ON SOME PROPERTIES OF PRETOPOLOGICAL CONVERGENCE STRUCTURES

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Abstract In this paper we introduce generalized $q$-interior operator and $n$-th pretopological modification of $q$. Furthermore we establish a characterization of $\pi_n(q) = \lambda(q)$.

1. Introduction

A convergence structure $q$ defined by Kent ([4]) is a correspondence between the filters on a given set $X$ and the subsets of $X$ which specifies that filters converge to points of $X$. For given convergence structure $q$ on a set $X$, Kent introduced convergence structures with $q$, which are called the pretopological modification and the topological modification. They are denoted by $\pi(q)$ and $\lambda(q)$, respectively.

A $q$-interior operator $I_q$ introduced by Choquet ([3]) is a set function which has all of the properties of topological interior operator except idempotency. In this paper, we introduce generalized $q$-interior operator and $n$-th pretopological modification of $q$. They are denoted by $I^n_q$ and $\pi_n(q)$, respectively. Also, we study some properties of them and obtain a characterization of $\pi_n(q) = \lambda(q)$.

2. Preliminaries

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Let \( X \) be a set. A nonempty collection \( \Phi \) of nonempty subsets of \( X \) is said to be a filter on \( X \) if it satisfies the following conditions:

1. \( A \in \Phi \) and \( B \in \Phi \) implies \( A \cap B \in \Phi \),
2. \( A \in \Phi \) and \( A \subseteq B \) implies \( B \in \Phi \).

For a nonempty set \( X \), \( F(X) \) denotes the set of all filters on \( X \) and \( P(X) \) the set of all subsets of \( X \).

A convergence structure \( q \) on a set \( X \) is defined to be a function from \( F(X) \) into \( P(X) \) satisfying the following conditions:

For each \( \Phi \) and \( \Psi \) in \( F(X) \),

1. \( x \in q(\hat{x}) \) for each \( x \in X \),
2. if \( \Phi \subset \Psi \), then \( q(\Phi) \subset q(\Psi) \),
3. if \( x \in q(\Phi) \), then \( x \in q(\Phi \cap \hat{x}) \),

where \( \hat{x} \) denotes the ultrafilter containing \( \{x\} \). In this case the pair \( (X, q) \) is said to be a convergence space. If \( x \in q(\Phi) \), we say that \( \Phi \) \( q \)-converges to \( x \). The filter \( V_q(x) \) obtained by intersecting all filters which \( q \)-converge to \( x \) is said to be a \( q \)-neighborhood filter at \( x \). If \( V_q(x) \) \( q \)-converges to \( x \) for each \( x \in X \), then \( q \) is said to be a pretopological convergence structure on \( X \), and \((X, q)\) a pretopological convergence space. The pretopological convergence structure \( q \) is said to be a topological convergence structure if for each \( x \in X \), the filter \( V_q(x) \) has a filter base \( B_q(x) \) with the following property:

\[
y \in G \in B_q(x) \implies G \in B_q(y).
\]

Let \( C(X) \) be the set of all convergence structures on \( X \), partially ordered as follows:

\[
q_1 \leq q_2 \iff q_2(\Phi) \subset q_1(\Phi) \text{ for all } \Phi \in F(X).
\]

If \( q_1 \leq q_2 \), then we say that \( q_1 \) is coarser than \( q_2 \) and \( q_2 \) is finer than \( q_1 \).

For any \( q \in C(X) \), we define the following related convergence structures \( \pi(q) \) and \( \lambda(q) \):

1. \( x \in \pi(q)(\Phi) \iff V_q(x) \subset \Phi \),
2. \( x \in \lambda(q)(\Phi) \iff U_q(x) \subset \Phi \),
where \( U_q(x) \) is the filter generated by the sets \( U \in V_q(x) \) which have the property: \( y \in U \) implies \( U \in V_q(y) \).

In this case \( \pi(q) \) and \( \lambda(q) \) are called the the pretopological modification and the topological modification of \( q \). Also, the pairs \((X, \pi(q))\) and \((X, \lambda(q))\) are called the pretopological modification and the topological modification of \((X, q)\), respectively.

