두개의 동일한 탐조등으로 볼록다각형을 비추는 알고리즘

(An Algorithm for Illuminating a Convex Polygon with Two Equal Floodlights)

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요 약 G를 꼭지점들이 일반 위치에 있는 n개의 꼭지점을 갖는 볼록다각형이라 하자. 본 논문에서는 두 개의 탐조등을 G의 경계선에 배치하여 G의 내부를 모두 비추는 알고리즘을 개발하는데, 두 탐조등의 최대각을 최소화하는 것이 목적이다. 확장된 real-RAM상에서 $O(n^2)$ 시간 알고리즘을 제시한다.

Abstract Let G be an n-vertex convex polygon whose vertices are in general position. We consider an illumination problem of locating two floodlights on the boundary of G, which illuminates G and whose larger angle is as small as possible. We present an $O(n^2)$ time algorithm under the extended real-RAM model.

1. Introduction

Let G be a convex polygon of n vertices in general positions, meaning no four or more vertices of G are co-circular. A *floodlight* is a light source illuminating the area within a cone with an angle. In this paper, we consider a floodlight illumination problem which is to illuminate G by two floodlights on the boundary of G so that the maximum of their angles is minimized (see Figure 1).

Estivill-Castro and Urrutia [1] presented an $O(n^2)$-time algorithm to locate two floodlights on the boundary of G so that the sum of their angles is minimized. Floodlight illumination problems are motivated by a practical restriction that illumination devices, e.g., light sources and guards, cannot illuminate or search in all directions, simultaneously. Previous results related to floodlight illumination problems are well summarized in [2].

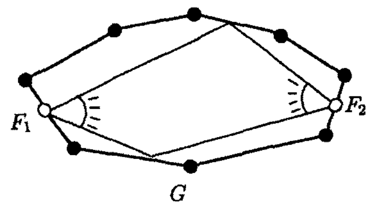

Fig. 1 Two-floodlight problem

In this paper, we consider a 2-floodlight illumination problem of locating two floodlights on the boundary of G so that the larger angle is minimized. We present an $O(n^2)$-time algorithm for the problem, which is the same time bound as that of an algorithm [1] minimizing the sum of angles. However, we do not know if our algorithm is optimal. As guessed in [1], we also suspect that our $O(n^2)$-time bound is optimal. The model of computation used throughout this paper is the extended real RAM (for details refer

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to [3]).

2. The Algorithm

Before describing the algorithm for the problem, we need to give some terminology and definitions. A floodlight F of angle α is a light source that illuminates a region within a cone of angle α. A pair of floodlights F_1 and F_2 is said to be an *illumination pair* of G if each of F_1 and F_2 has its apex on the boundary of G and the union of regions illuminated by them covers G. If F_1 and F_2 are an illumination pair such that the maximum of their angles is minimized, we call them an *optimal* illumination pair. As done in [1], we consider two kinds of illumination pairs: an *opposite* and a *dividing* illumination pair. We say F_1 and F_2 to be *opposite* if the intersection of the illuminated regions is a quadrilateral whose vertices are on the boundary of G; see Figure 2(a). We say F_1 and F_2 to be *dividing* if the interior of two regions illuminated by them are disjoint as shown in Figure 2(b). Then we can easily observe that an optimal illumination pair must be either opposite or dividing. Moreover, the following fact holds; its proof is very similar to that of [1].

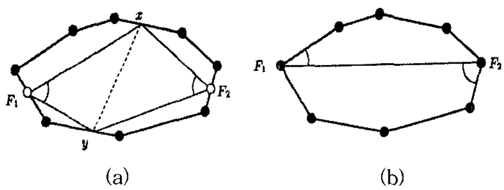

Fig. 2 (a) Opposite illumination pair (b) Dividing illumination pair

Lemma 1: Let F_1 and F_2 be an optimal illumination pair of G. Then the apexes of F_1 and F_2 must be located at vertices of G.

