Journal of the Korean Institute of Industrial Engineers
Vol. 26, No. 1, pp. 48-53, March 2000.

An Optimal Ordering Policy under the Condition of a Free Addition

Seong-Whan Shinn[1] · Chang-Yong Song[2]

[1]Department of Industrial Engineering, Halla Institute of Technology,
[2]Department of Business Administration, Tamna University

덤이 주어지는 상황하의 최적주문정책

신성환[1] · 송창용[2]

This paper deals with the problem of determining the optimal ordering quantity under the condition of a free addition. It is assumed that the supplier permits a fixed free addition depending on the amount of the quantity purchased by the customer. Investigation of the properties of an optimal solution allows us to develop an algorithm whose validity is illustrated through an example problem.

1. Introduction

The basic economic ordering quantity model has been extensively studied in the literature. One of the realistic extensions is based on the assumption that the supplier offers some incentive policy to the customers in order to stimulate the demand for the product he produces. In this regard, a number of research works appeared which deal with the quantity discounts model(Abad, 1988a, 1988b; Hadley and Within, 1963; Kim and Hwang, 1989). The traditional quantity discount models have analyzed solely the unit purchase price discount and considered two types of price discount, "all-units" and "incremental" discount. Recognizing another type of discount structure, Lee(1986) formulated the classical EOQ model with set up cost including a fixed cost and freight cost, where the freight cost has a quantity discount(economies of scale).

The common assumption held by the above authors is that the customer must pay for the items as soon as he receives them from a supplier. However, some supplier will allow a certain fixed period(credit period) for settling the amount the customer owes to him for the items supplied. According to Mehta(1968), a major reason for the supplier to offer a credit period to the customers is to stimulate the demand for the product he produces. Also, Fewings(1992) stated that the advantage of trade credit for the supplier is substantial in terms of influence on the customer's purchasing and marketing decisions. Based upon the above observations, some research papers dealt with the problem under trade credit. Chung (1998), Goyal(1985), and Hwang and Shinn(1997) examined the effects of trade credit on the optimal inventory policy. Also, Shinn *et al.*(1996) introduced the joint price and lot size determination problem under conditions of trade credit and quantity discounts for freight cost

However, in some supermarkets or discount stores, it has been observed that the supplier suggests a fixed free addition related to the bundle size for reasons of marketing policy. Therefore, the customers can get some extra with no additive cost depending on the amount of the quantity purchased. The availability of opportunity to get some extra with no cost effectively reduces the customer's total purchasing cost and it enables the customer to choose an optimal ordering quantity from another options.

This paper deals with the problem of determining the optimal ordering quantity when the supplier allows a fixed free addition depending on the amount of the quantity purchased by the customer. In Section 2, we formulate a relevant mathematical model. A solution algorithm is developed in Section 3 based on the properties of an optimal solution. A numerical example is provided in Section 4, which is followed by concluding remarks.

2. Model Formulation

In deriving the model, the following assumptions and notations are used:

1) The demand rate is known and constant.
2) No shortages are allowed.
3) The order size is considered as a discrete value.
4) The supplier allows a certain free addition depending on the amount of the quantity purchased.

D : Annual demand rate
C : Unit purchasing cost
Q : Order size
A : Ordering cost
H : Inventory carrying cost, excluding the capital opportunity cost.
R : Capital opportunity cost(as a percentage)
U : Bundle size for a free addition
α : Free addition rate(as a percentage of U)

In this situation, the supplier suggests a constant bundle size (U) and a certain free addition rate (α). For the first U, as products are purchased to $(1-\alpha)U$, the unit purchasing cost C is charged for each unit. Therefore, the total purchasing cost becomes CQ for $Q<(1-\alpha)U$. When the order size Q becomes $(1-\alpha)U$, the products are sold in a bundle of size U as the total purchasing cost is $C(1-\alpha)U$. Namely, there is no additive purchasing cost for αU units.

<Figure 1> illustrates the total purchasing cost to the order size and note that the feasible quantities of Q are $Q\in[(j-1)U, (j-1)U+(1-\alpha)U)$, $j=1, 2, \cdots$. Therefore, the total purchasing cost, $C(Q)$ is

$$C(Q)=C(1-\alpha)(j-1)U+C(Q-(j-1)U)$$
$$=C(Q-\alpha(j-1)U),$$
$$Q\in[(j-1)U, (j-1)U+(1-\alpha)U),$$
$$j=1,2,\cdots.$$

Note that when $\alpha=0$, the total purchasing cost $C(Q)$ reduce to CQ.