**Proposition 1 ([4]).** Let \((X, q)\) be a convergence space. If \((X, \pi(q))\) and \((X, \lambda(q))\) are the pretopological modification and the topological modification of \((X, q)\), respectively. Then the following statements hold:

1. \( \pi(q) \) is the finest pretopological convergence structure coarser than \( q \),
2. \( \lambda(q) \) is the finest topological convergence structure coarser than \( q \),
3. \( \lambda(q) \leq \pi(q) \leq q \).

Let \( f \) be a map from a convergence space \((X, q)\) to a convergence space \((Y, p)\). Then \( f \) is said to be continuous at a point \( x \in X \), if the filter \( f(\Phi) \) on \( Y \) \( p \)-converges to \( f(x) \) for every filter \( \Phi \) on \( X \) \( q \)-converging to \( x \). If \( f \) is continuous at every point \( x \in X \), then \( f \) is said to be continuous.

We define a set function \( I^\infty_q : P(X) \to P(X) \) for each \( n \in N \cup \{\infty\} \cup \{0\} \), where \( N \) is the set of all positive integers, as follows:

1. \( I^0_q(A) = A, \)
2. \( I^n_q(A) = I_q(A) = \{x \in X \mid A \in V_q(x)\}, \)
3. \( I^{n+1}_q(A) = I_q(I^n_q(A)), \) if \( n \in N, \)
4. \( I^\infty_q(A) = \bigcap \{I^n_q(A) \mid n \in N\}. \)

**Proposition 2 ([5]).** For each \( n \in N \cup \{\infty\} \cup \{0\} \), \( I^n_q \) has the following properties:

1. \( I^0_q(\emptyset) = \emptyset, I^n_q(A) \subset A, \)
2. \( I^0_q(X) = X, \)
3. \( I^n_q(A \cap B) = I^n_q(A) \cap I^n_q(B), \)
4. \( A \subset B \) implies \( I^n_q(A) \subset I^n_q(B) \)

for each \( A, B \subset X \).

But, in general \( I^n_q(I^n_q(A)) \neq I^n_q(A) \) for all \( A \subset X \).
Define $V_q^n(x) = \{ A \subset X \mid x \in I_q^n(A) \}$. Then $V_q^n(x)$ is a filter on $X$ for each $n \in N \cup \{ \infty \}$.

Also, we know that for each $n \in N \cup \{ \infty \}$

$$I_q^n(A) \supset I_q^{n+1}(A) \supset I_q^\infty(A)$$

and

$$V_q^n(x) \supset V_q^{n+1}(x) \supset V_q^\infty(x)$$

for each $x \in X$.

Define a structure $\pi_n(q)$ for each $n \in N \cup \{ \infty \}$ as follows:

$$x \in \pi_n(q)(\Phi) \iff V_q^n(x) \subset \Phi$$

for each $\Phi \in F(X)$.

While, since $V_q^n(x) \subset \hat{x}$, $x \in \pi_n(q)(\hat{x})$ for each $x \in X$. Also, $\Phi \subset \Psi \in F(X)$ implies $\pi_n(q)(\Phi) \subset \pi_n(q)(\Psi)$.

Let $x \in \pi_n(q)(\Phi)$. Then $V_q^n(x) \subset \Phi$. Since $V_q^n(x) \subset \hat{x}$, we obtain $V_q^n(x) \subset \Phi \cap \hat{x}$ and so $x \in \pi_n(q)(\Phi \cap \hat{x})$. Also, $x \in \pi_n(q)(V_q^n(x)) = \pi_n(q)(\pi_n(q)(x))$ for each $x \in X$. Thus $\pi_n(q)$ is a pretopological convergence structure on $X$.

In this case $\pi_n(q)$ is called the $n$-th pretopological modification of $q$.

Also, $(X, \pi_n(q))$ is called the $n$-th pretopological modification of $(X, q)$.

It is not difficult to show that for each $n \in N \cup \{ \infty \}$, the following statements hold:

1. $V_{\pi_n(q)}(x) = V_q^n(x)$ for all $x \in X$.
2. $I_{\pi_n(q)}(A) = I_q^n(A)$ for all $A \subset X$.
3. For each $n \in N$, $q \geq \pi_n(q) \geq \pi_{n+1}(q) \geq \pi_{\infty}(q)$.