From Lemma 1, we know that if an optimal illumination pair of G is dividing, then the optimal dividing pair can be trivially computed in $O(n^2)$ time because the number of possible vertex pairs of G is $O(n^2)$. Thus, in the remainder of this section, we shall explain how to find an optimal opposite illumination pair. Let us now reduce the number of pairs of vertices of G that can be apexes of an optimal opposite illumination pair. We call such pairs *candidate pairs* of G. Estivill-Castro and Urrutia [1] showed the number of the candidate pairs is $O(n)$; though they considered the problem of minimizing the sum of the angles of two floodlights, we can apply the result here without changes.

Lemma 2: [1] The number of candidate pairs of G is $O(n)$ and the pairs can be found in $O(n)$ time.

By Lemma 2, we have only $O(n)$ candidates for an optimal opposite illumination pair. Given a candidate pair, we shall show how to solve in linear time a minimax problem with the restriction that two floodlights are at the vertices of the pair.

Let (p, q) be a candidate pair of G. Suppose that two floodlights F_1 and F_2 are an illumination pair and their apexes are at p and q, respectively. Without loss of generality, assume that p is to the left of q. Then the pair divides the boundary of G into two chains: an upper chain G^U and a lower chain G^L. Consider the subchain of G^U that are commonly illuminated by F_1 and F_2. The subchain can be assumed to consist of a sinlge point, x; otherwise the larger angle of F_1 and F_2 can be further reduced. Similarly, the commonly-illuminated boundary of G^L consists of a single point, y (see Figure 2(a)).

Now, we will find two points, $x \in G^U$ and $y \in G^L$, such that $\max(\angle xpy, \angle xqy)$ is minimized. We shall denote such points by x^* and y^*. If $\angle xpy > \angle xqy$, we can slide x on G^U or y on G^L to the right toward q until the angles are equal. Similarly, if $\angle xpy < \angle xqy$, we can slide x on G^U or y on G^L to the left toward p until the angles are equal. Thus $\max(\angle xpy, \angle xqy)$ is minimized only if two are equal. That is, $\angle x^*py^* = \angle x^*qy^*$. Let us define $A(p, q) = \angle x^*py^* = \angle x^*qy^*$. As it will be proved below, two points x^* and y^* can be found in linear time by the technique of computing row minima in a totally monotone matrix [4]. Since there are $O(n)$ candidate pairs by Lemma 2, we can find an optimal illumination pair in $O(n^2)$ time.

Let M denote an $n_1 \times n_2$ $(n_1 \le n_2)$ matrix whose entries are real numbers. The matrix M is *totally monotone* if $M(i,l)+M(j,k) \le M(i,k)+M(j,l)$ for any $1 \le i < j \le n_1$ and $1 \le k < l \le n_2$. If each entry in M can be evaluated in $f(n_1, n_2)$ time, then a minimum entry in each row of M can be computed in time $O(n_2 f(n_1, n_2))$. Note that the matrix is defined implicitly -- an entry is evaluated only when needed.

We order the edges in G^U and in G^L from p to q, and view each chain as a sequence of edges, $G^U=(d_1, d_2, \cdots, d_{n_1})$ and $G^L=(e_1, e_2, \cdots, e_{n_2})$. Note that $n_1+n_2=n$ and assume that $n_1 \le n_2$. Let us define

$$B(d_i, e_j) = \min_{x \in d_i, y \in e_j} \max(\angle xpy, \angle xqy).$$

$B(d_i, e_j)$ is the minimum when the floodlights are at p and q, and x and y are restricted on d_i and e_j, respectively. We define an $n_1 \times n_2$ matrix M whose entry $M(i,j)$ represents $B(d_i, e_j)$, $1 \le i \le n_1$ and $1 \le j \le n_2$. Clearly,

$$A(p,q) = \min_{i,j} B(d_i, e_j) = \min_{i,j} M(i,j).$$

If M is totally monotone, then we can compute a minimum in each row of M in $O(nf(n))$ time, where $f(n)$ is the evaluation time of a matrix entry. Taking the minimum among these minima, say $M(s,t)$, we have $x^* \in d_s$, $y^* \in e_t$, and $A(p,q)=M(s,t)$. The two lemmas below show that M is totally monotone and $f(n)=O(1)$, respectively.

