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A NOTE ON VALUE DISTRIBUTION OF COMPOSITE ENTIRE FUNCTIONS

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ABSTRACT. We discuss the value distribution of composite entire functions including those of infinite order and estimate the number of Q-points of such functions for an entire function Q of relatively slower growth.

1. Introduction

Gross [3] and Ozawa [9] proved that if f and g are transcendental entire functions such that $f(g)$ is of finite order, then $f(g)$ has no finite Picard exceptional value.

Improving this result, Goldstein [2] proved that if f and g are transcendental entire such that $f(g)$ is of finite order, then $\sum_{a\neq\infty}\delta(a;f(g)) < 1$ where $a's$ are complex numbers.

Recently Langley [7] proved that if f and g are transcendental entire such that $f(g)$ is of finite order, then for any nonconstant rational function Q $\delta(0;f(g)-Q)<1$.

In all the above results, the order of $f(g)$ is assumed to be finite,which implies that the order of f is zero (p.53 [5]). In the paper, we prove a result on the value distribution of those composite entire functions whose left factor do not necessarily have the zero order. Throughout the paper, we use standard notations of value distribution theory and the theory of entire functions without any further explanation (cf. [5], [14]). In the paper, we denote by f and g two transcendental entire functions and by $\lambda(f)$ and $\rho(f)$, we denote the lower order and the order of f.

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2. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

LEMMA 1 (cf. p.72 [11]). *For all large values of r*

$$M\big(r, f(g)\big) \geq M\Big(\frac{1}{8}M(r/2, g) - \mid g(0) \mid, f\Big).$$

LEMMA 2 ([8]). *If* $M(r,g) > \frac{2+\epsilon}{\epsilon} \mid g(0) \mid$ *for given* $\epsilon(>0)$ *then* $T\big(r, f(g)\big) \leq (1+\epsilon)T\big(M(r,g), f\big)$. *In particular, if* $g(0) = 0$ *then* $T\big(r, f(g)\big) \leq T\big(M(r,g), f\big)$ *for all* $r > 0$.

LEMMA 3 ([4], [10]). *Let* $F_o, F_1, \ldots, F_m$ *be not identically vanishing entire functions and* $h_o, h_1, \ldots, h_m$ *be arbitrary meromorphic functions* $(m \geq 1)$. *Let* g *be a nonconstant entire function,* K *be a positive real number and* $\{r_j\}_{j=1}^{\infty}$ *be an unbounded monotone increasing sequence of positive real numbers such that for each* j,

$$T(r_j, h_p) \leq KT(r_j, g) \ (p = 0, 1, 2, \ldots, m)$$

and on $\{r_j\}_{j=1}^{\infty}$,

$$T(r, g') \leq \{1 + o(1)\}T(r, g).$$

If F_p *and* h_p $(p = 0, 1, 2, \ldots, m)$ *satisfy* $F_o(g)h_o + F_1(g)h_1 + \cdots + F_m(g)h_m \equiv 0$, *then there exist polynomials in z* $\phi_o, \phi_1, \ldots, \phi_m$ *not all identically zero, such that* $\phi_o F_o + \phi_1 F_1 + \cdots + \phi_m F_m \equiv 0$.

3. The main result

THEOREM 1. *Let* (i) $\rho(f) < \frac{1}{2}$ or $\rho(f)$ *be an irrational number and* (ii) $\lambda(g) < \infty$. *Then for any entire function* Q *with* $T(r,Q) = O\big(T(r,g)\big)$ *at least one of the following two cases holds :*
(a) *the exponent of convergence of the zeros of* $f(g) - Q$ *is not finite,*
(b) $\limsup\limits_{r\to\infty} \dfrac{N\big(r, 0; f(g) - Q\big)}{T(r,g)} = \infty.$