3. Main Results

By Proposition 1 and the definition of $\pi_n(q)$, we know that

$q \geq \pi(q) \geq \pi_2(q) \geq \cdots \geq \pi_n(q) \geq \pi_{n+1}(q) \geq \cdots \geq \pi_{\infty}(q) \geq \lambda(q)$.

**Theorem 3.** Let $(X, q)$ be a pretopological convergence space. Then the following are equivalent:

1. $q$ is a topological convergence structure.
2. $I_q$ is idempotent.
PROOF (1) ⇒ (2): It is clear that $I_q(I_q(A)) \subset I_q(A)$ for all $A \subset X$. We will show that $I_q(A) \subset I_q(I_q(A))$. Let $x \in I_q(A)$. Then $A \in V_q(x)$. Since $q$ is a topological convergence structure, there exists $G \in B_q(x)$ such that $G \subset A$, where $B_q(x)$ is a filter base of $V_q(x)$ which has the following property:

$$y \in H \in B_q(x) \text{ implies } H \in B_q(y).$$

Since $y \in G \Rightarrow G \in B_q(y) \subset V_q(y)$, we obtain $y \in I_q(G)$. Thus $I_q(G) = G$. Since $G = I_q(G) \subset I_q(A)$ and $V_q(x)$ is a filter, $I_q(A) \in V_q(x)$. Thus $x \in I_q(I_q(A))$ and so $I_q(A) = I_q(I_q(A))$. That is $I_q$ is idempotent.

(2) ⇒ (1): Take $B_q(x) = \{B \in V_q(x) \mid I_q(B) = B\}$ for each $x \in X$. Since $I_q(X) = X$, we obtain $B_q(x)$ is not an empty collection. Since $\emptyset \notin V_q(x)$, we obtain $\emptyset \notin B_q(x)$. Let $G_i \in B_q(x)$ for $i \in \{1, 2\}$. Then $G_i \in V_q(x)$ and $I_q(G_i) \subset G_i$ for $i \in \{1, 2\}$. Since $G_1 \cap G_2 = I_q(G_1) \cap I_q(G_2) = I_q(G_1 \cap G_2)$ and $V_q(x)$ is a filter, we obtain $G_1 \cap G_2 \in B_q(x)$. Also, let $A \in V_q(x)$. Since $I_q$ is idempotent, $I_q(A) = I_q(I_q(A))$ and $I_q(A) \in V_q(x)$. Thus $I_q(A) \in B_q(x)$. Since $I_q(A) \subset A$, $B_q(x)$ is a filter base of $V_q(x)$. Let $y \in H \in B_q(x)$. Since $H = I_q(H)$, we obtain $y \in I_q(H)$. Thus $H \in B_q(y)$. Therefore $q$ is a topological convergence structure.

PROPOSITION 4 Let $(X, q)$ be a convergence space. Then $\phi(q) = \lambda(q)$ if and only if $I_q$ is idempotent.

PROOF. Assume that $\pi(q) = \lambda(q)$. Since $\pi(q)$ is a pretopological convergence structure and $\pi(q) = \lambda(q)$, $\pi(q)$ is a topological convergence structure. By Theorem 3, $I_{\pi(q)}$ is idempotent. Since $I_{\pi(q)}(A) = I_q(A)$ for all $A \subset X$, $I_q$ is idempotent. Conversely, let $I_q$ be idempotent. By Theorem 3, $q$ is a topological convergence structure. It is clear that $\lambda(q) = q$ if $q$ is a topological convergence structure. We know that $q \geq \pi(q) \geq \lambda(q)$. Thus $q = \pi(q) = \lambda(q)$.