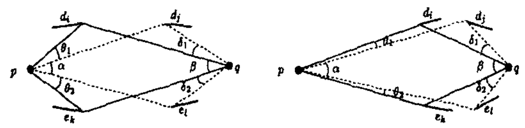

Fig. 3 An illustration of the proof of Lemma 3

Lemma 3: The matrix M is totally monotone.

Proof: It suffices to show that $B(d_i, e_l)+B(d_j, e_k) \le B(d_i, e_k)+B(d_j, e_l)$ for any $1 \le i < j \le n_1$ and $1 \le k < l \le n_2$. Let x^*_{ab} and y^*_{ab} denote two points on edges d_a and e_b, respectively, that give $B(d_a, e_b)$. We have two cases as shown in Figure 3.

Case 1: $\angle x^*_{ik} p y^*_{ik} > \angle x^*_{ik} q y^*_{ik}$ and $\angle x^*_{jl} p y^*_{jl} < \angle x^*_{jl} q y^*_{jl}$.

See Figure 3(a). Since $\angle x^*_{ik} p y^*_{ik} > \angle x^*_{ik} q y^*_{ik}$, x^*_{ik} and y^*_{ik} should be located at the right end-points of d_i and e_k, respectively. Similarly, x^*_{jl} and y^*_{jl} should be located at the left end-points of d_j and e_l, respectively. We define some angles as shown in Figure 3(a). Then, $B(d_i, e_k) = \angle x^*_{ik} p y^*_{ik} = \alpha + \theta_1 + \theta_2$ and $B(d_j, e_l) = \angle x^*_{jl} q y^*_{jl} = \beta + \delta_1 + \delta_2$. Now, consider $B(d_j, e_k)$ and $B(d_i, e_l)$. Clearly, $B(d_j, e_k) \le \max(\alpha+\theta_2, \beta+\delta_1)$, and $B(d_i, e_l) \le \max(\alpha+\theta_1, \beta+\delta_2)$.

Case 1.1: $\alpha+\theta_2 \le \beta+\delta_1$.

If $\alpha+\theta_1 \le \beta+\delta_2$, then $B(d_i, e_l) \le \beta+\delta_2$. So,

$$\begin{aligned} & B(d_i, e_l) + B(d_j, e_k) \\ & \le \beta+\delta_2+\beta+\delta_1 \\ & < \alpha+\theta_1+\theta_2+\beta+\delta_1+\delta_2 \\ & = B(d_i, e_k) + B(d_j, e_l). \end{aligned}$$

The second equality comes from $\angle x^*_{ik} q y^*_{ik} = \beta < \alpha+\theta_1+\theta_2 = \angle x^*_{ik} p y^*_{ik}$. Otherwise (i.e., if $\alpha+\theta_1 > \beta+\delta_2$), then $B(d_i, e_l)+B(d_j, e_k) \le \beta+\delta_1+\alpha+\theta_1 \le B(d_i, e_k)+B(d_j, e_l)$.

Case 1.2: $\alpha+\theta_2 \ge \beta+\delta_1$.

This can be handled in a symmetric way.

Case 2: $\angle x^*_{ik} p y^*_{ik} \le \angle x^*_{ik} q y^*_{ik}$.

