Kangweon-Kyungki Math. Jour. **8** (2000), No. 1, pp. 53–61

COMPACT KÄHLER-EINSTEIN 4-MANIFOLD

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Abstract. The object of this paper is to find the 4-dimensional compact Einstein manifold with negative Ricci curvature r.

1. Introduction

Let M be a smooth compact 4-dim. manifold. An Einstein metric g on M satisfies that

$$r = \lambda g$$

where r is th Ricci curvature of g and λ is real constant. Therefore, every Einstein manifold M always admits constant scalar curvature s. In particular, for 4-dimensional case, $s = 4\lambda g$, and then we get that $r = (s/4)g$. This means that to get a constant scalar curvature metric on M is a necessary condition. Actually, by Friedman's classification, $R = CP_2\#8\overline{CP}_2$, which is the standard smooth structure and the Barlow surface S, which is a simply-connected minimal surface of general type with $q = p_g = 0$ and $K^2 = 1$ are homeomorphic but not diffeomorphic.

But it was proved by Catanese and LeBrun[CL] that $R \times R$, $S \times S$ are diffeomorphic. And $R = CP_2\#8\overline{CP}_2$ is Kähler-Einstein manifold with positive scalar curvature.

From the above three facts, we can get a possibility that S has Kähler-Einstein metric with negative Ricci curvature. Actually, the second fact is the counter-example of Besse's conjecture[Be] for 8-dimensional case that no smooth compact n-manifold can ever admit Einstein metrics with different signs of λ. But that conjecture still stands in dimension 4. By the way, Lohkamp[L], generalizing an earlier result of Gao and Yau[GY], has shown that absolutely every smooth

Received November 30, 1999.
1991 Mathematics Subject Classification: 53C25.
Key words and phrases: Eistein manifold.
Supported by Korean Research Foundation in 1997 Post-Doc. program

manifold of dim. $n \geq 3$ admits Riemannian metrics of negative Ricci curvature. But that metric induces locally negative Ricci curvature, that is, doesn't imply negative constant Ricci curvature on the entire manifold. Therefore that metric is not Einstein.

2. On Ricci Curvatures for CP_2, $S^2 \times S^2$, S^4

At first, what we can do is to find 4-dim. Einstein manifold and calculate Ricci curvature directly CP_2, $S^2 \times S^2$, S^4 are well-known examples of 4-dimensional compact Einstein manifold. On the other hand, there are few examples of compact 4-manifolds, which don't admit Einstein metric manifold. If M is an Einstein manifold, then M must have Euler characteristic $\chi \geq 0$ with equality, iff M is Flat, and so that $T^4 \# T^4 \& S^1 \times S^3$ don't admit Einstein metrics. For the well-known examples CP_2,$S^2 \times S^2$, S^4, we will check whether their Ricci curvatures have negative sign or not, case by case.

For CP_2, Yau[Y1] proved that the only Kähler-Einstein metric on CP_2 is th usual Fubini-Study metric with positive sectional curvature $c = 4$. Therefore, there is no negative Ricci curvature of Einstein metric on CP_2.

For $S^2 \times S^2$, as Riemannian manifold, we consider the general product manifold $(S_p(a) \times S_q(b), a^2 ds_p^2 + b^2 ds_q^2)$. For the orthonormal frame F_i, $i = 1, 2, 3, 4$

$$Ric(F_i) = \begin{cases} (1/a^2)(p-1)F_i(i \leq p) \\ (1/b^2)(q-1)F_i(i > P) \end{cases}$$

If take $p = q = 2$, $a = b = 1$, i.e. $S^2 \times S^2$, then the product metric is Einstein with positive constant. As complex manifold, since S^2 is the underlying differential manifold of CP_1, the complex structure of S^2 is 1-dim. complex projective space. The first fact that CP_1 has only the usual Fubini-Study metric and its metric has constant positive Ricci curvature and the second one that $M \times M$ is Einstein manifold with the same constant for Einstein manifold M with a constant λ have induced that $S^2 \times S^2$ is Einstein manifold with positive constant Ricci curvature.

For S^4, using the theorem proved by Borel & Serre[BS] that S^n, for $n \neq 2, 6$ doesn't admit almost complex structure, S^4 has no almost complex structure metric. By Ziller[Z], S^4 has only one homogeneous Einstein Riemannian metric and its Ricci curvature is positive. Then we have to decide whether S^4 has nonhomogeneous Einstein Riemannian metric. But since S^4 is symmetric, S^4 is homogeneous from ch.7 in [Be]. Only remaining study is to construct compact complex surface that admits Kähler- -Einstein manifold with negative Ricci curvature.

3. Einstein metrics on compact complex surfaces

At first, I would like to introduce the famous several theorems for existence of Einstein metric.

