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ADAPTIVE STABILIZATION OF NON NECESSARILY INVERSELY STABLE CONTINUOUS-TIME SYSTEMS BY USING ESTIMATION MODIFICATION WITHOUT USING HYSTERESIS FUNCTION

M. De la Sen

Abstract. This note presents a an indirect adaptive control scheme for first-order continuous-time systems. The estimated plant model is controllable and then the adaptive scheme is free from singularities. The singularities are avoided through a modification of the estimated plant parameter vector so that its associated Sylvester matrix is guaranteed to be nonsingular. That property is achieved by ensuring that the absolute value of its determinant does not lie below a positive threshold. A modification scheme based on the achievement of a modified diagonally dominant Sylvester matrix of the parameter estimates is also given as an alternative method. This diagonal dominance is achieved through estimates modification as a way to guarantee the controllability of the modified estimated model when a controllability measure of the 'a priori' estimated model fails. In both schemes, the use of a hysteresis switching function for the modification of the estimates is not required to ensure the nonsingularity of the Sylvester matrix of the estimates.

I. Introduction

The adaptive stabilization and control of linear continuous and discrete systems has been successfully developed in the last years, [1-4]. Usually, the plant is assumed to be inversely stable and its relative degree and its high-frequency gain sign are assumed to be known together with an absolute upper-bound for that gain in the discrete case. Attempts of relaxing those assumptions have been recently made for continuous systems, [4-5]. The assumption on the knowledge of the order can be relaxed by assuming

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a nominal known order and considering the exceeding modes as unmodelled dynamics, [9-12] and [15]. The assumption on the knowledge of the high frequency gain has been removed in [4] and [13] and the assumption of the plant being inversely stable has been successfully removed in the discrete case and, more recently, in the continuous one [7-11]. The problem is solved by using either excitation of the plant signals (see [4]-[5]) or a modification of the least-squares estimation by either using excitation of the plant signals or exploiting the properties of the standard least-squares covariance matrix, [8-12] and [14]. In a set of papers, the assumption of the plant being inversely stable has been removed by using either excitation of the plant signals or estimates modification by using hysteresis switching functions which generate the controllability of the estimated plant model while exploiting the properties of the covariance matrix, [8-12]. This paper presents an adaptive stabilization algorithm for continuous-time systems which can have unstable zeros. The adaptive scheme uses a parameter modification scheme which neither involves hysteresis switching nor takes advantage of the properties of the covariance matrix while guarantees numerically that the absolute value of the determinant of the Sylvester matrix associated with the parameter estimates is bounded from below by a positive threshold. An alternative modification procedure based upon the achievement of a diagonally dominant Sylvester matrix of the modified estimates is also proposed as an alternative method in the case when a sufficiency test on maintenance of controllability fails. Such a test consists of guaranteeing through the manipulation of matrix norms that the maximum absolute eigenvalue of the Sylvester matrix of the 'a priori' estimated model is bounded above by a finite real finite constant while the minimum one is bounded from below by a positive real constant. The boundedness and convergence of all the estimates and controller parameters is guaranteed in both the ideal perfectly modelled case and when the wide class of unmodelled dynamics and bounded disturbances considered in [11-12] and [15] are present. The plant input and output are bounded and converge to zero in the ideal perfectly modelled case while they are bounded in the above mentioned non-ideal situation. Section II is devoted to the synthesis of the adaptive stabilizer in the perfectly modeled case for unknown continuous-time plants. The used 'a priori' estimation scheme is of least-squares type. The two estimation modification procedures are also given. Section III presents the convergence and stability properties of the proposed scheme. Robustness issues against the presence of unmodelled dynamics and bounded disturbances are discussed in section IV. The mechanism used to guarantee robustness is the variation of the basic 'a priori' estimation scheme by adding a relative dead zone so that the estimation and covariance matrix adaptation are frozen when the adaptation error is small compared to an absolute overbounding function of the contribution to the uncertainties to the output. The modification procedures that ensure controllability of the estimated model are kept as in the ideal case. Section IV is devoted to the scheme's modifications to operate in the case of presence of unmodelled dynamics and/or bounded disturbances. A numerical example is given in section IV and, finally, conclusions end the paper. The mathematical proofs are developed in Appendices A and B.

II. Adaptive stabilizer for a continuous-time plant

In the sequel, the time-argument is suppressed unless confusion can arise and the constant parameters are denoted by a superscript '$*$'. Consider the following continuous-time controllable system

$$(1) \qquad A^*(D)y(t) = B^*(D)u(t) \; ; \; D^i y(0) = y_0^{(i)} \quad (i = 0, 1, \dots, n-1)$$

where $D^i \equiv \frac{d^i}{dt^i}$ $(i = 0, 1, \dots, n-1)$ is the i-th time-derivative operator, $A^*(D) = D^n + \sum_{i=1}^{n} a_i^* D^{n-i}$ and $B^*(D) = \sum_{i=0}^{m} b_i^* D^{m-i}$ with $n \geq m$. Since (1) is controllable then its associated Sylvester matrix

$$S(\theta_0^*) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_1^* & 1 & \cdots & \vdots & b_0^* & \cdots & 0 \\ \vdots & a_1^* & \cdots & 0 & b_1^* & \cdots & \vdots \\ a_n^* & \vdots & \cdots & 1 & \vdots & \cdots & b_0^* \\ 0 & a_n^* & \cdots & a_1^* & b_m^* & \cdots & b_1^* \\ \vdots & 0 & \cdots & \vdots & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & a_n^* & 0 & \cdots & b_m^* \end{bmatrix}$$
$$\qquad\qquad\qquad \uparrow_{m+1} \qquad\qquad \uparrow_{n}$$

is nonsingular. *It is not assumed that* (1) *is inversely stable.* Define the filtered signals:

$$(2) \qquad E^*(D)u_f = u \; ; \; E^*(D)y_f = y \; ; \; E^*(D) = D^n + \sum_{i=1}^{n-1} e_i^* D^{n-i}$$