THEOREM 5 Let $(X, q)$ be a convergence space. Then for each $n \in N \cup \{\infty\}$, the following statements are equivalent:

1. $\pi_n(q) = \lambda(q)$,
2. $I_q^n$ is idempotent.
PROOF. (1) ⇒ (2): Assume that \( \pi_n(q) = \lambda(q) \). We will show that \( I^n_q \) is idempotent. Let \( A \subset X \) and \( x \in I^n_q(A) \). Then \( A \in V^n_q(x) \). Since \( \pi_n(q) \) is a topological convergence structure, there exists \( G \in B^n_q(x) \) such that \( G \subset A \), where \( B^n_q(x) \) is a filter base of \( V^n_q(x) \) which has the following property:

\[
y \in H \in B^n_q(x) \text{ implies } H \in B^n_q(y).
\]

Thus \( I^n_q(G) = G \). Since \( G = I^n_q(G) \subset I^n_q(A) \) and \( V^n_q(x) \) is a filter, we obtain \( I^n_q(A) \in V^n_q(x) \). Thus \( x \in I^n_q(I^n_q(A)) \) and so \( I^n_q(A) = I^n_q(I^n_q(A)) \). That is \( I^n_q \) is idempotent.

(2) ⇒ (1): Assume that \( I^n_q \) is idempotent. Let \( B^n_q(x) = \{ G \in V^n_q(x) \mid I^n_q(G) = G \} \) for each \( x \in X \). Since \( I^n_q(X) = X \), we obtain \( X \in B^n_q(x) \). Since \( \emptyset \notin V^n_q(x) \), we obtain \( \emptyset \notin B^n_q(x) \). Let \( G_i \in B^n_q(x) \) for \( i \in \{1, 2\} \). Since \( G_1 \cap G_2 = I^n_q(G_1) \cap I^n_q(G_2) = I^n_q(G_1 \cap G_2) \) and \( V^n_q(x) \) is a filter, we obtain \( G_1 \cap G_2 \in B^n_q(x) \). Also, let \( A \in V^n_q(x) \). Since \( I^n_q \) is idempotent, \( I^n_q(A) = I^n_q(I^n_q(A)) \) and \( I^n_q(A) \in V^n_q(x) \). Thus \( I^n_q(A) \in B^n_q(x) \). Since \( I^n_q(A) \subset A \), \( B^n_q(x) \) is a filter base of \( V^n_q(x) \). Let \( y \in G \in B^n_q(x) \). Since \( H = I^n_q(H) \), we obtain \( y \in I^n_q(H) \). Thus \( G \in B^n_q(y) \). Therefore \( \pi_n(q) \) is a topological convergence structure. Since \( \lambda(q) \) is the finest topological convergence structure coarser than \( q \). That is \( \pi_n(q) = \lambda(q) \).

In that case \( n = \infty \), the proof is similar to in the case \( n \in N \).

DEFINITION 6. Let \((X, q)\) be a convergence space. The length of \( q \) is defined by the smallest positive integer \( n \) satisfying \( I^n_q + 1(A) = I^n_q(A) \) for each \( A \subset X \). We denote \( l(q) = n \).

If \( l(q) \neq n \) for all \( n \in N \) and \( I_q(I_q^\infty(A)) = I_q^\infty(A) \) for all \( A \subset X \), then we denote \( l(q) = \infty \).

THEOREM 7. Let \((X, q)\) be a convergence space and \( n \in N \cup \{\infty\} \). Then the following statements are equivalent:

(1) \( I^n_q \) is idempotent and \( I^m_q \) is not idempotent for \( m < n \).
(2) \( l(q) = n \).
At first we will prove in the case \( n \in N \).

(1) \( \Rightarrow \) (2): Assume that for each \( A \subseteq X \), \( I_q^n(I_q^n(A)) = I_q^n(A) \) and \( I_q^m(I_q^m(B)) \neq I_q^m(B) \) for some \( B \subseteq X \) if \( m < n \). By the definition of \( I_q^n \):

\[
I_q(A) \supset I_q^2(A) \supset \cdots \supset I_q^n(A) \supset I_q^{n+1}(A) \supset \cdots \supset I_q^\infty(A)
\]

Since \( I_q^n(I_q^n(A)) = I_q^n(A) \), we obtain \( I_q^{n+1}(A) = I_q^n(A) \). Suppose that \( I_q^{m+1}(A) = I_q^m(A) \) for \( m < n \). Then \( I_q^n(I_q^m(A)) = I_q^m(A) \) and so \( I_q^m \) is idempotent. This is a contradiction. Thus \( l(q) = n \).