THEOREM A (CALABI[CAL] & YAU[Y2]). *Let M be a compact Kählermanifold, ω its Kählerform, $c_1(M)$ the real first Chern class of M. Any closed(real) 2-form of type(1.1) belonging to $2\pi c_1(M)$ is the Ricci form of one and only one Kählermetric in the Kählerclass of ω.*

THEOREM B (AUBIN[AUB] & YAU[Y2]). *Any compact complex manifold with negative first Chern class admits a Kähler- Einstein metric with negative scalar curvature. This metric is unique up to homothety.*

This metric is called Aubin-Calabi-Yau metric. And the statement analogous to theorem B, when the first Chern class is assumed to be positive, is false from [Be].

PROPOSITION A. *The sign of the (constant) scalar curvature s of a Kähler-Einstein metric -if any- on given (compact) complex manifold M is determined by the complex structure of M. Moreover, the value of s is then determined by*

$$Vs^m = ((4\pi m)^m/(m!))c_1^m(M)$$

where $c_1^m(M)$ denotes the Chern number associated with the m-th power of the 1^{st} Chern class of M, depending on the complex structure only, and V the total volume.

As we just saw, a necessary condition for a given (compact) complex manifold to admit a Kähler-Einstein metric is that its first Chern class have a sign, negative, zero or positive.

Theorem C. *The compact complex manifolds with positive (negative) first Chern class are exactly the compact complex manifolds admitting a Kählermetric with positive (negative) Ricci form.*

Theorem D. *Let M be a compact complex manifold of dimension at least 2, such that -$c_1(M)$ can be represented by a Kählerform. Then M admits a Kähler-Einstein metric.*

Theorem E. *If on the 2-dimensional compact, connected complex manifold X with $c_1^2(X) = 3c_2(X)$, there exists a Kähler-Einstein metric, then the holomorphic sectional curvature is constant.*

Since compact complex surface X with constant sectional curvature admits Kähler-Einstein metric g, the following necessary and sufficient condition for existence of Kähler-Einstein metric.

Theorem F. *A compact complex surface X with $c_1^2 = 3c_2$ has Kähler-Einstein metric if and only if X has constant sectional curvature.*

To approach the Barlow surface that is homeomorphic to $CP_2\#8\overline{CP}_2$ we will consider the problem of classifying up $CP_2\#k\overline{CP}_2 = \Sigma_k$.

For $k = 0$, there is Yau's result that any complex surface homeomorphic to CP_2 is diffeomorphic to CP_2. For $k = 1$, there are the Hirzebruch surfaces $\Sigma_n(n = \text{odd})$ which are known to be diffeomorphic to the standard $CP_2\#\overline{CP}_2$. On the other hand, we have the result of Friedman and Morgan[FM] that $CP_2\#k\overline{CP}_2$ has infinitely many smooth structures underlying algebraic surfaces if $k \geq 9$. By the Enrigues-Kodaira classification of complex surfaces, the only other surfaces possibly homeomorphic to some $CP_2\#k\overline{CP}_2$ with $0 < k < 9$ are surfaces of general type. It can be proved ([Hit] or [Y3]) that the complex surfaces Σ_k obtained from CP_2 by blowing up k distinct points, $0 \leq k \leq 8$, do have a positive first Chern class, whenever those points are in general position, that is; no 3 of them lie on a same line, no 6 of them lie on a same conic curve, and if $k = 8$, they are all simple points of each cubic curve passing through all eight of them.

Moreover, the manifolds Σ_k are the only (compact) complex surfaces having positive first Chern class, with CP_2, $CP_1 \times CP_1$. Consequently, those are the only compact complex surfaces on which the existence of a Kähler-Einstein metric with positive scalar curvature can be expected.

On the other hand, the complex surfaces Σ_1, Σ_2, Σ_3 are proved to admit no Kähler-Einstein metric, by showing that their connected group of complex automorphisms is not reductible by Theorem G. Concerning this question, nothing is known for the other complex surfaces Σ_k, $4 \leq k \leq 8$ by 1987.

Theorem G. *The identity component $U^0(M)$ of the automorphism group of a compact complex manifold carrying a Kähler-Einstein metric is reductible.*

For $1 \leq k \leq 6$, the manifolds Σ_k are known as Del Pezzo surfaces (of degree $9 - r$). Among them, two families only can be realized as a complete intersection in a complex projective space; Σ_6 which is (complex) hypersurfaces of degree 3 in CP_3 and Σ_5 which is the intersection of two quadrics (complex hypersurface of degree 2) of CP_4 (loc. cit.). (recall that $CP_1 \times CP_1$ itself is realized as a quadric of CP_3). Since for $d > N$, M can admit a Kähler-Einstein metric where M is a complex hypersurface of deg d in CP_N, two cases Σ_5, Σ_6 can't admit Kähler-Einstein metrics. But in 1990, Tian[T], building on his joint work with Yau[Y5], has shown that $CP_2 \# 8\overline{CP}_2$ admits an Kähler-Einstein metric g of scalar curvature $+1$.