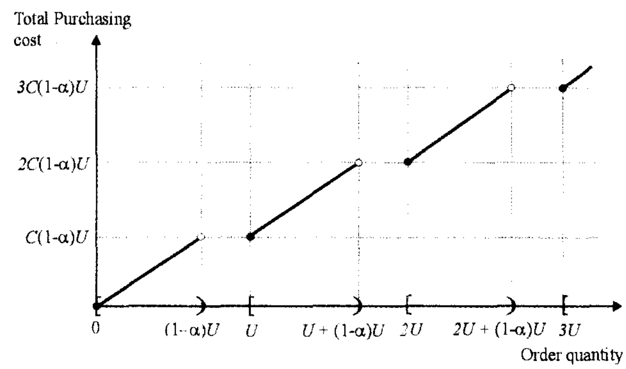

Figure 1. Total purchasing cost vs. Order quantity.

The objective of this model is to minimizes the annual total cost $TC(Q)$ and the annual total cost consists of the following four elements.

1) Annual ordering cost $= A\frac{D}{Q}$.
2) Annual inventory carrying cost $= \frac{1}{2}HQ$.
3) Annual capital opportunity cost $= \frac{1}{2}RC(Q)$
$$=\frac{1}{2}RC(Q-\alpha(j-1)U),$$
$$Q\in[(j-1)U, (j-1)U+(1-\alpha)U).$$
4) Annual purchasing cost $= C(Q)\frac{D}{Q}$
$$=C(Q-\alpha(j-1)U)\frac{D}{Q},$$
$$Q\in[(j-1)U, (j-1)U+(1-\alpha)U).$$

Then, the annual total cost $TC(Q)$ can be expressed as

$TC(Q)$ = Ordering cost + Inventory carrying cost + Capital opportunity cost + Purchasing cost.

Depending on the relative size of Q to U,

$$TC_j(Q)=A\frac{D}{Q}+\frac{1}{2}HQ+\frac{1}{2}RC(Q-\alpha(j-1)U)$$
$$+C(Q-\alpha(j-1)U)\frac{D}{Q},$$
$$Q\in[(j-1)U, (j-1)U+(1-\alpha)U),$$
$$j=1,2,\cdots. \quad (1)$$

Note that if $\alpha=0$, then equation (1) reduce to the total cost function for the classical EOQ model.

3. Determination of optimal policy

The problem is to find an optimal ordering quantity Q^*, which minimizes $TC_j(Q)$.

For $j<1+A/(C\alpha U)$, $TC_j(Q)$ is a convex function for every j and thus, there exists a unique value Q_j, which minimizes and they are:

$$Q_j=\sqrt{\frac{2D(A-C\alpha(j-1)U)}{H+RC}}. \tag{2}$$

For $j\geq 1+A/(C\alpha U)$, $TC_j(Q)'>0$ and so $TC_j(Q)$ is an increasing function of Q.

Q_j and $TC_j(Q)$ can be shown to have the following properties.

Property 1.
$Q_{j-1}>Q_j$ holds for $j<1+A/(C\alpha U)$.

Property 2.
For any Q, $TC_j(Q)>TC_{j+1}(Q)$, $j=1,2,\cdots$.

Property 1 indicates that if Q_j exists, then the value of Q_j is strictly decreasing as j increases. Also, Property 2 implies that $TC_j(Q)$ is strictly decreasing for any fixed Q as j increases. From Property 1, we have the following useful property.

Property 3.
There exists at least one $Q_j\geq(j-1)U$.

