유한 셀룰러 오토마타 규칙 15에 대한 카테고리적 분석

(Categorical Analysis for Finite Cellular Automata Rule 15)

박 정 희 † 이 현 열 ††

(Jung-Hee Park)(Hyen-Yeal Lee)

요 약 두가지 상태값 (0, 1)과 서로 다른 네가지 경계조건을 갖는 1차원 셀룰러 오토마타 규칙 15의 상태전이그래프를 자기 재생시킬 수 있는 재귀식을 카테고리적 접근법으로 발견하였다. 카테고리적 접근법은 서로 다른 도메인을 갖는 오토마타들 간의 매핑을 가능케하므로 오토마타의 진화과정을 쉽게 표현할 수 있도록 한다.

Abstract The recursive formulae, which can self-reproduce the state transition graphs, of one-dimensional cellular automata rule 15 with two states (0 and 1) and four different boundary conditions were founded by categorical access. The categorical access makes the evolution process for cellular automata be expressed easily since it enables the mapping of automata between different domains.

1. 서 론

background. Cellular automata were first introduced by Neumann in early 1950's as models of biological systems in order to investigate the logical basis of self-reproduction mechanism[1]. After that, Neumann began to work with a cellular environment in which space considered of discrete cells. Each cell could be in one of a small set of states and the whole lattice of cells changed state synchronously in discrete time steps. Every cell had the same set of simple rules for state transitions which were conditional on the states of the cells in a small region around that cell. These systems were given various names such as cellular automata or cellular spaces, tessellation automata or tessellation spaces, iterative arrays and homogeneous structures. The motivation for this system is based on the idea for the design and construction of a general purpose computing.

Then in 1990's, attempts to find the recursive formulae which can self-reproduce the state transition diagram of the nearest-neighbor cellular automata with two states (0,1) and four different boundary conditions have been made with particular tree expressions by Lee[2, 3] or transition functions [4] in algebraic method because the recursive formulae could make the dynamical behaviors of cellular automata easily identified. Many rules have still been remained unexplored since it is not easy to find the recursive formulae of even one-dimensional cellular automata.

Aim. This paper tries to find the recursive formulae for rule 15 of such automata with particular morphisms ϕ, ψ, τ and χ which will be defined in chapter 2. The aim of this study is to find the recursive formulae which can self-reproduce the state transition diagrams for rule 15 of finite cellular automata from the cell size 0 (empty string) to given cell size m by categorical

· 이 논문은 99년 부산대학교 컴퓨터 및 정보통신연구소 일반기초과제 연구비 지원에 의해 작성되었음.
† 종신회원 : 양산대학 컴퓨터인터넷정보과 교수
jhpark@mail.yangsan.ac.kr
†† 종신회원 : 부산대학교 전자전기정보컴퓨터공학부 교수
hylee@hyowon.cc.pusan.ac.kr
논문접수 : 1999년 12월 30일
심사완료 : 2000년 6월 12일

method. The categorical access is more efficient than the algebraic access with regard to the representation of the recursive formulae since the relations of automata for different size of m can be easily explained with categorical representation. On the other hand, the algebraic access can not explain the relations of automata directly for different size of *m*.

2. A Categorical Access for the State Transition Graphs

2.1 Finite Cellular Automata with Four Different Boundary Conditions

First of all, we will consider a cellular automata $(X_m, \delta_{a-b}^{m,R})$ and define the finite cellular automata A(m), B(m), C(m) and D(m) with four different boundary conditions.

One-dimensional cellular automata for any rule R, whose boundary condition is a-b (a and b take 0 or 1) and the cell size (namely the size of the string at a site) is m, are a dynamical system $(X_m, \delta_{a-b}^{m,R})$. Here X_m is the set of states for the cellular automata with cell size *m*, i.e., $X_m = \{x_1x_2\cdots x_{m-1}x_m | x_i \in \{0,1\}\}$ and $X_0 = \varepsilon$ where *m* is a positive integer.

Definition 1. The state transition function $\delta_{a-b}^{m,R}$ is defined as $\delta_{a-b}^{m,R}: X_m \rightarrow X_m$ such as

$$\delta_{a-b}^{m,R}(x_1x_2\cdots x_{m-1}x_m) = f_R(ax_1x_2)f_R(x_1x_2x_3)\cdots f_R(x_{m-1}x_mb) \tag{1}$$

and

$$\delta_{a-b}^{0,R}(\varepsilon) = \varepsilon$$

where f_R is the triplet local transition function[5].

