}{G}{}^{\alpha j\beta l}=\overset{1}{M}{}^{\alpha j\beta l}-\overset{0}{M}{}^{\alpha j3k}(\overset{0}{M}{}^{\beta m3k})^{-1}\overset{1}{M}{}^{\beta m\beta l}-\overset{1}{M}{}^{\alpha j3k}(\overset{0}{M}{}^{\beta m3k})^{-1}\overset{0}{M}{}^{\beta m\beta l}+\overset{0}{M}{}^{\alpha j3k}(\overset{0}{M}{}^{\beta m3k})^{-1}\overset{1}{M}{}^{\beta m3p}(\overset{0}{M}{}^{\beta n3p})^{-1}\overset{0}{M}{}^{\beta n\beta l},\ \text{etc.}$$

Now, specially, we apply this general theory to our current cylindrical shell bifurcation problem, noting that the characteristic length of the current configurartion $\mathcal{L}\equiv r_i$.

Boundary Condition

For lubricated free ends, we should have two boundary conditions: the one is the constant vertical displacement condition (essential boundary condition) (volume controlled experiment), the other is the no shear traction condition (natural boundary condition). From these boundary conditions we may obtain the boundary conditions for the asymptotic expansion of the bifurcation mode Δu_l successively. For example, for the lowest order mode, we have, at $x(\equiv z/r_i)=0$, x_o $(\equiv l/r_i)$

$$\overset{0}{v}_z=0 \quad \text{(essential B.C.)}$$

and (38)

$$\frac{\partial\overset{0}{v}_\theta}{\partial x}=0,\quad\frac{\partial\overset{0}{v}_r}{\partial x}=0 \quad \text{(natural B.C.)}$$

Solution

Now in the volume controlled experiment, e.g., under the lubricated free ends condition, usually the waveform bifurcation modes in the angular direction are found (Horton et al., 1965). Hence, the mode is periodic in the angular direction without loss of generality, and so the solution should be the following form

$$\begin{aligned}\overset{0}{v}_\theta&=\overset{0}{u}_n\sin n\theta\\ \overset{0}{v}_z&=\overset{0}{v}_n\cos n\theta\\ \overset{0}{v}_r&=\overset{0}{w}_n\cos n\theta\end{aligned} \tag{39}$$

where $\overset{0}{u}_n$, $\overset{0}{v}_n$, $\overset{0}{w}_n$ are functions of x and n is the angular wavenumber and integer. Inserting these mode assumption into the differential equation Eq. (36), we will obtain the following partial differential equations (coupled)

$$\begin{aligned}&\frac{n^2}{1-\nu^2}\overset{0}{u}_n-\frac{1}{2}\Big(\frac{1}{1+\nu}-\ln\overset{0}{\lambda}\Big)\frac{\partial^2\overset{0}{u}_n}{\partial x^2}+\frac{n}{2}\Big(\frac{1}{1-\nu}+\ln\overset{0}{\lambda}\Big)\frac{\partial\overset{0}{v}_n}{\partial x}+\frac{n}{1-\nu^2}\overset{0}{w}_n=0,\\ &\frac{n}{2}\Big(\frac{1}{1-\nu}+\ln\overset{0}{\lambda}\Big)\frac{\partial\overset{0}{u}_n}{\partial x}-\frac{n^2}{2}\Big(\frac{1}{1+\nu}+\ln\overset{0}{\lambda}\Big)\overset{0}{v}_n+\Big(\frac{1}{1-\nu^2}+\ln\overset{0}{\lambda}\Big)\frac{\partial^2\overset{0}{v}_n}{\partial x^2}+\frac{\nu}{1-\nu^2}\frac{\partial\overset{0}{w}_n}{\partial x}=0,\\ &\frac{n}{1-\nu^2}\overset{0}{u}_n+\frac{\nu}{1-\nu^2}\frac{\partial\overset{0}{v}_n}{\partial x}+\frac{1}{1-\nu^2}\overset{0}{w}_n+(\ln\overset{0}{\lambda})\frac{\partial^2\overset{0}{w}_n}{\partial x^2}=0\end{aligned} \tag{40}$$

And the boundary conditions are, using the assumed modes

$$\overset{0}{v}_n=0,\ \frac{\partial\overset{0}{u}_n}{\partial x}=\frac{\partial\overset{0}{w}_n}{\partial x}=0 \text{ at } x=0,\ x_0(=l/r_i) \tag{41}$$