Proof. Because $A/(C\alpha U)>0$, $TC_1(Q)$ must be a convex function and thus, there exists at least one Q_j. Also, from Property 1, if all $Q_j<(j-1)U$ for every j, then $Q_1<0$ holds, which contradicts the feasibility of Q, i.e., $0<Q<\infty$. $Q.E.D.$

Now, we are going to investigate the characteristics of the annual total cost at $Q=jU$, $j=1,2,\cdots$. With $Q=jU$ as a function of j, the following single variable function is obtained:

$$TC^0(j)=A\frac{D}{jU}+\frac{1}{2}(H+RC(1-\alpha))jU+C(1-\alpha)D. \tag{3}$$

For the first and second order conditions with respect to j, we have

$$\frac{dTC^0(j)}{dj}=-\frac{1}{j^2}\frac{AD}{U}+\frac{1}{2}(H+RC(1-\alpha))U, \tag{4}$$

$$\frac{d^2TC^0(j)}{dj^2}=\frac{1}{2}\frac{1}{j^3}\frac{AD}{U}. \tag{5}$$

So, $TC^0(j)$ is a convex function of j, and there exists a unique value j^*, which minimizes $TC^0(j)$ as follows:

$$j^*=\sqrt{\frac{2AD}{U^2(H+RC(1-\alpha))}}. \tag{6}$$

Therefore, the annual total cost at $Q=jU$, $j=1,2,\cdots$, satisfy the following inequality

$$TC_{j+1}(jU)\geq\min\{TC_{j^++1}(j^+U),\ TC_{j^-+1}(j^-U)\},$$
$$\text{where } j^+=\lceil j^*\rceil \text{ and } j^-=\lfloor j^*\rfloor. \tag{7}$$

Now, from the above properties and the inequality (7), we can make the following observations about the characteristics of $TC_j(Q)$ for $Q\in I_j=\{Q\mid(j-1)U\leq Q<(j-1)U+(1-\alpha)U\}$, $j=1,2,\cdots$. These observations simplify our search process such that only a finite number of candidate values of Q need to be considered to find an optimal value Q^*. Let m be the largest index such that $Q_m\geq(m-1)U$ and $Q^0=(m-1)U+(1-\alpha)U-1$.

Observation 1.
(i) If $Q_m\geq Q^0$, then we only have to consider $Q=Q^0$ for $Q\in I_m$ as candidate for Q^* and $Q^*\geq Q^0$.
(ii) Else if $Q_m\geq(m-1)U$, then we only have to consider $Q=\lceil Q_m\rceil, \lfloor Q_m\rfloor$ for $Q\in I_m$ as candidate for Q^* and $Q^*\geq\lfloor Q_m\rfloor$.

Proof.
(i) Because $Q_m\geq Q^0$, $TC_m(Q)$ is a decreasing function for $Q\in I_m$ and so, we have

$$TC_m(Q^0)\leq TC_m(Q) \text{ for } Q\in I_m.$$

Therefore, if $Q_m\geq Q^0$, then $Q=Q^0$ yields the minimum total cost for $Q\in I_m$.

Also, by Property 2

$$TC_m(Q)<TC_j(Q),\ j<m.$$

Hence,

$$Q^* \geq Q^0.$$

(ii) Since $(m-1)U \leq Q_m < Q^0$, $TC_m(Q)$ is a convex function for $Q \in I_m$. So, by definition of Q_m, we have

$$TC_m(Q_m) \leq TC_m(Q) \text{ for } Q \in I_m.$$

Therefore, if $(m-1)U \leq Q_m < Q^0$, then the total cost becomes the minimum at $Q = \lceil Q_m \rceil, \lfloor Q_m \rfloor$.

And by Property 2

$$TC_m(Q) < TC_j(Q),\ j < m.$$

Hence,

$$Q^* \geq \lfloor Q_m \rfloor.$$

Q.E.D.

Also, by definition of the index m, $TC_j(Q)$ is increasing in $Q \in I_j$ for $j > m$. And so we only need to consider $Q = (j-1)U$, $j > m$, in finding an optimal value Q^* for $Q \geq mU$. But, by the following observation, most of the candidate values can be dropped from consideration in search of Q^*.

Observation 2.
For $Q \geq mU$,
(i) if $j^* \leq m$, then we only need to consider $Q = j^+U$ in finding Q^*.
(ii) if $j^* > m$, then we only need to consider $Q = j^+U$, j^-U in finding Q^*.

Proof.
(i) Because a is a positive number representing the free addition rate,

$$j^*U = \sqrt{\frac{2AD}{H + RC(1-a)}} > Q_1 = \sqrt{\frac{2AD}{H + RC}}$$

So, if $Q_1 \in [(k-1)U, kU)$, then $j^* > k-1$. Also, from definition of the index m and k, if $j^* \leq m$, then $m = k$ and $m-1 < j^* \leq m$. Therefore, by the inequality (7) and Observation 1, $j^+U (= mU)$ becomes the only candidate value for Q^*.