with $E^*(D)$ being a strictly Hurwitz polynomial. The filtered control law for a known plant (1) is generated as

$$S^*(D)u_f = -R^*(D)y_f, \tag{3}$$

where $S^*(D) = D^n + \sum_{i=1}^{n} s_i^* D^{n-i}$, $R^*(D) = D^n + \sum_{i=0}^{m-1} r_i^* D^{m-i-1}$ satisfy the diophantine equation: $A^*(D)S^*(D) + B^*(D)R^*(D) = C^*(D)$, where $C^*(D) = D^n + \sum_{i=1}^{n^*-1} c_i^* D^{n^*-i}$ of prefixed degree fulfilling the constraint $n^* \le n + \deg\big(S^*(D)\big) \le 2n$ is a strictly Hurwitz polynomial(i.e., with roots in $|z| \le 1$) which defines the suited closed-loop dynamics. $S^*(D)$ and $R^*(D)$ are the unique solution to the above diophantine equations since $A^*(D)$ and $B^*(D)$ are coprime because of the controllability of (1) and the constraints $\deg\big(S^*(D)\big) \le \deg\big(E^*(D)\big) \le n$ and $\deg\big(R^*(D)\big) < \deg\big(A^*(D)\big)$. [In Particular, if $E^*(D)$ satisfies $\deg\big(E^*(D)\big) \le n-1$ then its appropriate coefficients in (2) are zeroed]. Eqn. 3 is equivalent to its unfiltered version:

$$u = \big(E^*(D) - S^*(D)\big)u_f - R^*(D)y_f. \tag{4}$$

The control objective in the adaptive case for unknown plant is to update the controller parameters s_i and r_j $(i = 1, 2, \dots, n; j = 0, 1, \dots, m)$ in an adaptive way so that the plant (1), subject to the control law (4) when replacing the parameters by their estimates, is exponentially stable in the large (i.e., globally asymptotically stable for any bounded initial condition of (1)) according to a prescribed nominal closed-loop dynamics specified by the strictly Hurwitz polynomial. Simple direct calculus with (1)-(2) yields for filtered signlas:

$$D^n y_f = \theta^{*^T} \varphi \tag{5}$$

with

$$\begin{aligned} \theta^* &= [\theta_0^{*^T} : \varepsilon_0^{*^T}]^T \\ &= [\theta_1^*, \theta_2^*, \dots, \theta_{n+m+1}^* : \theta_{n+m+2}^*, \theta_{n+m+3}^*, \dots, \theta_{2n+m+1}^*]^T \\ &= [b_0^*, b_1^*, \dots, b_m^*, a_1^*, a_2^*, \dots, a_n^* : \varepsilon_{01}^*, \varepsilon_{02}^*, \dots, \varepsilon_{0n}^*]^T \end{aligned} \tag{6.a}$$

$$\begin{aligned} \varphi(t) &= [\varphi_0^T(t), i_\varphi^T(t)]^T \\ &= [D^m u_f, D^{m-1}u_f, \dots, u_f, -D^{n-1}y_f, -D^{n-2}y_f, \dots, \\ &\quad - y_f, i_1, i_2, \dots, i_n]^T, \end{aligned} \tag{6.c}$$

where $g(t) = \varepsilon_0^T(t)i(t)$ is an exponentially decaying term that depends on initial conditions and each $i_j(t)$ is known and it has the form $t^l e^{\lambda_k^* t}$ for $l = 0, 1, ..., m_k - 1$ with m_k being the multiplicity of the root λ_k^* of $C^*(D)$. There are m_k terms $i_{(.)}(t)$ of such a form for each λ_k^*. The parameter vector θ^* is estimated 'a priori' by using an standard least-squares algorithms of covariance matrix $P(t)$ and estimated vector $\theta(t) = \left(\theta_0^T(t), \varepsilon_0^T(t)\right)^T$ with $\varepsilon_0(t)$ being the estimation of the initial conditions of ε_0^*. The estimation algorithm consists of an 'a priori' estimation and an 'a posteriori' modification of the 'a posteriori' estimates as follows:

'A priori' Estimation

(7.a) $$e = D^n y_f - \theta^T \varphi \quad \text{(prediction error)}$$

(7.b) $$\dot{\theta} = P\varphi e$$

(7.c) $$\dot{P} = -P\varphi\varphi^T P \; ; \; P(0) = P^T(0) > 0$$

The basic 'a posteriori' modification of the estimated plant model is performed when necessary to maintain the controllability of the estimated model in the sense that $\left|\mathrm{Det}\left(S(\bar{\theta}_0)\right)\right| \geq \rho \geq 0$ even if $\left|\mathrm{Det}\left(S(\theta_0)\right)\right| < \rho$ for as prefixed positive real constant ρ while the Sylvester matrices of the 'a priori' and modified estimates are :

$$S(\theta_0) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_1 & 1 & \cdots & \vdots & b_0 & \cdots & 0 \\ \vdots & a_1 & \cdots & 0 & b_1 & \cdots & \vdots \\ a_n & \vdots & \cdots & 1 & \vdots & \cdots & b_0 \\ 0 & a_n & \cdots & a_1 & b_m & \cdots & b_1 \\ \vdots & 0 & \cdots & \vdots & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & a_n & 0 & \cdots & b_m \end{bmatrix}$$

$$\begin{matrix} \uparrow & & \uparrow \\ m+1 & & n \\ \downarrow & & \downarrow \end{matrix}$$

$$S(\bar{\theta}_0) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \bar{a}_1 & 1 & \cdots & \vdots & \bar{b}_0 & \cdots & 0 \\ \vdots & \bar{a}_1 & \cdots & 0 & \bar{b}_1 & \cdots & \vdots \\ \bar{a}_n & \vdots & \cdots & 1 & \vdots & \cdots & \bar{b}_0 \\ 0 & \bar{a}_n & \cdots & \bar{a}_1 & \bar{b}_m & \cdots & \bar{b}_1 \\ \vdots & 0 & \cdots & \vdots & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & \bar{a}_n & 0 & \cdots & \bar{b}_m \end{bmatrix}.$$