Now, we can solve the above coupled partial differential equation in a straightforward manner. The solution should be surely of the form

$$\begin{aligned}\overset{0}{u}_n&\sim\overset{0}{u}e^{rx}\\ \overset{0}{v}_n&\sim\overset{0}{v}e^{rx}\\ \overset{0}{w}_n&\sim\overset{0}{w}e^{rx}\end{aligned} \tag{42}$$

Inserting this solution form into the equation Eq. (40), we may obtain the characteristic equation of r for the nontrivial solution $(\overset{0}{u},\overset{0}{v},\overset{0}{w})$ as

$$r^2=0 \text{ and } Ar^4+Br^2+C=0 \tag{43}$$

where A, B, C are functions of $\overset{0}{\lambda}$, n and ν. Hence, we have six roots of r (r=0 of them is double root), and the corresponding solution is

$$\begin{aligned}\overset{0}{u}_n&=\overset{0}{u}_0(A_0+B_0x)+\overset{0}{u}_1(A_1\cosh\sqrt{t_1}x+B_1\sinh\sqrt{t_1}x)\\&\quad+\overset{0}{u}_2(A_2\cosh\sqrt{t_2}x+B_2\sinh\sqrt{t_2}x)\\ \overset{0}{v}_n&=\overset{0}{v}_0(A_0+B_0x)+\overset{0}{v}_1(A_1\sinh\sqrt{t_1}x+B_1\cosh\sqrt{t_1}x)\\&\quad+\overset{0}{v}_2(A_2\sinh\sqrt{t_2}x+B_2\cosh\sqrt{t_2}x)\\ \overset{0}{w}_n&=A_0+B_0x+A_1\cosh\sqrt{t_1}x+B_1\sinh\sqrt{t_1}x\\&\quad+A_2\cosh\sqrt{t_2}x+B_2\sinh\sqrt{t_2}x\end{aligned} \tag{44}$$

where t_1 and t_2 are two different solutions of $At^2+Bt+C=0(t\equiv r^2)$. Applying the boundary condition Eq. (41), we may obtain the values of six unknown constants (A_0, B_0, A_1, B_1, A_2, B_2) as $A_0=B_0=B_1=A_2=B_2=0$ and $A_1\neq 0$(arbitrary). Also, we should have the characteristic equation

$$\sinh\sqrt{t_1}x_0=0 \quad \text{or} \quad t_1=-\omega^2 \tag{45}$$

where $\omega\equiv p\pi r_i/l$ (p is the axial wavenumber and integer). And the corresponding solution is

$$\begin{aligned}\overset{0}{u}_n&=\overset{0}{u}_1A_1\cos\omega x\\ \overset{0}{v}_n&=\overset{0}{v}_1\, i\, A_1\sin\omega x\\ \overset{0}{w}_n&=A_1\cos\omega x\end{aligned} \tag{46}$$

where $\overset{0}{u}_1$, $\overset{0}{v}_1$ are functions of n, ω, $\overset{0}{\lambda}$ and ν, and A_1 is arbitray nonzero constant. Hence, the complete solution may be written as

$$\begin{aligned}\overset{0}{v}_\theta&=\overset{0}{u}_1\sin n\theta\cos\omega x\\ \overset{0}{v}_z&=\overset{0}{v}_1\, i\cos n\theta\sin\omega x\\ \overset{0}{v}_r&=\cos n\theta\cos\omega x\end{aligned} \tag{47}$$

Since these solutions are valid for all n and p, the linear combination of these solutions for n and p will be the general solution for the bifurcation modes in the limit of zero thickness of the shell. And these are the exactly the waveform type bifurcation modes, which coincides with the classical experimental results (Horton, 1965), e.g., periodic diamond type or Yoshimura's periodic triangle type bifurcation mode. Now, to obtain the characteristic equation for $\overset{0}{\lambda}$, we should note that $t_1=-w^2$ is the one of roots of the quadratic equation $At^2+Bt+C=0(t\equiv r^2)$. Hence, we get the following characterisitic equation for $\overset{0}{\lambda}$

$$\begin{aligned}&\omega^2(n^2+\omega^2)(\ln\overset{0}{\lambda})^3+\left\{\frac{\nu\omega^4}{1-\nu^2}-\frac{n^2\omega^2}{1+\nu}-\frac{\omega^2}{1-\nu^2}\right.\\&\left.-\frac{n^2(n^2+1)}{1-\nu^2}\right\}(\ln\overset{0}{\lambda})^2-\left\{\frac{\omega^4}{(1+\nu)(1-\nu^2)}\right.\\&+\frac{2n^2\omega^2}{(1+\nu)(1-\nu^2)}+\frac{\nu\omega^2}{(1+\nu)(1-\nu^2)}\\&\left.+\frac{n^2(n^2+1)}{(1-\nu^2)(1+\nu)}\right\}(\ln\overset{0}{\lambda})+\frac{\omega^2}{(1+\nu)(1-\nu^2)}=0\end{aligned} \tag{48}$$