(ii) By the inequality (7), if $j^* > m$, then we only need to consider $Q = j^+U$, j^-U in finding an optimal value Q^*.

Q.E.D.

To facilitate the explanation of Observations 1 and 2, we present <Figure 2> which shows the shape of $TC_j(Q)$ of an example problem introduced in Section 4. Note that $TC_j(Q)$ in solid lines satisfies the above properties. Applying Observations 1 and 2 to problem, it is found that $m=6$, $j^+ = 12$ and $j^- = 11$. Hence, the candidates for an optimal Q are $(6-1)U + (1-a)U - 1$, $11U$ and $12U$, and the optimal ordering quantity becomes $11U$ with its minimum annual total cost of \$4125.727 .

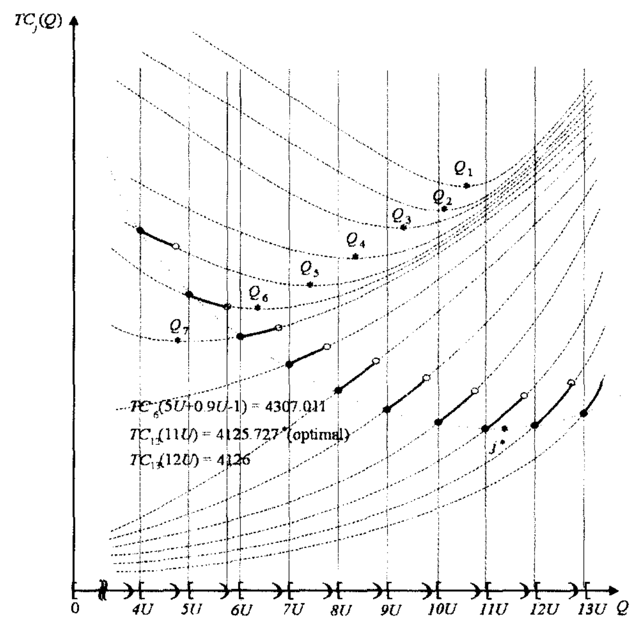

Figure 2. $TC_j(Q)$ of an example problem in Section 4.

Based on the above observations, we develop the following solution algorithm for determining an optimal ordering quantity Q^*.

Solution Algorithm
Step 1. Compute Q_1 by equation (2) and find the index k such that $Q_1 \in [(k-1)U, kU)$.
Step 2. Find the largest index l such that $l < 1 + \frac{A}{CaU}$.
Step 3. Compute Q_j, $j \leq \min\{k, l\}$ by equation (2) and find the largest index m such that $Q_m \geq (m-1)U$.
Step 4. If $Q_m \geq (m-1)U + (1-a)U - 1$, then compute the annual total cost for $Q = (m-1)U + (1-a)U - 1$.
Otherwise, compute the annual total cost for $Q = \lceil Q_m \rceil$, $\lfloor Q_m \rfloor$.
Step 5. Compute j^* by equation (6) and find the

Table 1. Results with various values of α

α	$Q=(m-1)U+(1-\alpha)U-1$			$Q=\lceil Q_m \rceil$			$Q=\lfloor Q_m \rfloor$			$Q=j^+U$			$Q=j^-U$		
	m	Q	$TC_m(Q)$	m	Q	$TC_m(Q)$	m	Q	$TC_m(Q)$	j^+	Q	$TC_{j^++1}(Q)$	j^-	Q	$TC_{j^-+1}(Q)$
0.00	-	-	-	11	2191*	4547.723*	11	2190*	4547.723*	12	2200	4547.728	-	-	-
0.05	8	1589	4393.010	-	-	-	-	-	-	13	2400	4338	12	2200*	4336.728*
0.10	6	1179	4307.011	-	-	-	-	-	-	13	2400	4126	12	2200*	4125.727*
0.15	5	969	4232.964	-	-	-	-	-	-	13	2400*	3914*	12	2200	3914.728
0.20	4	759	4240.978	-	-	-	-	-	-	13	2400*	3702*	12	2200	3703.727
0.50	2	299	4696.272	-	-	-	-	-	-	16	3000	2425	15	2800*	2424.286*

* Optimal solution for the example problem.

index j^+ and j^- such that $j^+=\lceil j^* \rceil$ and $j^-=\lfloor j^* \rfloor$.