(2) \( \Rightarrow \) (1): Assume that \( l(q) = n \). Then \( I_q^n(A) = I_q^{n+1}(A) = I_q(I_q^n(A)) = I_q^2(I_q^n(A)) = \cdots = I_q^\infty(I_q^n(A)) \). Thus \( I_q^n \) is idempotent. Also, by the definition of \( l(q) = n \), \( I_q^m \) is not idempotent for \( m < n \). In that case \( n = \infty \). By the definition of \( l(q) = \infty \), it is clear that (1) \( \Leftrightarrow \) (2).

**Corollary 8** Let \((X, q)\) be a convergence space and \( n \in N \cup \{\infty\} \). Then \( \pi_n(q) = \lambda(q) \) and \( \pi_m(q) \neq \lambda(q) \) for \( m < n \) iff \( l(q) = n \).

**Proof** By Theorem 5 and Theorem 7.

**Proposition 9** Let \((X, q)\) and \((Y, p)\) be convergence spaces and \( f : (X, q) \rightarrow (Y, p) \) be a map. Then for each \( n \in N \cup \{\infty\} \), the following statements are equivalent:

1. \( f(V_q^n(x)) = V_p^n(f(x)) \) for all \( x \in X \).
2. \( I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B)) \) for each \( B \subseteq Y \).

**Proof** (1) \( \Rightarrow \) (2): Assume that \( f(V_q^n(x)) = V_p^n(f(x)) \) for all \( x \in X \). Let \( x \in I_q^n(f^{-1}(B)) \). Then \( f^{-1}(B) \in V_q^n(x) \) and so \( B \in f(V_q^n(x)) \). Since \( f(V_q^n(x)) = V_p^n(f(x)) \), \( B \in V_p^n(f(x)) \). Thus \( f(x) \in I_p^n(B) \) and so \( x \in f^{-1}(f(x)) \in f^{-1}(I_p^n(B)) \). Therefore \( I_q^n(f^{-1}(B)) \subseteq f^{-1}(I_p^n(B)) \). The reverse inequality is proved by the counter-order.

(2) \( \Rightarrow \) (1): Assume that \( I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B)) \) for each \( B \subseteq Y \). Let \( B \in V_p^n(f(x)) \). Then \( f(x) \in I_p^n(B) \) and so \( x \in f^{-1}(I_p^n(B)) \). Since \( I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B)) \), \( x \in I_q^n(f^{-1}(B)) \). Thus \( f^{-1}(B) \in V_q^n(x) \) and so \( B \in f(V_q^n(x)) \). Therefore \( V_p^n(f(x)) \subseteq f(V_q^n(x)) \). The reverse inequality is proved by the counter-order.
Proposition 10. Let \((X, q)\) and \((Y, p)\) be convergence spaces. Let \(f : (X, q) \rightarrow (Y, p)\) be a map. Then the following statements are equivalent:

1. \(V_p(f(x)) = f(V_q(x))\).
2. \(V_p^n(f(x)) = f(V_q^n(x))\) for each \(n \in \mathbb{N} \cup \{\infty\}\).

Proof. (2) \(\Rightarrow\) (1): It is clear.

(1) \(\Rightarrow\) (2): We will use the mathematical induction to prove above Proposition. Assume that \(V_p^k(f(x)) = f(V_q^k(x))\) and let \(B \in V_p^{k+1}(f(x))\). Then \(f(x) \in I_p^{k+1}(B) = I_p(I_p^k(B))\) and so \(I_p^k(B) \subseteq V_p(f(x)) = f(V_q(x))\).

By assumption and Proposition 9, \(f^{-1}(I_p^k(B)) = I_q^k(f^{-1}(B)) \subseteq V_q(x)\). Thus \(x \in I_q^k(I_q^k(f^{-1}(B)) = I_q^{k+1}(f^{-1}(B))\) and so \(f^{-1}(B) \subseteq V_q^{k+1}(x)\).