The modification scheme to calculate $\bar{\theta}$ from θ is implemented according to the following scheme:

Basic 'A posteriori' Modification of the Estimation

The plant parameter estimates are modified 'a posteriori' as follows. First define the nonnegative scalars δ_α and α :

(8.a)
$$\delta_\alpha = \begin{cases} \dfrac{\rho+\varepsilon-\left|\mathrm{Det}\big(S(\theta_0)\big)\right|\mathrm{Sign}(\bar{C})\mathrm{Sign}\big(\mathrm{Det}(S(\theta_0))\big)}{|\bar{C}|} & \text{if} \quad |\mathrm{Det}S(\theta_0)| < \rho \\ 0 & \text{if} \quad |\mathrm{Det}S(\theta_0)| \geq \rho \end{cases}$$

(8.b)
$$\alpha = \begin{cases} \delta_\alpha & \text{if } \delta_\alpha \geq 1 \\ \delta_\alpha^{\frac{1}{n-m}} & \text{if } \delta_\alpha < 1 \end{cases}$$

any prefixed real constants ε and ρ and

(8.c)
$$\begin{aligned} \bar{C} = \Big\{ & C(\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_{n+m+1}) \\ & : |C(\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_{n+m+1})| = \max_{\sigma_i \in \{0,-1,1\}} |C(\sigma_1, \sigma_2, \cdots, \sigma_{n+m+1})| \Big\} \end{aligned}$$

$$\begin{aligned} & (\bar{\sigma}_1, \bar{\sigma}_2, \cdots, \bar{\sigma}_{n+m+1}) \\ & \quad = \Big\{ \mathrm{Arg}(\sigma_1, \sigma_2, \cdots, \sigma_{n+m+1}) : \bar{C} = C(\sigma_1, \sigma_2, \cdots, \sigma_{n+m+1}) \\ & \text{and} \quad \sigma_i \in \{0,-1,1\} \ ; \ i = 1, 2, \ldots, n+m+1 \Big\} \end{aligned}$$

(8.d)
$$\begin{aligned} & C(\sigma_1, \sigma_2, \cdots, \sigma_{n+m+1}) \\ & = \sum_{k=1}^{n+m} \sum_{i_1, i_2, \cdots, i_k = 1}^{n+m+1} \frac{1}{k!} \mathrm{Trace}\big(S_{\theta_{i_1}}(\theta_0) \tilde{S}_{\theta_{i_1} \ldots \theta_{i_k}}(\theta_0)\big) \prod_{j=i_1}^{i_k} [\sigma_j], \end{aligned}$$

where $\tilde{S}(\theta_0)$ is the matrix of cofactors of $S(\theta_0)$, and the first-order derivatives with respect to the parameter estimates are:

(8.e)
$$\begin{aligned} & S_{a_i}(\theta_0) = \frac{dS}{da_i}\Big|_{\theta_0} \\ & = \begin{bmatrix} 0_i \times (m+n+1) \\ \cdots \\ I_{m+n+1-i} 0_{(m+n+1-i)x(m+n+1)} \end{bmatrix} \leftarrow (i+1)\text{th row} \quad (i = 1, \cdots, n), \\ & S_{b_j}(\theta_0) = \frac{dS}{db_j}\Big|_{\theta_0} \\ & = \begin{bmatrix} 0_{(j+1)\times(m+n+1)} \\ \cdots \\ 0_{(m+n-j)\times(m+n+1)} I_{m+n-j} \end{bmatrix} \leftarrow (j+2)\text{th row} \quad (j = 0, 1, \cdots, m), \end{aligned}$$

$$\bar{\theta} = \theta + \bar{\delta}, \tag{9.a}$$

$$\begin{aligned}\bar{\delta} &= [\delta\theta_1,\ \delta\theta_2,\ \cdots,\ \delta\theta_{n+m+1},\ 0,\ \cdots,\ 0]^T \\ &= [\delta b_0,\ \delta b_1,\ \ldots,\ \delta b_m,\ \delta a_1,\ \delta a_2,\ \ldots,\ \delta a_n,\ 0,\ \ldots,\ 0]^T,\end{aligned} \tag{9.b}$$

$$\begin{aligned}&\bar{a}_i = a_i + \delta a_i = a_i + \alpha\bar{\sigma}_i\ ; \\ &\bar{b}_j = b_j + \delta b_j = b_j + \alpha\bar{\sigma}_j\ ; \\ &i = 1,2,\ldots,n\ ;\ j = 0,1,\ldots,m.\end{aligned} \tag{9.c}$$

Note that the modification of the estimates (8)-(9) ensures that all the parameters maintain their signs and vary the same percentage after modification with respect to their 'a priori' values provided such values are nonzero. Eventual zero 'a priori' estimates are all modified to yield the same positive 'a posteriori' values. Note that in (8.c), obtained from (8.d), is calculated for all the possible combinations of plus /minus signs and nonzero /zero values for all the 'a priori' estimates in order to include all the tentatively possible modifications being either zero or of modulus α. Such a philosophy leads to the calculation of the smaller α in (8.a) which is used in (9.a)-(9.c) for modification the 'a priori' estimates. The above modification procedure basically operates as follows. Assume that θ_i is any 'a priori' estimate $a(.)$ or $b(.)$. If $\sigma = 0$ then such a parameter does not contribute to the maximum C(i.e., to $\bar{C}$). That means that if the parameter were accounted for in (8.c) for eventual parameter modification with $\sigma = 1$, then C would have less absolute value. If $\sigma = 1$, then the parameter contributes to $\bar{C}$, i.e., if it is accounted for to calculate $\bar{C}$ which reaches a larger absolute value than for any other possibilities for accounting or not for all the remaining parameter estimates. The choices of δ_α and α versus time in eqns. (8.a)-(8.b) have the role of guaranteeing a technical step in the proof of Proposition 1 supplied in Appendix A. At the end of the modification procedure, all the estimates whose corresponding $\bar{\sigma}_{(.)}$ is 1 become modified while those ones whose corresponding $\bar{\sigma}_{(.)}$ is zero remain unmodified as a result of the proposed modification scheme. It is proved in Appendix A as an intermediate step in the proof of the subsequent controllability result that $\bar{C} \neq 0$ at all time because not all the derivatives in (8.e) with respect to the estimates evaluated at the 'a priori' estimated parameter vector are zero. This feature makes possible that the Sylvester determinant of the modified estimates can always be modified with respect to its 'a priori' value. It becomes obvious from the above modification philosophy that $|\bar{C}|$ can be replaced by any value of $|C|$ which be bounded from below by a positive constant. Then, the modification procedure also works at the expense of computing large estimates variations from the modification procedure. The above procedure ensures the controllability of the 'a posteriori' *estimated plant model as emphasized in the subsequent result :*