Step 6. If $j^* \le m$, then compute the annual total cost for $Q=j^+U$. Otherwise, compute the annual total cost for $Q=j^+U$ and j^-U.

Step 7. Select the one that yields the minimum annual total cost as Q^* and stop.

4. Numerical Example

To illustrate the solution algorithm, the following problem is considered.

D = 2,000 units,
C = \$2,
A = \$300,
H = \$0.05,
R = 0.1(=10%),
U = 200 units,
α = 0.1(=10%).

An optimal solution can be obtained through the following steps.

Step 1. Since $Q_1=2190.9$, $k=11$.

Step 2. Since $1+A/(C\alpha U)=8.5$, $l=8$.

Step 3. Since $\min\{11,8\}=8$, compute Q_j. for $j \le 8$. And since $Q_8=565.7<7U$, $Q_7=979.8<6U$ and $Q_6=1264.9>5U$, $m=6$.

Step 4. Since $Q_6 \ge (6-1)U+(1-\alpha)U-1=1179$, compute $TC_6(1179)$.

Step 5. Since $j^*=11.4$, $j^+=12$ and $j^-=11$.

Step 6. Since $j^*=11.4>m=6$, compute $TC_{13}(2400)$ and $TC_{12}(2200)$.

Step 7. Since $TC_{12}(2200)=4125.727=\min\{TC_6(1179), TC_{12}(2200), TC_{13}(2400)\}$, an optimal order quantity becomes 2200 with its minimum annual total cost of \$4125.727.

For example problem, <Table 1> shows the results obtained from the various values of α. Note that for the case with $\alpha=0$, the algorithm generates the same results as those by the classical EOQ model.

5. Conclusions

This paper dealt with an optimal ordering quantity when the supplier allows a fixed free addition depending on the amount of the quantity purchased by the customer. In some supermarkets or discount stores, it is not uncommon that a supplier offers some free addition to a certain degree expecting that he can make more profit by stimulating the customer demand. In this regard, we think that the model presented in this paper may be one of the realistic extensions for the classical EOQ model.

After formulating the mathematical model, we proposed the solution algorithm that leads to an optimal ordering quantity. With an example, the validity of the algorithm is illustrated and the effect of the supplier's free addition rate (α) is examined on the annual total cost and the customer's order size. The results show that the total cost seems quite sensitive to α and the model presented in this paper equals to the classical EOQ model when $\alpha=0$.

References

Abad, P. L. (1988a), Determining optimal selling price and lot size when the supplier offers all-unit quantity discounts, *Decision Sciences*, 19, 622-634.

Abad, P. L. (1988b), Joint price and lot-size determination when supplier offers incremental quantity discounts, *Journal of Operational Research Society*, 39(6), 603-607.

Chung, K. J. (1998), A theorem on the determination of economic order quantity under conditions of permissible delay in payments, *Computers and Operations Research*, 25(1), 49-52.

Fewings, D. R. (1992), A credit limit decision model for inventory floor planning and other extended trade credit arrangement, *Decision Sciences*, 23(1), 200-220.

Goyal, S. K. (1985), Economic order quantity under conditions of permissible delay in payments, *Journal of Operational Research Society*, 36(4), 335-338.

Hadley, G. and Within, T. M. (1963), *Analysis of inventory systems*, Prentice-Hall, Englewood Cliffs, NJ.

Hwang, H. and Shinn, S. W. (1997), Retailer's pricing and lot sizing policy for exponentially deteriorating products under the condition of permissible delay in payments, *Computers and Operations Research*, 24(6), 539-547.

Kim, K. H. and Hwang, H. (1989), Simultaneous improvement of supplier's profit and buyer's cost by utilizing quantity discount, *Journal of Operational Research Society*, 40(3), 255-265.

Lee, C. Y. (1986), The economic order quantity for freight discount costs, *IIE Transactions*, 18, 318-320.

Mehta, D. (1968), The formulation of credit policy models, *Management Science*, 15(2), B30-B50.

Shinn, S. W., Hwang, H. and Park, S. S. (1996), Joint price and lot size determination under conditions of permissible delay in payments and quantity discounts for freight cost, *European Journal of Operational Research*, 91, 528-542.