Proof. If possible, let the cases (a) and (b) do not hold. Let β be the canonical product formed with the zeros of $f(g) - Q$, where we choose $\beta \equiv 1$ if $f(g) - Q$ has no zero. Then β is of finite order and there exists an entire function D such that

$$f(g) = Q + \beta \exp(D). \tag{1}$$

From Lemma 1, we get for all large values of r

$$\begin{aligned} M\big(2r, \exp(D)\big) &\geq M\big(\frac{1}{8}M(r,D) - \mid D(0) \mid, \exp(z)\big) \\ &= \exp\big\{\frac{1}{8}M(r,D) - \mid D(0) \mid\big\} \\ &\geq \exp\big\{\frac{1}{9}M(r,D)\big\}, \end{aligned}$$

i.e., $M(r,D) \leq 9 \log M(2r, \exp(D))$.

Now from (1), we get by the first fundamental theorem for all large values of r

$$\begin{aligned} \exp\{T(r,D)\} &\leq M(r,D) \\ &\leq 9 \log M\big(2r, \exp(D)\big) \\ &\leq 45 T\big(3r, \exp(D)\big), \end{aligned}$$

i.e.,

(2) $\quad \exp\{T(r,D)\} \leq 45\big\{T\big(3r, f(g)\big) + T(3r,Q) + T(3r,\beta) + O(1)\big\}.$

In Lemma 2, we put $\epsilon = 1$ and since $M(r,g) > 3 \mid g(0) \mid$ for all large values of r, it follows that

(3) $$T\big(r, f(g)\big) \leq 2T\big(M(r,g), f\big).$$

Since $\rho(f) < \infty$ and $\rho(\beta) < \infty$, it follows from (2) and (3) that for all large values of r

$$\begin{aligned} T(r,D) &\leq \log^+ T\big(3r, f(g)\big) + \log^+ T(3r,Q) + \log^+ T(3r,\beta) + O(1) \\ &\leq \log T\big(M(3r,g), f\big) + O\big(\log T(3r,g)\big) + O(\log r) \\ &\leq \big(\rho(f) + 1\big) \log M(3r,g) + O\big(\log T(3r,g)\big) \\ &= O\big(\log M(3r,g)\big), \end{aligned}$$

i.e.,

(4) $$T(r,D) = O\big(T(4r,g)\big).$$

Let $\lambda(r)$ be a lower proximate order of g. Then
(i) $\lambda(r)$ is nonnegative and continuous for $r \geq r_o$, say,
(ii) $\lambda(r)$ is differentiable for $r \geq r_o$ except possibly at isolated points at which $\lambda'(r-0)$ and $\lambda'(r+0)$ exist,
(iii) $\lim_{r\to\infty} \lambda(r) = \lambda(g)$,
(iv) $\lim_{r\to\infty} r\lambda'(r) \log r = 0$, and
(v) $\liminf_{r\to\infty} \dfrac{T(r,g)}{r^{\lambda(r)}} = 1$.

The existence of such a function $\lambda(r)$ can be proved in the line of [6]. By (v), there exists an unbounded monotone increasing sequence $\{r_j\}_{j=1}^{\infty}$ of positive real numbers such that

$$T(8r_j, g) < 2(8r_j)^{\lambda(8r_j)} = \frac{2(8r_j)^{\{\lambda(g)+1\}}}{(8r_j)^{\{\lambda(g)+1-\lambda(8r_j)\}}}$$

for $j = 1, 2, \ldots$.

Since for all large values of r, $r^{\{\lambda(g)+1-\lambda(r)\}}$ is an increasing function of r (cf. Corollary 1 [6]), we get for $j = 1, 2, 3, \ldots$

$$T(8r_j, g) \leq \frac{2(8r_j)^{\{\lambda(g)+1\}}}{(r_j)^{\{\lambda(g)+1-\lambda(r_j)\}}} = 2^{(3\lambda(g)+4)}(r_j)^{\lambda(r_j)}. \tag{5}$$