For example, let us consider the transition function $\delta_{a-b}^{m,R}$ for rule 15. If the boundary condition is 0-0 and the cell size is 5, for a string 11001 $\in X_5$

$$\delta_{0-0}^{5,15}(11001) = f_{15}(011)f_{15}(110)f_{15}(100)f_{15}(001)f_{15}(010) \tag{2}$$

By eq.(5), the right hand side of eq.(2) becomes 10011. In other word, the string 11001 at a site is transitted to the state string 10011 as shown in Figure 1.

11001 → 10011

Fig. 1. A part of the state transition diagram with m=5 and R=15

Definition 2. The cellular automata *A(m), B(m), C(m) and D(m)* with four different boundary conditions such as 0-0, 0-1, 1-0 and 1-1 for each rule R define as following, respectively:

$$A(m) = (X_m, \delta_{0-0}^{m,R}),$$
$$B(m) = (X_m, \delta_{0-1}^{m,R}),$$
$$C(m) = (X_m, \delta_{1-0}^{m,R}),$$
$$D(m) = (X_m, \delta_{1-1}^{m,R})$$

Here *A(0), B(0), C(0) and D(0)* is an empty graph.

2.2 The Recursive Formulae for rule 15 by Categorical Representation

The recursive formulae which can self-reproduce the state transition graph of the cellular automata $(X_m, \delta_{a-b}^{m,R})$ from the previous system $(X_{m-1}, \delta_{a-b}^{m-1,R})$ for rule R=15 are founded by the categorical representation in this section[6, 7, 8, 9]. With this, the self-evolution system for rule 15 of finite cellular automata can be made when the cell size m increases.

Definition 3. The category CA of the evolution system of finite cellular automata has the dynamical

system such as $(X_m, \delta_{a-b}^{m,R})$ as objects. Its morphism is the map:$(X_{m-1}, \delta_{a-b}^{m-1,R}) \rightarrow (X_m, \delta_{a-b}^{m,R})$ which satisfies the commutative diagram[10]:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{a-b}^{m-1,R}} & X_{m-1} \\ 0 \downarrow & \circlearrowright & \downarrow 0 \\ X_m & \xrightarrow[\delta_{a-b}^{m,R}]{} & X_m \end{array}$$

Now we define some morphisms φ, ϕ, τ and χ to express the recursive formulae. Let X_m^0, X_m^1 be the set of states on cellular automata with cell size *m*, satisfied with

$$X_m = X_m^0 \cup X_m^1 \quad \text{and} \quad X_m^0 \cap X_m^1 = \varnothing, \tag{3}$$

where

$X_m^0 = \{0x_1x_2\cdots x_{m-1} | x_1x_2\cdots x_{m-1} \in X_{m-1}\}$ and

$X_m^1 = \{1x_1x_2\cdots x_{m-1} | x_1x_2\cdots x_{m-1} \in X_{m-1}\}$

Then the maps **0** and **1** are defined as

$$\mathbf{0}: X_{m-1} \to X_m^0 (: x_1x_2\cdots x_{m-1} \mapsto 0x_1x_2\cdots x_{m-1}) \qquad (4)$$

$$1: X_{m-1} \to X_m^1 (: x_1x_2\cdots x_{m-1} \mapsto 1x_1x_2\cdots x_{m-1})$$

Definition 4. A morphism φ is defined as

$$\varphi: (X_{m-1}, \delta_{a-b}^{m-1,R}) \to (X_m^0, \delta_{a-b}^{m,R})$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{a-b}^{m-1,R}} & X_{m-1} \\ 0\downarrow & \circlearrowright & \downarrow 0 \\ X_m^0 & \xrightarrow[\delta_{a-b}^{m,R}]{} & X_m^0 \end{array}$$

This diagram means that the strings which prefix 0 to the strings in X_{m-1} are transited to the strings which prefix 0 to the images of X_{m-1} mapped by $\delta_{a-b}^{m-1,R}$.

Definition 5. A morphism ψ is defined as

$$\psi : (X_{m-1}, \delta_{a-b}^{m-1,R}) \to (X_m^1, \delta_{a-b}^{m,R})$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{a-b}^{m-1,R}} & X_{m-1} \\ 1\downarrow & \circlearrowright & \downarrow 0 \\ X_m^1 & \xrightarrow[\delta_{a-b}^{m,R}]{} & X_m^0 \end{array}$$

This diagram means that the strings which prefix 1 to the strings in X_{m-1} are transited to the strings which prefix 0 to the images of X_{m-1} by $\delta_{a-b}^{m-1,R}$.