PROPOSITION 1. *The modified estimation scheme* (8)-(9) *of the 'a priori' estimated plant model from* (7) *fulfills at all time* $|\mathrm{Det}(S(\theta_0))| \geq \rho \geq 0$ *so that such a model is controllable.*

REMARK 1. A first direct variation of the above estimation modification rule of (8)-(9) consists of modifying the calculation of δ_α in (8.a) so that the sign of the Sylvester determinant of the modified estimates is the same as that of the 'a priori' one when this one is nonzero without constraining the maintenance of the signs of the estimates after the estimates modification. Eqn. (8.a) is replaced with

$$\text{(8.a')} \qquad \delta_\alpha = \frac{(\rho+\varepsilon)\bar{s}(\mathrm{Det}(S(\theta_0))) - \mathrm{Det}(S(\theta_0))}{\bar{C}}$$

where $\bar{s}$ is a real function which is zero if its argument is zero and defined by the sign of its argument otherwise. The above choice guarantees trivially that $|\mathrm{Det}(S(\bar{\theta}_0))| = \rho$ in the case when $|\mathrm{Det}(S(\theta_0))| \leq \rho$ Proposition 1 holds under a very similar proof, which is omitted, with the replacement of (8.b) by (8.b').

A second variation of the above estimation modification rule of (8)-(9) is given below by modifying the algorithm rules (8) and (9.c). It is based on ensuring that the Sylvester matrix of the modified estimates is diagonally dominant in the case that obtained from 'a priori' estimates is not guaranteed to be controllable by the application of a sufficiency test. Such a test is based on the evaluation of matrix norms of $S(\theta_0)$ and it does not requires the computation of its eigenvalues. First establish Condition 1 for controllability test purposes of the 'a priori' estimated model.

Condition 1. $(n+m+1)^{\frac{1}{2}}||S(\theta_0)||_\infty \geq \frac{1}{\varepsilon_0'}$ and $(n+m+1)^{-\frac{1}{2}}||S(\theta_0)||_1 \leq \frac{1}{\varepsilon_0}$ for some design prefixed real constants $\varepsilon_0' \geq \varepsilon_0 > 0$.

Condition 1 guarantees that all the absolute values eigenvalues of the Sylvester matrix of the 'a priori' estimated model are positive and upper-bounded by a finite constant. As a result, Condition 1 guarantees that $|\mathrm{Det}(S(\theta_0))|$ is nonzero. These key points will be then established in Proposition 2.

Alternative 'A posteriori' Modification of the Estimation

The parameter estimates are now modified as follows. Modify (9.c) as follows :

(10.a)

$$\delta_{a_i} = \begin{cases} 0 & \text{if Condition 1 holds} \\ -\alpha a_i & \text{otherwise} \end{cases} \; ; \; \delta_{b_j} = \begin{cases} 0 & \text{if Condition 1 holds} \\ -\alpha b_j & \text{otherwise} \end{cases}$$

$(i = 0,\ 1,\ \cdots,\ n\ ;\ j = 0,\ 1,\ \cdots,\ m-1)$,

(10.b)

$$\delta_{b_m} = \begin{cases} 0 & \text{if Condition 1 holds} \\ \beta b_m & \text{if Condition 1 does not hold and} \quad |b_m| \geq \varepsilon \\ \beta' \mathrm{sign}(b_m) & \text{if Condition 1 does not hold and} \quad 0 < |b_m| < \varepsilon \\ \beta' & \text{if Condition 1 does not hold and} \quad b_m = 0, \end{cases}$$

where

$$\text{(11.a)} \qquad \alpha = \frac{\sum_{i=1}^{n} |a_i| + \sum_{i=0}^{m} |b_i| + \rho_\alpha - 1}{\sum_{i=1}^{n} |a_i| + \sum_{i=0}^{m} |b_i|},$$

(11.b)

$$\beta = \begin{cases} \frac{1}{|b_m|}\{(1-\alpha)[\sum_{i=1}^{n} |a_i| + \sum_{i=1}^{m-1} |b_i| + \gamma|b_0|] + \rho_\beta\} - 1 \\ \qquad \text{if} \quad \varepsilon < |b_m| \leq \sum_{i=1}^{n} |a_i| + \sum_{i=1}^{m-1} |b_i| + \gamma|b_0| + \rho_\beta \\ 0 \\ \qquad \text{if} \quad |b_m| > \mathrm{Max}\Big(\varepsilon,\ \sum_{i=1}^{n} |a_i| + \sum_{i=1}^{m-1} |b_i| + \gamma|b_0|] + \rho_\beta\Big) \end{cases}$$

$$\text{(11.c)} \quad \beta' = \sum_{i=1}^{n} |a_i| + \sum_{i=1}^{m-1} |b_i| + \gamma|b_0| + \rho'_\beta - |b_m| \ ; \ \gamma = \begin{cases} 0 & \text{if} \quad m = n \\ 1 & \text{if} \quad m < n \end{cases}$$

for prefixed given constants $\rho_\alpha \in (0,1)$; $\rho_\beta > 0$; $\varepsilon'_0 > \varepsilon_0 > 0$; $\rho'_\beta \geq \varepsilon > 0$; $\alpha \in [0,1]$; $\beta > 0$ and $\beta' > 0$.