Theorem H([Y4]). *Let M be a Kählerian manifold of complex dimension m with negative $c_1(M)$. Then*

$$(-1)^m 2(m+2)c_2 {c_1}^{(m-2)} - m{c_1}^m \geq 0$$

.

Equality holds iff the holomorphic sectional curvature of the Aubin-Calabi-Yau metric is constant(negative) so that M is (holomorphically) covered by the unit ball in C^m.

4. Existence for Einstein metrics with negative Ricci curvature on compact 4-dim. manifolds

We got two examples for compact 4-dim. Einstein manifolds with negative Ricci curvature. The first one is Barlow surface as algebraic aspect. The second is E_χ, the complex line bundle CP_1 with Euler number χ as partial differential equation's one. Now, we introduce the

Barlow surface. Let F be a quintic surface in CP_3 with exactly 20 nodes and no other singularities. Thus F naturally carries the structure of a complex orbifold. We will be interested in the case in which F is a global orbifold;

$$F = Y/Z_2$$

for some compact complex 2-manifold Y with a holomorphic involution. Using the theory of Hilbert modular surfaces, the specific 20 nodal quintic surface F

$$\sum_{j=1}^{5} z_j^5 = (5/4)(\sum_{j=1}^{5} z_j^2)(\sum_{j=1}^{5} z_j^3)$$

$$\sum_{j=1}^{5} z_j = 0$$

is a hyperplane of CP_4.

The symmetric group S_5 acts on F, and in particular we have an action of the dihedral group $D_{10} \subset S_5$ of pentagonal isometries, generated by (25)(34) and (12345). This action lifts to Y in such a way that the cyclic subgroup $Z_5 \subset D_{10}$ generated by (12345) acts freely on Y and such that the involution (25)(34) acts with exactly 20 fixed points. The so-called Catanese surface $X = Y/Z_5$ is therefore nonsingular, and comes equipped with an involution α, we obtain a surface X/Z_2 whose only singularities are four nodes. The Barlow surface is by definition the minimal resolution S of X/Z_2. with an involution $\alpha : X \to X$ with exactly 4 fixed points. Dividing X by the action of the involution α, we obtain a surface X/Z_2 whose only singularities are four nodes. The Barlow surface is by definition the minimal resolution S of X/Z_2. One can show [B] that S is a minimal, simply connected complex surface of general type, with $c_1^2(S) = 1$, $q = p_g = 0$. Specially,since $-c_1(Y)$ is represented by a positive 2-form in Y, it is also, by averaging, represented by a positive Z_5-invariant 2-form; such a form descends to X, and represents -$c_1(X)$. In the same vein, -$c_1(S)$ is represented by a positive 2-form in Barlow surface S. Therefore, the first Chern number $c_1(S)$ in Barlow surface is negative form. From Theorem D and Theorem B, Barlow surface S admits Kähler-Einstein metric with negative scalar curvature.

THEOREM. *The Barlow surface S is a compact Kähler-Einstein manifold with negative Ricci curvature.*

For the second example, we will discuss metrics of form

$$dr^2 + \Phi^2(r)\sigma_1^2 + \Psi^2(r)(\sigma_2^2 + \sigma_3^2)$$

where σ_i is the standard orthonormal frame on S^3 satisfying $d\sigma_i = -2\sigma_{i+1} \wedge \sigma_{i+2}$ (all indices are mod 3). $U(2)$ acts by isometries on the metrics. Denote by $\Theta_0 = dr$, $\Theta_1 = \Psi\sigma_1$,$\Theta_2 = \Psi\sigma_2$, $\Theta_3 = \Psi\sigma_3$ the standard orthonormal coframe on $I \times S^3$ with respect to the metric and $\partial/\partial r$, ξ_1, ξ_2, ξ_3 corresponding orthonormal frame. By calculating curvature directly,

$$S_{23} = \frac{4\Psi^2 - 3\Phi^2}{\Psi^4} - \frac{\dot{\Psi}^2}{\Psi^2},$$

$$S_{10} = -\frac{\ddot{\Phi}}{\Phi},$$

$$S_{i0} = -\frac{\ddot{\Psi}}{\Psi}, i = 2, 3,$$

$$S_{1i} = \frac{\Phi^2}{\Psi^4} - \frac{\dot{\Phi}\dot{\Psi}}{\Phi\Psi}, i = 2, 3,$$

$$< R(\xi_1, \xi_2)\xi_3, \xi_0 >= \frac{\dot{\Phi}\Psi - \Phi\dot{\Psi}}{\Psi^3} = K,$$

where $S_{ij} = \sec(\xi_i, \xi_j)$

Then the curvature operator R is

$$R = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$$

$$B = 1/2diag(-S_{01} + S_{23}, -S_{02} + S_{13}, -S_{03} + S_{12})$$

$$A = 1/2diag(S_{01} + S_{23} - 4K, S_{02} + S_{13} + 2K, S_{03} + S_{12} + 2K)$$

$$D = 1/2diag(S_{01} + S_{23} + 4K, S_{02} + S_{13} - 2K, S_{03} + S_{12} - 2K)$$