Definition 6. A morphism τ is defined as

$$\tau: (X_{m-1}, \delta_{a-b}^{m-1,R}) \to (X_m^0, \delta_{a-b}^{m,R})$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{a-b}^{m-1,R}} & X_{m-1} \\ 0\downarrow & \circlearrowright & \downarrow 1 \\ X_m^0 & \xrightarrow[\delta_{a-b}^{m,R}]{} & X_m^1 \end{array}$$

This diagram means that the strings which prefix 0 to the strings in X_{m-1} are transited to the strings which prefix 1 to the images of X_{m-1} by $\delta_{a-b}^{m-1,R}$.

Definition 7. A morphism χ is defined as

$$\chi : (X_{m-1}, \delta_{a-b}^{m-1,R}) \to (X_m^1, \delta_{a-b}^{m,R})$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{a-b}^{m-1,R}} & X_{m-1} \\ 1\downarrow & \circlearrowright & \downarrow 1 \\ X_m^1 & \xrightarrow[\delta_{a-b}^{m,R}]{} & X_m^1 \end{array}$$

This diagram means that the strings which prefix 1 to the strings in X_{m-1} are transited to the strings which prefix 1 to the images of X_{m-1} by $\delta_{a-b}^{m-1,R}$.

The recursive formulae for self-reproducing the state transition graph of rule 15 are proven from now on. The triplet local transition function for rule 15 is written as following:

$$\begin{array}{cccccccc} 111 & 110 & 101 & 100 & 011 & 010 & 001 & 000 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \qquad (5)$$

In the following theorem, *A(m)* for rule 15 is partitioned by the digraphs derived by *A(m-1) and C(m-1).* Thus for convenience, we consider τ and χ as following:

$$\tau : (X_{m-1}, \delta_{0-0}^{m-1,15}) \to (X_m^0, \delta_{0-0}^{m,15}) \qquad (6)$$

which satisfies the commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{0-0}^{m-1,15}} & X_{m-1} \\ 0\downarrow & \circlearrowright & \downarrow 1 \\ X_m^0 & \xrightarrow[\delta_{0-0}^{m,15}]{} & X_m^1 \end{array}$$

and

$$\chi : (X_{m-1}, \delta_{1-0}^{m-1,15}) \to (X_m^1, \delta_{0-0}^{m,15}) \qquad (7)$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{1-0}^{m-1,15}} & X_{m-1} \\ 1\downarrow & \circlearrowright & \downarrow 1 \\ X_m^1 & \xrightarrow[\delta_{0-0}^{m,15}]{} & X_m^1 \end{array}$$

By definition 2 and eq.(3), eq.(6) and eq.(7) can be rewritten as

$$\tau : A(m-1) \hookrightarrow A(m) \qquad (8)$$

$$\psi : C(m-1) \hookrightarrow A(m) \qquad (9)$$

since X_m^0 and X_m^1 are subsets of X_m.

Theorem 1. With rule 15, *A(m)* can be partitioned by *(A(m-1), τ)* and *(C(m-1), χ).*

We denote it as $A(m) = \tau \cdot A(m-1) + \chi \cdot C(m-1)$ for short.

Proof. By eq.(6) and eq.(7), dynamic systems

(A(m-1), τ) and (C(m-1), χ) are the same as $(X_m^0, \delta_{0-0}^{m,15})$ and $(X_m^1, \delta_{0-0}^{m,15})$, respectively. By definition 2, *A(m)* is equal to $(X_m, \delta_{0-0}^{m,15})$. Thus Theorem 1 means that $(X_m, \delta_{0-0}^{m,15})$ is partitioned by $(X_m^0, \delta_{0-0}^{m,15})$ and $(X_m^1, \delta_{0-0}^{m,15})$. By eq.(3), since $X_m = X_m^0 \cup X_m^1$ and $X_m^0 \cap X_m^1 = \emptyset$, our theorem is trivial.□

Now let us consider τ and χ as

$$\tau: (X_{m-1}, \delta_{0-1}^{m-1,15}) \rightarrow (X_m^0, \delta_{0-1}^{m,15})$$

which satisfies the following commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{0-1}^{m-1,15}} & X_{m-1} \\ {\scriptstyle 0}\downarrow & \circlearrowright & \downarrow{\scriptstyle 1} \\ X_m^0 & \xrightarrow[\delta_{0-1}^{m,15}]{} & X_m^1 \end{array}$$

and

$$\chi : (X_{m-1}, \delta_{1-1}^{m-1,15}) \rightarrow (X_m^1, \delta_{0-1}^{m,15})$$

which satisfies the commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{1-1}^{m-1,15}} & X_{m-1} \\ {\scriptstyle 1}\downarrow & \circlearrowright & \downarrow{\scriptstyle 0} \\ X_m^1 & \xrightarrow[\delta_{0-1}^{m,15}]{} & X_m^0 \end{array}$$

Then these lead to Theorem 2 by the similar way shown in Theorem 1.