The above alternative modification procedure (9.a)-(9.b), subject to (10)-(11), [instead that to (8) and (9.c)] to the estimation modification procedure (8)-(9) ensures the suitable fact that the 'a priori' b_m-parameter estimate maintains its sign after eventual modification provided that its 'a priori' estimation is nonzero. It is always guaranteed that the Sylvester matrix of the modified estimates satisfies $|\mathrm{Det}(S(\bar{\theta}_0))| > 0$ at all time since it is diagonally dominant irrespective of its 'a priori' value in the case that the 'a priori' $|\mathrm{Det}(S(\theta_0))|$ is not ensured to be nonzero. Note that Condition 1 could be directly replacing the modification algorithm by the alternative condition $|\mathrm{Det}(S(\bar{\theta}_0))| > \rho > 0$ which is implied by Condition 1. A good practical strategy to apply coherently Condition 1 is the use of very large values for ε_0' and very small ones for ε_0. However, this last condition is numerically more cumbersome to test especially for high-order systems.

The subsequent result is proved in Appendix A:

PROPOSITION 2. *If* Condition 1 *holds then the 'a priori' estimated plant model is controllable and its associate* Sylvester *matrix is nonsingular. The alternative modification scheme* (10)-(11) *of the 'a priori' parameter estimates* (7) *satisfies* Proposition 1.

Stabilizing Adaptive Control Law Introducing (9.a) into (7.a), we obtain :

$$\begin{aligned} D^n y_f &= e + \theta^T \varphi \\ &= e + (\bar{\theta}^T - \bar{\delta}^T)\varphi \\ &= e + A(D,t)y_f + B(D,t)u_f + \varepsilon_0^T(t) i_\varphi(t) \end{aligned} \tag{12}$$

with $A(D,t)$ and $B(D,t)$ being time-varying polynomials of the 'a priori' estimates which define the 'a priori' estimated model of the plant and whose adjustable parameters are the components of the ' a priori' estimated vector θ. *The filtered and unfiltered control inputs are generated from the adaptive version of* (3)-(4),

$$S(D,t)u_f = -R(D,t)y_f \tag{13}$$

$$u = \big(E^*(D) - S(D,t)\big)u_f - R(D,t)y_f \tag{14}$$

so that the following closed-loop diophantine equation is satisfied by the controller polynomials $R(D)$ and $S(D)$ which are calculated from modified parameter estimates:

$$\bar{A}(D,t)S(D,t) + \bar{B}(D,t)R(D,t) = C^*(D) \tag{15.a}$$

with $\bar{A}(D,t) = A(D,t) + \delta A(D,t)$ and $\bar{B}(D,t) = B(D,t) + \delta B(D,t)$ and $\delta A(D,t) = \sum_{i=1}^{n} \delta a_i D^{n-i}$ and $\delta B(D,t) = \sum_{i=0}^{m} \delta b_i D^{m-i}$ with $n \geq m$.

The solution is unique since the modified plant parameter estimated model is controllable at all time what implies that the time-varying polynomials and $\bar{A}(D,t)$ and $\bar{B}(D,t)$ are coprime at all time.

Calculation of the Parameters of the Adaptive Stabilizer

The expression (15.a) is equivalent to the following algebraic linear system

(15.b) $$S(\bar{\theta}_0)v = c^*$$

for all time with

(15.c) $$v = \left[1,\ s_1,\ \dots,\ s_n,\ r_0,\ r_1,\ \dots,\ r_{m-1}\right]^T$$
$$c^* = \left[1,\ c_1^*,\ c_2^*,\ \dots,\ c_{n^*}^*\right]^T$$

which is uniquely solvable with updated parameters at all time in $s_{(.)}$ and $r_{(.)}$ which are used to generate the filtered plant input (3) so that the reference closed-loop dynamics characteristic equation is $C^*(D) = 0$.

III. Stability Results

The properties of the adaptive control scheme are given in the subsequent main result of this paper which is proved in Appendix A:

THEOREM 1. *The adaptive control law* (13)-(14), *subject to the estimation scheme* (7)-(9) [*or* (7), (9.a) *and* (10)-(11)] *and* (15), *has the following properties when applied to the plant* (1) *provided that* Assumption 1 *holds:*
(i) θ *and* $\bar{\theta}$ *are uniformly bounded and the modified estimated plant model is controllable at all time.*
(ii) e *and* $P\varphi$ *are in* L_2.
(iii) $\theta, P, \bar{\theta}, s_i$ *and* r_j $(i = 1,2,\dots,n\ ;\ j = 0,1,\dots,m-1)$ *converge asymptotically for any bounded initial conditions for the plant and the estimation algorithm.*
(iv) $D^i u_f, D^i y_f$ $(i = 0,1,\dots,n-1)$ *and* u *and* y *are uniformly bounded and converge asymptotically to zero.*

Note that $e \in L_2 \cap L_\infty$ so that $e \to 0$ as $t \to \infty$ and $\theta \varepsilon L_\infty$ and converges to a finite limit. These features guarantee that $\mathrm{Det}(S(\theta_0))$ and θ_0 care bounded and converge to finite limits so that the modification $\bar{\delta}$ is bounded and converges for both proposed modification schemes (8)-(9) and (9.a) and (10)-(11). The result also stands if (8.a') replaces (8.a).

