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ASSOCIATED PRIME IDEALS OF A PRINCIPAL IDEAL

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Abstract. Let R be an integral domain with identity. We show that each associated prime ideal of a principal ideal in $R[X]$ has height one if and only if each associated prime ideal of a principal ideal in R has height one and R is an S-domain.

Krull's principal ideal theorem [7, Theorem 142] states that for a nonunit element x of a Noetherian ring R, if P is a prime ideal of R which is minimal over xR, then the height of P is at most one. Thus if R is a Noetherian domain then each minimal prime ideal of a nonzero principal ideal has height one. In [2], Barucci-Anderson-Dobbs studied integral domains in which each prime ideal over a nonzero principal ideal has height one. As [2], we say that an integral domain R satisfies the *principal ideal theorem* (PIT) if each prime ideal over a nonzero principal ideal of R has height one.

Let R be an integral domain with identity. A prime ideal P of R is called an *associated prime ideal* of a principal ideal in R if there exist some elements $a, b \in R$ such that P is minimal over $aR : bR = \{x \in R | xb \in aR\}$. Consider an integral domain R with the following property:

APIT: each associated prime ideal of a principal ideal in R has height one.

One can easily show that R satisfies APIT if and only if $R = \cap_{P \in X^1(R)} R_P$ where $X^1(R)$ is the set of all height one prime ideals of R (cf. [6, Ex. 22, p.52]). The purpose of this paper is to show that $R[X]$ satisfies APIT if and only if R satisfies APIT and R is an S-domain. (Recall that an integral domain R is an S-domain if for each height one prime ideal P of R, the expansion $P[X]$ of P to $R[X]$ has again height one.) All rings considered in this paper are commutative integral domains with identity.

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If $a \in R$, then $aR = aR : R$ and so a minimal prime ideal of a nonzero principal ideal is an associated prime ideal of a principal ideal. Hence R satisfies APIT then R satisfies PIT. The following example shows that the converse does not hold.

EXAMPLE 1. *Let R be the field of real numbers and let $R[[x, y]] = R + M$ be the power series ring over R, where $M = (x, y)R[[x, y]]$. Let $\overline{Q}$ be the algebraic closure of the field Q of rational numbers in R. Let $D := \overline{Q} + M$, then (1) D is a 2-dimensional quasi-local Mori domain with maximal M, (2) M is an associated prime ideal of a principal ideal in D, and hence D does not satisfies APIT, and (3) D satisfies PIT.* (see [4, Example 8]) .

In [5, Theorem 4], Chang proved that if R is integrally closed, then $R[X]$ satisfies PIT if and only if R satisfies PIT and R is an S-domain. The following theorem is an APIT-analog of that fact.

THEOREM 2. *$R[X]$ satisfies APIT if and only if R satisfies APIT and R is an S-domain.*

Proof. ($\Rightarrow$) Since $R[X]$ satisfies APIT, $R[X]$ satisfies PIT, and R is an S-domain [2, Proposition 6.1]. Let P be an associated prime ideal of a principal ideal in R, i.e., P is minimal over $aR : bR$ for some $a, b \in R$. Then $P[X]$ is minimal over $(aR : bR)R[X]$. Since $(aR : bR)R[X] = aR[X] : bR[X]$, $P[X]$ is an associated prime ideal of a principal ideal in $R[X]$. Hence $\text{ht}(P[X]) = 1$, and so $\text{ht}P = 1$.

($\Leftarrow$) Let Q be an associated prime ideal of a principal ideal in $R[X]$. If $Q \cap R = 0$, then $\text{ht}Q = 1$ [7, Theorem 36]. If $Q \cap R (:= P) \neq 0$, then $Q = P[X]$ and P is an associated prime ideal of a principal ideal in R [3, Corollary 8]. Hence $\text{ht}Q = \text{ht}P = 1$. □

Since $R[X]$ is an S-domain [1, Theorem 3.2], it follows directly from Theorem 2 that $R[X_1, \ldots, X_n]$ satisfies APIT if and only if $R[X_1]$ satisfies APIT, where $\{X_1, \ldots, X_n\}$ is a finite set of indeterminates over R. It is easy to show that for nonzero elements $a, b \in R$, $aR[\{X_\alpha\}] \cap R = aR$, $(aR : bR)R[\{X_\alpha\}] = aR[\{X_\alpha\}] : bR[\{X_\alpha\}]$ and $(aR[\{X_\alpha\}] : bR[\{X_\alpha\}]) \cap R = aR : bR$ where $\{X_\alpha\}$ is a set of indeterminates over R. Using this and the proof of [2, Proposition 6.4], we have

COROLLARY 3. *$R[\{X_\alpha\}]$ satisfies APIT if and only if R satisfies APIT and R is an S-domain.*

Given a fractional ideal I of an integral domain R, we define $I_v = (I^{-1})^{-1}$ and $I_t = \cup\{J_v | J$ is a finitely generated subideal of $I\}$. An ideal A of R is said to be divisorial (resp. t-ideal) if $A_v = A$ (resp. $A_t = A$). Recall that an integral domain R is an H-domain if each maximal t-ideal P of R is divisorial. Examples of H-domains include discrete valuation domains, Mori domains, Krull domains and Noetherian domains. It is clear that each prime t-ideal of R has height one, then R satisfies APIT (in fact, R satisfies PIT). The following theorem shows that if R is an H-domain, the converse also holds.

Theorem 4. *Let R be an H-domain. Then R satisfies APIT if and only if each prime t-ideal of R is a maximal t-ideal.*

Proof. Suppose that R satisfies APIT. Let A be the set of associated prime ideals of principal ideals in R. Then $R = \cap_{P \in A} R_P$ [3, Proposition 4]. Let M be a maximal t-ideal of R. Since M is divisorial (note that R is an H-domain), $R \subset M^{-1}$. Hence $M \subseteq P$ for some $P \in A$ [8, Theorem1]. Hence $M = P$, and each maximal t-ideal of R is an associated prime ideal of a principal ideal. Since R satisfies APIT, each prime t-ideal is a maximal t-ideal. The converse is clear. $\square$

Corollary 5. *If R is a Noetherian ring, then R satisfies APIT if and only if each prime t-ideal of R is a maximal t-ideal.*

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