Finally, \(B \subseteq f(V_q^{k+1}(x))\). This means \(V_p^{k+1}(f(x)) \subseteq f(V_q^{k+1}(x))\). The reverse inequality is proved by the counter-order.

In that case \(n = \infty\), let \(B \in V_p^\infty(f(x))\). Then \(f(x) \in I_p^\infty(B)\) and so \(f(x) \in I_p^n(B)\) for each \(n \in \mathbb{N}\). Thus \(B \in V_p^n(f(x)) = f(V_q^n(x))\) for each \(n \in \mathbb{N}\). \(B \in \bigcap\{f(V_q^n(x)) \mid n \in \mathbb{N}\} = f(\bigcap\{V_q^n(x) \mid n \in \mathbb{N}\}) = f(V_q^\infty(x))\). Finally, \(V_p^\infty(f(x)) \subseteq f(V_q^\infty(x))\). The reverse inequality is proved by the counter-order.

Definition 11 ([16]). Let \((X, q)\) and \((Y, p)\) be convergence spaces. An onto map \(f : (X, q) \rightarrow (Y, p)\) is said to be open if satisfies the following condition: whenever an ultrafilter \(\Psi\) on \(Y\) \(p\)-converges to \(y\), then for each \(x \in f^{-1}(y)\) there is a filter \(\Phi\) which maps on \(\Psi\) and \(q\)-converges to \(x\).

Proposition 12. Let \((X, q)\) and \((Y, p)\) be convergence spaces. If a map \(f : (X, q) \rightarrow (Y, p)\) is onto, continuous and open, then \(V_p(f(x)) = f(V_q(x))\) for each \(x \in X\).

Proof. Since \(f\) is continuous, \(f(\Phi)\) \(p\)-converges to \(f(x)\) whenever \(\Phi\) \(q\)-converges to \(x\). Thus \(f(V_q(x)) = f(\cap\{\Phi \mid x \in q(\Phi)\}) = \cap\{f(\Phi) \mid x \in q(\Phi)\} \supseteq \cap\{f(\Phi) \mid f(x) \in p(f(\Phi))\} \supseteq f(V_p(f(x)))\). Also, we will claim that \(f(V_q(x)) \subseteq V_p(f(x))\). Let \(B \subseteq f(V_q(x))\). Then \(B = f(A)\) for some \(A \subseteq V_q(x)\). Let \(\Psi\) be an ultrafilter which \(p\)-converges to \(f(x)\). Since \(f\) is open, there is a filter \(\Phi\) such that \(\Phi\) \(q\)-converges to \(x\) and
Since $A \in \Phi$, we obtain $B = f(A) \in f(\Phi) = \Psi$. Thus $B$ is in each ultrafilter which $p$-converges to $f(x)$ and so $B \in V_p(f(x))$. Therefore $f(V_q(x)) \subset V_p(f(x))$.

**Theorem 13** Let $(X, q)$ and $(Y, p)$ be convergence spaces. Let a map $f : (X, q) \to (Y, p)$ be onto, continuous and open. If $I_q^n$ is idempotent, then $I_p^n$ is idempotent.

**Proof** Let $B \subset Y$. Then $f^{-1}(B) \subset X$. Since $I_q^n$ is idempotent, $I_q^n(I_q^n(f^{-1}(B))) = I_q^n(f^{-1}(B))$. By Proposition 9 and Proposition 12, $I_q^n(I_p^n(f^{-1}(B))) = I_p^n(I_q^n(f^{-1}(B))) = f^{-1}(I_p^n(I_q^n(B)))$ and $I_q^n(f^{-1}(B)) = f^{-1}(I_p^n(B))$. Thus $f^{-1}(I_p^n(I_q^n(B))) = f^{-1}(I_p^n(B))$ and so $I_p^n(I_q^n(B)) = I_p^n(B)$. Therefore $I_p^n$ is idempotent.

**Corollary 14** Let $(X, q)$ and $(Y, p)$ be convergence spaces. Let a map $f : (X, q) \to (Y, p)$ be onto, continuous and open. Then $f$ preserves the length of convergence structure.

**Proof** By Corollary 8 and Theorem 13.

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