IV. Robustness Issues

If the plant is not perfectly modelled and/ or it is subject to bounded disturbances then, it is modelled as

$$A^*(D)y = B^*(D)u + \eta \tag{16}$$

which after filtering through $E^*(D)$ becomes once included the effect of initial conditions :

$$A^*(D)y_f = B^*(D)u_f + \eta_f + \varepsilon_0^T(t)i_\varphi(t) \tag{17}$$

with $\eta_f = \frac{1}{E^*(D)}\eta(t)$. The following standard hypothesis on (16)-(17) is given (see, for instance, [11-12] and [15]):

ASSUMPTION 1. *Assume that* (1) *is controllable when* $\eta \equiv 0$ *and that an overbounding function* $\bar{\eta}_f$ *of the absolute value of the filtered unmodelled dynamics contribution* $\eta_f(t) = \frac{1}{E^*(D)}\eta(t)$ *to the filtered output is known.*

$|\bar{\eta}_f(t)| \geq \eta_f(t)$ for all $t \geq 0$ can be calculated under the form $\bar{\eta}_f(t) = \varepsilon_1\rho(t) + \varepsilon_2$ for some nonnegative real constants ε_i $(i = 1,2)$ which are assumed to be known when $\eta_f(t)$ is the sum of a bounded term plus a term related to u_f by a strictly proper exponentially stable transfer function with $\rho(t) = K \sup_{o\leq\tau\leq t}\left\{||v(\tau)e^{-\sigma_0(t-\tau)}||\right\}$ for positive real constants K and σ_0 and $v^T(t) = \left[D^{n-1}u_f, D^{n-2}u_f, \ldots, Du_f, u_f, D^{n-1}y_f, D^{n-2}y_f, \ldots, Dy_f, y_f\right]$ (see [11-12] and [15]). In the case when ε_i $(i = 1,2)$ are unknown they can be estimated by extending the number of parameters in the estimation scheme, [20]. The 'a priori' estimation scheme is modified with respect to eqns. 7 as follows:

$$e = D^n y_f - \theta^T\varphi = -\tilde{\theta}^T\varphi + \eta_f \tag{18.a}$$

$$\dot{\theta} = bP\phi e \tag{18.b}$$

(18.c) $\dot{P} = -bP\phi\phi^T P$ [Covariance adaptation] $P(0) = P^T(0) > 0$

$$b := \frac{gs}{1 + \gamma\phi^T P\phi} \tag{18.d}$$

with $g : [0,\infty) \to [g_1, g_2]$; $\gamma : [0,\infty) \to [\gamma_1,\gamma_2]$ for some arbitrary real constants g_i, γ_i $(i = 1,2)$ with $0 < g_1 \leq g_2 < \infty$; $0 < \gamma_1 \leq \gamma_2 < \infty$, and

$$\text{(18.e)} \quad s := \begin{cases} 0 & \text{if} \quad t\varepsilon I_1 := \{t\varepsilon R_0^+ : |e| < \mu\bar{\eta}_f\} \\ f(\mu\bar{\eta}_f, e)/e & \text{otherwise} \quad \text{(i.e.,} \quad \text{for} \quad t\varepsilon I_2 := R_0^+ - I_1), \end{cases}$$

where $\mu : [0, \infty) \to [\mu_1, \mu_2]$ for some arbitrary real constants μ_i $(i = 1, 2)$ fulfilling $\infty > \mu_2 \geq \mu_1 \geq 1$; and $f : [0, \infty) \to R$ is defined by

$$\text{(18.f)} \qquad f(\sigma, e) := \begin{cases} e - \sigma & \text{if} \quad e > \sigma \\ 0 & \text{if} \quad |e| \leq \sigma \\ e + \sigma & \text{if} \quad e < -\sigma. \end{cases}$$

The modifications schemes to guarantee the controllability of the estimated palnt model are identical to those given in Section III (see (8) to (11)). The above scheme is based on an adaptation relative dead zone to freeze the 'a priori' parameter estimates and covariance matrix in the case when the adaptation error is smaller than the absolute overbounding function for the unmodelled dynamics contribution to the plant output (see (18.d)-(18.f)). This technique guarantees boundedness of the 'a priori' and modified estimates as well as closed-loop stability. The set of results given in Theorem 1 for the ideal case are now modified as given in the subsequent result:

THEOREM 2. *The subsequent propositions hold:*
(i) $\theta \in L_\infty$ *and* $\bar{\theta} \in L_\infty$
(ii) $\bar{f} = \frac{f}{\sqrt{1+\gamma\varphi^T P\varphi}} \in L_2 \cap L_\infty$; $||\dot{\theta}|| \in L_\infty \cap L_2$ *and* θ *converges to a finite limit.*
(iii) $P\varphi\varepsilon L_2$ *and* $b|\eta_f^2 - e^2| \in L_1 \cap L_\infty$
(iv) $D^i u_f$ *and* $D^i y_f$ *and* $(i = 0, 1, \ldots, n-1)$ *and* u *and* y *are uniformly bounded.*

The proof of Theorem 2 is very similar to that of Theorem 1 and it is outlined in Appendix B.

V. Numerical Example

A numerical example is now tested for a nominally unstable and inversely unstable plant (1) parameterized by $A^*(D) = 1 + D + 2D^2 + 3D^3 + 4D^4$ and $B^*(D) = 1 + D + 2D^2 + 3D^3$ with initial conditions $(-5, -7, 0, 0)^T$ with filter parameter $E^*(D) = (D + 6.93)^2$. The unmodelled dynamics is defined by a second-order differential equation $\ddot{\eta} + 0.12\eta - 7.8 = 7.8u$. The estimation-modification algorithm used is that of (7)-(9) with the replacement of (8.b) with (8.b'). The determinant threshold for parameter modification of the estimates is $\rho = 0.15$ and $\varepsilon = 10^{-5}$. The adaptive stabilizer satisfies the constraints $\deg(R(D)) = \deg(S(D)) - 1 = 1$. The estimation algorithm of the adaptive regulator is initialized as follows :