Also (v) implies that for all large values of r, $T(r, g) > \frac{1}{2}r^{\lambda(r)}$. So from (4) and (5), we get for $j = 1, 2, \ldots$

$$T(2r_j, D) = O(T(r_j, g)).$$

Since for the entire function g we know that $T(r, g') \leq T(r, g) + o(T(2r, g))$ (cf. Lemma 2.3 [5]), it follows from (5) and the fact that $T(r, g) > \frac{1}{2}r^{\lambda(r)}$ for all large values of r

$$T(r_j, g') \leq \{1 + o(1)\}T(r_j, g) \quad (j = 1, 2, \ldots). \tag{6}$$

Again $T(r, D') \leq T(r, D) + o(T(2r, D)) + O(\log r)$ (cf. Lemma 2.3 [5]) implies that

$$T(r_j, D') = O(\log r_j) + O(T(2r_j, D)) = O(T(r_j, g)) \quad (j = 1, 2, \ldots). \tag{7}$$

Differentiating (1), we get

$$g' f'(g) = Q' + (\beta' + \beta D') \exp(D). \tag{8}$$

From (1) and (8), we obtain

$$g' f'(g) - \{(\beta'/\beta) + D'\}f(g) + Q\{(\beta'/\beta) + D'\} - Q' \equiv 0. \tag{9}$$

Since β is of finite order and the case (b) does not hold, we get for all large values of r

$$\begin{aligned} T(r, \frac{\beta'}{\beta} + D') &\leq T(r, \frac{\beta'}{\beta}) + T(r, D') + O(1) \\ &= N(r, \frac{\beta'}{\beta}) + T(r, D') + O(\log r) \\ &\leq \overline{N}(r, 0; f(g) - Q) + T(r, D') + O(\log r) \\ &= O(T(r, g)) + T(r, D'). \end{aligned}$$

So by (7), we get for $j = 1, 2, \ldots$

$$T(r_j, \frac{\beta'}{\beta} + D') = O(T(r_j, g)). \tag{10}$$

Again we get for all large values of r

$$\begin{aligned} T\big(r, Q(\tfrac{\beta'}{\beta} + D') - Q'\big) &\leq T(r, Q) + T(r, Q') + T(r, \tfrac{\beta'}{\beta} + D') + O(1) \\ &= O\big(T(2r, Q)\big) + T(r, \tfrac{\beta'}{\beta} + D') + O(\log r) \\ &= O\big(T(2r, g)\big) + T(r, \tfrac{\beta'}{\beta} + D'). \end{aligned}$$

Since by (5) and the fact that $2T(r, g) > r^{\lambda(r)}$ for all large values of r it follows that $T(2r_j, g) = O\big(T(r_j, g)\big)$, we get from above in view of (10) that for $j = 1, 2, \ldots$

$$T\Big(r_j, Q\{\frac{\beta'}{\beta} + D'\} - Q'\Big) = O\big(T(r_j, g)\big).$$

This together with (6) and (10) gives from (9) that

$$\phi_1 f' + \phi_2 f + \phi_3 \equiv 0 \tag{11}$$

by Lemma 3, where ϕ_1, ϕ_2, ϕ_3 are polynomials in z, not all identically zero.

Since in view of [12] and [13] (see also [15]), the order of any entire transcendental solution of any first order algebraic differential equation is a rational number not less than $\frac{1}{2}$, (11) shows that $\rho(f)$ is a rational number not less than $\frac{1}{2}$, which contradicts condition (i). This proves the theorem. □

REMARK 1. The condition (i) of the theorem is necessary. For, let $f = z\exp(z) + \psi$, where ψ is a polynomial, $g = (z-1)\exp(z)$ and $Q = \psi(g)$. Then $\rho(f) = 1$ and $T(r, Q) = O\big(T(r, g)\big)$ (cf. p.39 [1]) but $f(g) - Q = (z-1)\exp(g+z)$ has only one zero.

REMARK 2. If in the theorem we suppose that $\rho(g) < \infty$ then it can be proved that $\limsup_{r\to\infty} \frac{N(r,0;f(g)-Q)}{T(r,g)} = \infty$ for any entire function Q with $T(r, Q) = O\big(T(r, g)\big)$.

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