From the above equation, we have the critical load parameter as

$$\begin{aligned}&\overset{0}{\lambda}_{cr}=(\overset{0}{\lambda})_{\min}=1 \text{ at } \omega^2=0 \text{ or } \frac{1}{\omega^2}=0\\&(i.e., \ \omega=0 \text{ or } \frac{1}{\omega}=0)\end{aligned} \tag{49}$$

And the corresponding critical axial stress is zero, i.e.,

$$(\overset{0}{\sigma}_2)_{cr}=-E(\overset{0}{\lambda}_{cr})^{\frac{1-\alpha}{1+\alpha}}\ln\overset{0}{\lambda}_{cr}=0 \tag{50}$$

Hence, in the limit of zero thickness of the circular cylindrical shell, as soon as the axial compressive load is exerted on the shell's edge surfaces, the shell tends to buckle. And the corresponding axial wavenumber parameter ω is zero or a very large number. The case in which ω is zero is the long cylindrical shell buckling case (i. e., $l\rightarrow$very large). In such case the shell will not buckle as bifurcation modes of waveform pattern (i.e., $\omega=0$), but it will buckle like Euler column. However, the shell with finite length tends to buckle as bifurcation modes of waveform pattern ($1/\omega=0$ or $\omega\rightarrow$very large value). The load parameter $\overset{0}{\lambda}$ decreases as the angular wavenumber n increases. Hence, in the limit of zero thickness of the shell with finite length, the shell tends to buckle into many waveform bifurcation modes (in both angular and axial directions). This agrees very well with the classical experimental observations (Horton et al., 1965). And the result on the critical axial stress in the limit of zero thickness also coincides with the classical theory (Hoff, 1966).

4. Conclusions

In this work, we investigated the bifurcation modes in the limit of zero thickness of the axially compressed circular cylindrical shell, adopting the general stability theory developed by Triantafyllidis and Kwon (1987) and Kwon (1992). Solving the lowest order solution of the general stability theory, the bifurcation modes ($\overset{0}{v}_\theta$, $\overset{0}{v}_z$, $\overset{0}{v}_r$) in the limit of zero thickness of the shell are obtained as

$$\begin{aligned} \overset{0}{v}_\theta &= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{np} \sin n\theta \cos\frac{p\pi z}{l} \\ \overset{0}{v}_z &= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} B_{np} \cos n\theta \sin\frac{p\pi z}{l} \\ \overset{0}{v}_r &= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty} C_{np} \cos n\theta \cos\frac{p\pi z}{l} \end{aligned} \quad (51)$$

where n and p are wavenumbers in angular and axial direction respectively, and A_{np}, B_{np} are functions of n, ω, ν and $C_{np} \equiv 1$. At the critical condition, the axial wavenumber parameter ω ($= p\pi r_i/l$) is zero or a very large number ($\omega=0$ or $1/\omega=0$), and the corresponding critical load parameter $\overset{0}{\lambda}_{cr}$ is one (i.e., $\overset{0}{\lambda}_{cr} = (\overset{0}{\lambda})_{\min} = 1$ or $(\overset{0}{\sigma_2})_{cr} = 0$). This result exactly coincides with the classical theory and observations. In the classical theory the critical axial load for the axially compressed cylindrical shell (Hoff, 1966) is obtained as

$$(\sigma_2)_{cr} = [3(1-\nu^2)]^{-\frac{1}{2}} E(h/a) \cong 0.6E(h/a) \quad (52)$$

Hence,

$$\lim_{h \to 0} (\sigma_2)_{cr} = 0 \quad (53)$$

which exactly agrees with our result Eq. (50). Many experimental results in the classical research (Evensen, 1964; Almroth et al., 1964; Horton et al., 1965) show that large number of bifurcation modes are observed in the form of wave for very thin shell, which agrees with the result of Eq. (49). Another interesting result is that ω is zero for long cylinder. Hence, the long circular cylindrical shell will buckle as Euler colunm as soon as the load is exerted.

In this paper, only the lowest order solution which is very satisfactory comparing with classical result is obtained. However, even though it is not calculated here, the higher order solutions may be obtained as many as required for accuracy using the methodology adopted here, but this calculation is not possible in the classical two dimensional nonlinear instability theory as shown in Eq. (52). Hence, the methodology used in this paper is very powerful, and the computaions performed in this paper can be extended to obtain the higher order solution in further study.

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