In order to get a Einstein metrics, we need to solve that $B = 0$. That is,

$$S_{01} = S_{23} : -\frac{\ddot{\Phi}}{\Phi} = \frac{4\Psi^2 - 3\Phi^2}{\Psi^4} - \frac{\dot{\Psi}^2}{\Psi^2},$$

$$S_{02} = S_{13}, \text{ or } S_{03} = S_{12} : -\frac{\ddot{\Psi}}{\Psi} = \frac{\dot{\Phi}\dot{\Psi}}{\Phi\Psi}.$$

The Einstein constant is $Ric(\xi_0, \xi_0) = -\frac{\ddot{\Phi}}{\Phi} - \frac{2\ddot{\Psi}}{\Psi} = \lambda$. We will denote by E the complex line bundle over CP_1 with Euler number. For Kählerian case, the necessary and sufficient condition for existence of Einstein metric with negative constant λ is the existence of solutions Φ, Ψ such that

$$\Phi = \Psi\dot{\Psi}$$

$$\dot{\Psi}^2 = 1 - (\lambda/6)\Psi^2 + ((\lambda/6) - 1)\Psi^{-4}$$

with $\Psi(0) > 0$, $\dot{\Psi}(0) = 0$. When $\lambda \leq 0$, the right side of the above second equation has no zero except at $t = o$. Thus the solution exists for all time. For $\lambda < 0$ ($\lambda = 2(2 - \chi)$ for $\chi > 2$), the metric induced by such Φ, Ψ is Kähler-Einstein metric with negative Ricci curvature on E_χ.

References

[Aub] T.Aubin, *Meilleurs constantes dans le theoreme d'inclusion de Sobolev et un theoreme de fredholm non lineaire pour la transformation conforme de la courbure scalaire*, J.Func.Anal. **32** (1979), 148–174.

[B] R.N.Barlow, *A simply connected surface of general type with $p_g = 0$*, Invent. Math. **79** (1985), 293–301.

[Be] A. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

[BS] A. Borel and J.P. Serre, *Determination des p-puissances reduites de steenrod dans la cohomologie des groupes classiques*, Applications C.R. Acad. Sci. Paris **233** (1951), 680–682.

[Cal] E. Calabi, *Algebraic geometry and topology, a symposium in honor of S.*

[CL] F. Catanese and C. LeBrun, *On the scalar curvature of Einstein manifolds* (to appear).

[GY] L.Z.Gao and S.T. Yau, *The existence of negatively Ricci curved metrics on three-manifolds*, Inven.math. **85** (1986), 637–652.

[Hit] N. Hitchin, *On the curvature of retional surfaces*, Differential Geometry, Proc. of Sym. Pure math. **27** (1975), 65-80.

[K] D. Kotschick, *On manifolds homeomorphic to $CP_2\#8\overline{CP}_2$*, Invent. Math. **95** (1989), 591–600.

[L] J.Lohkamp, *Metrics of nagative Ricci curvature*, Ann. Math. **140** (1994), 655-683.

[T] G. Tian, *On Calabi's conjecture for complex surfaces with positive first. Chern class*, Inv. math. **101** (1990), 101-172.

[L] J. Lohkamp, *Metrics of negative Ricci curvature*, Ann. Math. **140** (1994), 655-683.

[T] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Inv. Math. **101** (1990), 101-172.

[Y1] S.T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Pro.Nat.Acad. USA **74** (1977), 1798–179.

[Y2] S.T.Yau, *On the Ricci curvature of a compact Kählermanifold and the complex Monge-Ampere equation I*, Com. Pure and Appl. Math. **31** (1978), 339–411.

[Y3] S.T.Yau, *On the curvature of compact Hermitian manifolds*, Inventiones Math. **25** (1974), 213–239.

[Y4] S.T.Yau, *On Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA **74** (1977), 1798–1799.

[Y5] S.T.Yau and G.Tian, *Kähler-Einstein metrics on complex surfaces with* $c_1 > 0$, Comm. Math. Phys **112** (1987), 175–203.

[Z] W.Ziller, *Homogeneous Einstein metrics on spheres and projective spaces*, Math. Ann. **259** (1982), 351–358.

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