$$b_0(0) = 1,\ b_1(0) = -0.8,\ b_2(0) = -0.3,$$
$$a_1(0) = 0.5,\ a_2(0) = -0.5,\ a_3(0) = 0.$$

b_3 is assumed to be known and thus deleted from the estimation algorithm. The estimates of the initial conditions of the plant (1) are all zero. The covariance matrix is initialized to $P(0) = \text{Diag}(10^6)$ and $g = \gamma = 1, \mu = 1.04$. The absolute overbounding of the unmodelled dynamics contribution is computed with constants $K = 1, \varepsilon_1 = 1,\ \varepsilon_2 = 10^{-5}$, and $\sigma_0 = 0.1$. The output and input versus time are shown on Figs. 1-2, respectively. Figures 3a and 3b show the 'a priori' and modified parameter estimates of b_2, respectively. Figure 4 displays the dead-zone adaptation function $s(t)$, i.e., no adaptation takes place when $s(t)$ is zero since the estimated level of unmodelled dynamics contribution is large against the adaptation error value.

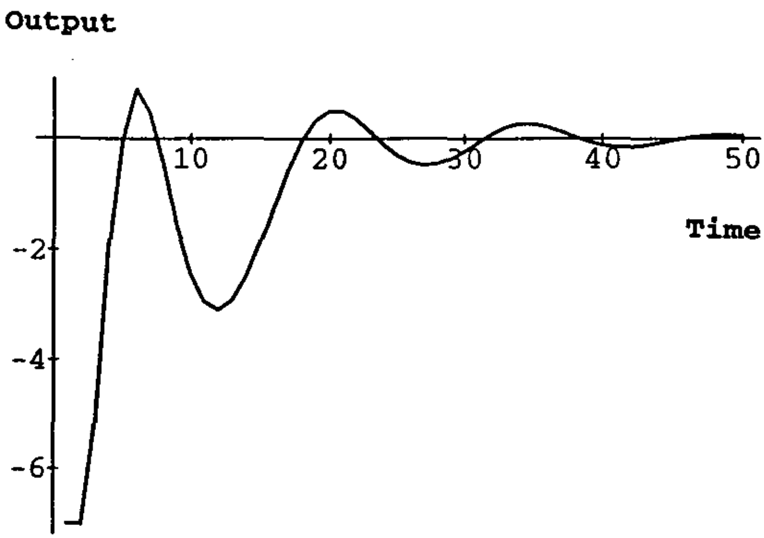

Fig. 1.

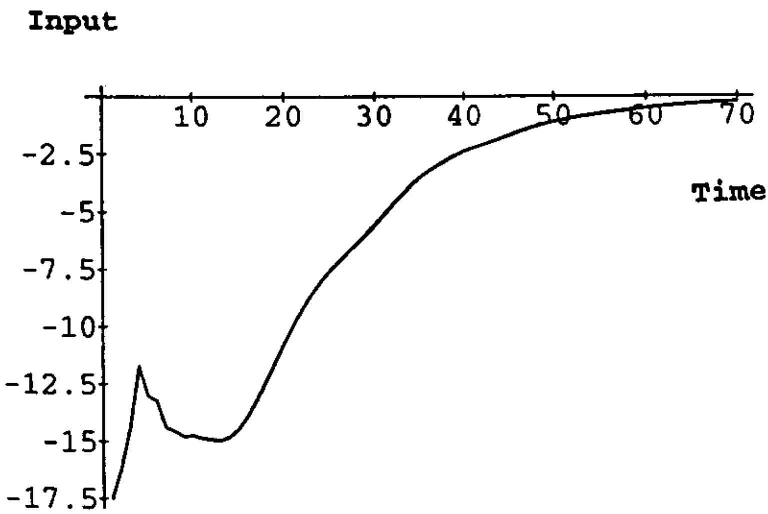
Input
10
20
30
40
50
60
70
-2.5
-5
-7.5
-10
-12.5
-15
-17.5
Time

Fig. 2.

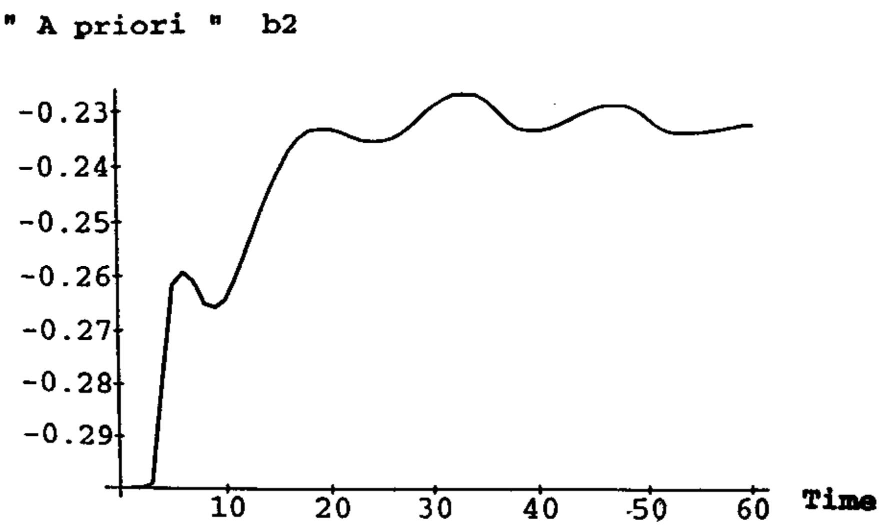
" A priori " b2
-0.23
-0.24
-0.25
-0.26
-0.27
-0.28
-0.29
10
20
30
40
50
60
Time

Fig. 3a.