Since $\angle x^*_{ik} p y^*_{ik} \le \angle x^*_{ik} q y^*_{ik}$, we must have $\angle x^*_{jl} p y^*_{jl} \le \angle x^*_{jl} q y^*_{jl}$. If we move either of x^*_{ik} and y^*_{ik} in the right direction (toward q), then $\angle x^*_{ik} q y^*_{ik}$ strictly increases. Thus, x^*_{jl} and y^*_{jl} that give $B(d_j, e_l)$ should be located at the left end-points of d_j and e_l, respectively. Define some angles as shown in Figure 3(b). Then $B(d_i, e_k) = \angle x^*_{ik} q y^*_{ik} = \beta$ and $B(d_j, e_l) = \beta+\delta_1+\delta_2$. Since $\alpha+\theta_1+\theta_2 \le \beta$ from the case assumption that $\angle x^*_{ik} p y^*_{ik} \le \angle x^*_{ik} q y^*_{ik}$, it holds that $B(d_i, e_l) \le \beta+\delta_2$ and $B(d_j, e_k) \le \beta+\delta_1$. Hence, $B(d_i, e_l) + B(d_j, e_k) \le \beta+\delta_1+\beta+\delta_2 = B(d_i, e_k)+B(d_j, e_l)$.

Case 3: $\angle x^*_{ji}py^*_{ji} \geq \angle x^*_{ji}qy^*_{ji}$.

This is a symmetric case of Case 2. □

Lemma 4: Let $d_i \in G^U$ and $e_j \in G^L$ and let $x^*_{ij} \in d_i$ and $y^*_{ij} \in e_j$ be two points that give $B(d_i, e_j)$. Then we can compute x^*_{ij}, y^*_{ij}, and $B(d_i, e_j)$ in $O(1)$ time.

Proof: Let $d_i = (u_1, u_2)$ and $e_j = (v_1, v_2)$. Assume that u_1 is closer to p than u_2, and v_1 is closer to p than v_2. If $\angle u_1pv_1 > \angle u_1qv_1$ and $\angle u_2pv_2 > \angle u_2qv_2$, then $B(d_i, e_j) = \angle u_2pv_2$, $x^*_{ij} = u_2$, and $y^*_{ij} = v_2$. If $\angle u_1pv_1 < \angle u_1qv_1$ and $\angle u_2pv_2 < \angle u_2qv_2$, then $B(d_i, e_j) = \angle u_1qv_1$, $x^*_{ij} = u_1$, and $y^*_{ij} = v_1$. Now consider the remaining case where $\angle u_1pv_1 \geq u_1qv_1$ and $\angle u_2pv_2 \leq u_2qv_2$.

In this case, we shoud have $\angle x^*_{ij}py^*_{ij} = \angle x^*_{ij}qy^*_{ij}$. If we regard points on d_i as vectors, then a point $\vec{x}$ on d_i can be expressed as $\vec{x} = \vec{u_1} + (\vec{u_2} - \vec{u_1})s$ for some real parameter $0 \leq s \leq 1$. Similarly, a point $\vec{y}$ on e_j is $\vec{y} = \vec{v_1} + (\vec{v_2} - \vec{v_1})t$ for some real parameter $0 \leq t \leq 1$. Then, for a value s, it is easy to find a value t so that $\angle xpy = \angle xqy$. Actually, t can be expressed as a function of s, i.e., $t = H(s)$. Since $\angle xpy = \angle xqy$ is less than π, minimizing $\angle xpy$ is equivalent to maximizing $\cos(\angle xpy)$. Note that $\cos(\angle xpy) = \frac{\vec{px} \cdot \vec{py}}{\|\vec{px}\| \cdot \|\vec{py}\|}$, which is a rational expression of two polynomials with a constant degree on the parameter s. Its maximum and the value of s at which the maximum occurs can be obtained in $O(1)$ time on the extended real RAM model. □

Combining Lemma 3 and Lemma 4, we have the following result.

Theorem 1: For a convex polygon of n vertices in general positions, an optimal illumination pair can be found in $O(n^2)$ time.

3. Concluding Remarks

We have given an $O(n^2)$ time algorithm to locate two floodlights on the boundary of a convex polygon so that the larger angle is minimized. We have considered convex polygons whose vertices are in general positions. Thus, an immediate open question is to develop an algorithm fot arbitrary convex polygons. Another open qestion is to reduce the time bound or to give a lower bound. It would be also interesting to consider illuminating convex polygons with three floodlights or more.

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