Theorem 2. For rule 15, *B(m)* is partitioned by *(B(m-1), τ) and (D(m-1), χ)*. We denote it as $B(m) = \tau \cdot B(m-1) + \chi \cdot D(m-1)$.□

Again let us consider φ and ψ as

$$\varphi: (X_{m-1}, \delta_{0-0}^{m-1,15}) \rightarrow (X_m^0, \delta_{1-0}^{m,15})$$

which satisfies the commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{0-0}^{m-1,15}} & X_{m-1} \\ {\scriptstyle 0}\downarrow & \circlearrowright & \downarrow{\scriptstyle 0} \\ X_m^0 & \xrightarrow[\delta_{1-0}^{m,15}]{} & X_m^0 \end{array}$$

and

$$\psi : (X_{m-1}, \delta_{1-0}^{m-1,15}) \rightarrow (X_m^1, \delta_{1-0}^{m,15})$$

which satisfies the commutative diagram:0

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{1-0}^{m-1,15}} & X_{m-1} \\ {\scriptstyle 1}\downarrow & \circlearrowright & \downarrow{\scriptstyle 0} \\ X_m^1 & \xrightarrow[\delta_{1-0}^{m,15}]{} & X_m^0 \end{array}$$

Then these lead to Theorem 3 by the similar way shown in Theorem 1.

Theorem 3. For rule 15, C(m) is partitioned by (A(m-1), φ) and (C(m-1), ψ). We denote it as

$$C(m) = \varphi \cdot A(m-1) + \psi \cdot C(m-1). \ \square$$

Let us consider φ and ψ as

$$\varphi: (X_{m-1}, \delta_{0-1}^{m-1,15}) \rightarrow (X_m^0, \delta_{1-1}^{m,15})$$

which satisfies the commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{0-1}^{m-1,15}} & X_{m-1} \\ {\scriptstyle 0}\downarrow & \circlearrowright & \downarrow{\scriptstyle 0} \\ X_m^0 & \xrightarrow[\delta_{1-1}^{m,15}]{} & X_m^0 \end{array}$$

and

$$\psi : (X_{m-1}, \delta_{1-1}^{m-1,15}) \rightarrow (X_m^1, \delta_{1-1}^{m,15})$$

which satisfies the commutative diagram:

$$\begin{array}{ccc} X_{m-1} & \xrightarrow{\delta_{1-1}^{m-1,15}} & X_{m-1} \\ {\scriptstyle 1}\downarrow & \circlearrowright & \downarrow{\scriptstyle 0} \\ X_m^1 & \xrightarrow[\delta_{1-1}^{m,15}]{} & X_m^0 \end{array}$$

Then these lead to Theorem 4 by the similar way shown in Theorem 1.

Theorem 4. For rule 15, D(m) is partitioned by (B(m-1), φ) and (D(m-1), ψ). We denote it as

$D(m) = \varphi \cdot B(m-1) + \psi \cdot D(m-1)$. □

The proofs of Theorem 2, 3 and 4 are word for word the same as the proof of Theorem 1, so are not reproduced.

Theorem 1, 2, 3 and 4 now gives a following Corollary sequently.

Corollary. The state transition graphs of A(m), B(m), C(m) and D(m) for rule 15 can be self-reproduced by the interconnection of the four recursive formulae as following:

(i) $A(m) = \tau \cdot A(m-1) + \chi \cdot C(m-1)$

(ii) $B(m) = \tau \cdot B(m-1) + \chi \cdot D(m-1)$

(iii) $C(m) = \varphi \cdot A(m-1) + \psi \cdot C(m-1)$

(iv) $D(m) = \varphi \cdot B(m-1) + \psi \cdot D(m-1)$

For example let us consider the self-reproduction of the state transition graphs of *A(m)* for rule 15. *A(2)* is partitioned by the subgraphs *(A(1), τ) and (C(1), χ)* by Theorem 1. In other word, *(A(1), τ)* is derived by transitting the strings obtained by prefixing 0 to each string in *A(1)* such as 00 and 01 to the strings obtained by prefixing 1 to $\delta_{0-0}^{1,15}(0)=1$ and $\delta_{0-0}^{1,15}(1)=1$ such as 11. *(C(1), χ)* is also constructed by transitting the strings obtained by prefixing 1 to each string in C(1) such as 10 and 11 to the strings obtained by prefixing 1 to $\delta_{1-0}^{1,15}(0)=0$ and $\delta_{1-0}^{1,15}(1)=0$ such as 10. These are shown in Figure 2. A(2) in Figure 3 is composed of the subgraphs *(A(1), τ) and (C(1), χ)* in Figure 2.

The state transition graphs of *A(m)* and *B(m)* for arbitrary cell size m can be reproduced by repeating this process. In a similar way, B(m) and D(m) for rule 15 can be self-reproduced.

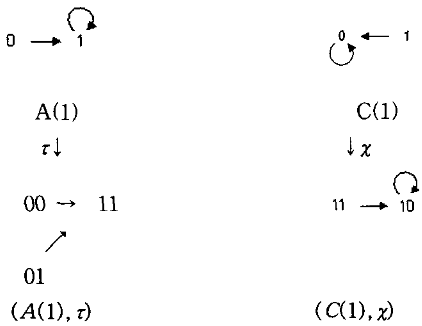


Fig. 2 The evolution process by the morphisms τ and χ

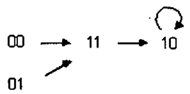


Fig. 3 The reproduction of A(2) from A(1) and C(1)

3. Conclusion

In this paper, it has been tried to find the recursive formulae for rule 15 of one-dimensional cellular automata by the categorical representation. As a result, the four recursive formulae of *A(m), B(m), C(m) and D(m)* with four different boundary conditions such as 0-0, 0-1, 1-0 and 1-1, respectively have been found and proven. We can easily identify the dynamical behaviour of rule 15 by a simple programming with these recursive formulae. The morphism φ, ϕ, τ and χ considered the transition of only the first digit of state string on automaton configuration. The study on the transition of more than first digit remains as further work.

References

[1] J. V. Neumann, *"Theory of Self-Reproducing Automata,"* A.W.Burks,ed. 1966.

[2] H. Y. Lee, *"Studies on Dynamical Behaviors of Finite Cellular Automata,"* Kyushu Univ. Ph.D Thesis, 1995.

[3] H. Y. Lee and Y. Kawahara, "Transition Diagrams of Finite Cellular Automata," *Bulletin of Informatics and Cybernetics,* Vol.28, No. 1, pp. 47-69, 1996.

[4] J. H. Park and H. Y. Lee, "A Study on the State Transition Graphs of Finite Cellular Automata," *Journal of KISS(A),* Vol. 25, No. 10, pp. 1180-1187, 1998.

[5] E. Jen, "Invariant Strings and Pattern-Recognizing Properties of One Dimensional Cellular Automata," *Journal of Statistical Physics,* Vol.43, No. 1/2, p.243, 1986.

[6] J. H. Park and H. Y. Lee, "A Categorical Representation of the State Transition Graph of Finite Cellular Automata," *Proceedings of Complex Systems'98,* Vol.4, pp.150-161, 1998.

[7] J. H. Park, *"Self-Reproduction of the State Transition Graphs for Finite Cellular Automata,"* Pusan National Univ. Ph.D Thesis, 1999.

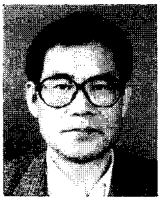
[8] T. Toffoli and N. Margolus, *"Cellular Automata Machines,"* The MIT Press, 1987.

[9] H. Herrlich and G. E. Strecker *"Category Theory,"* Herdermann Verlag Berlin, 1979.

[10] M. A. Arbib and E. G. Manes, *"ARROWS, STRUCTURES, AND FUNCTORS,"* Academic Press Inc, 1975.

박 정 희

1981년 부산대학교 수학과(이학사). 1982년 ~ 1983년 부산대학교 계산통계학과 조교. 1984년 ~ 1985년 부산대학교 전자계산소 프로그램머. 1986년 ~ 1988년 Scotland, Univ. of Glasow 전자계산학과(석사). 1992년 ~ 1995년 부산대학교 전자계산학과 박사과정 수료. 1992년 ~ 2000년 현재 양산대학 컴퓨터인터넷정보과 조교수. 관심분야는 오토마타, 암호학



이 현 열

1971년 부산대학교 수학과 (학사). 1978 일본 구주대학교 수학과(석사). 1996년 일본 구주대학교 정보학과 (박사). 1981 년 ~ 1983년 일본 구주대학교 객원 연구원. 1987년 ~ 1988년 일본 동경대학교 객원 연구원. 1988년 ~ 1989년 미국 버컬리 대학교 객원 연구원. 1989년 ~ 1991년 일본 구주대학교 객원 연구원. 1978년 ~ 2000년 현재 부산대학교 전자전기정보컴퓨터공학부 교수. 관심부야는 오토마타, 프랙탈.