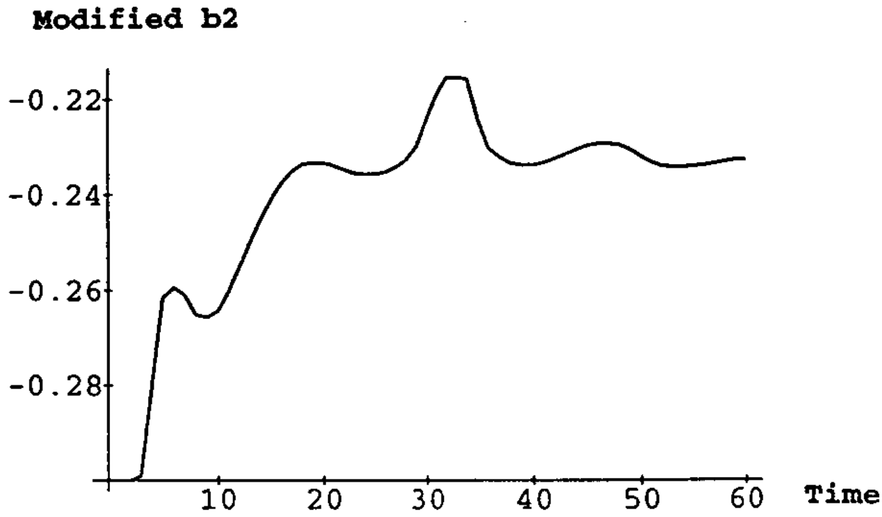
Modified b2
-0.22
-0.24
-0.26
-0.28
10
20
30
40
50
60
Time

Fig. 3b.

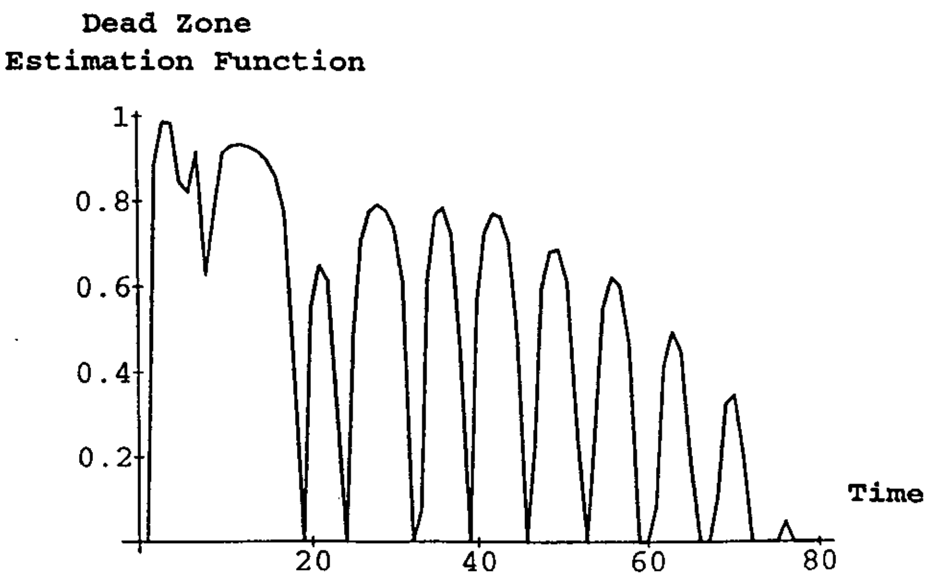
Dead Zone
Estimation Function
1
0.8
0.6
0.4
0.2
20
40
60
80
Time

Fig. 4.

VI. Conclusions

An adaptive stabilizer for a continuous-time plant possibly possessing a unstable inverse has been proposed without assuming the inverse stability of the plant, 'a priori' knowledge on the plant parameters and knowledge of the high-frequency gain sign. The adaptive stabilizer is of pole-placement type. It consists of an 'a priori' parameter estimation algorithm with covariance matrix adaptation with a subsequent 'a posteriori' parameter estimation modification of the parameter estimates. Two modification schemes have been proposed which ensure the controllability of the modified estimated plant model. The mechanism which guarantees the controllability of the modified estimated plant model consists basically of the perturbation of some of the 'a priori' estimated plant parameters so that the resulting modified Sylvester matrix becomes nonsingular when the 'a priori' fails against some appropriate numerical test about nonsingularity. The estimation scheme has suitable stability and convergence properties and the resulting closed-loop scheme is asymptotically stable in the large.

Appendix A

Proof of Proposition 1. First, note that the first-order derivatives of the determinant with respect to any parameter estimate are calculated as follows from elementary algebra (see, for instance, [19]):

$$\text{(A.1)} \qquad \frac{\partial}{\partial \theta_i}\text{Det}(S(\theta_0)) = \text{Trace}\Big(\frac{\partial S(\theta_0)}{\partial \theta_i}\tilde{S}(\theta_0)\Big)$$

which holds when taking derivatives of determinants with respect to any 'a priori' parameter estimate θ_i for $i = 1, 2, \dots, n+m+1$. The derivatives are evaluated at θ_0. However, it is clear from (8.e) that $S_{\theta_{i_1},\dots,\theta_{i_k}} = \frac{\partial^k S(\theta_0)}{\partial^k \theta_{i_1} \dots \partial^k \theta_{i_k}} = 0$; $k = 2, 3, \dots, n+m+1$ with all the partial derivatives being evaluated at θ_0. Also, since $\tilde{S}(\theta_0)$ is a matrix of cofactors, it contains products of at most $(n+m)$ parameters at each one of its entries so that $\tilde{S}_{\theta_{i_1},\dots,\theta_{i_k}}(\theta_0) = 0$ if $k > n+m$ for any integers $i_j \geq 1$ for $j = 1, 2, \dots, k$. Now, $\text{Det}(S(\bar{\theta}_0))$ is expanded in Taylor series around $\text{Det}(S(\theta_0))$ by taking successive derivatives with respect to parameter components evaluated at θ_0 by starting with (A.1) while zeroing any derivatives of higher-order than $(n+m)$. One gets directly

$$\text{(A.2a)} \qquad \text{Det}(S(\bar{\theta}_0)) = \text{Det}(S(\theta_0)) + \Delta(\theta_0, \bar{\theta}_0)$$

with
(A.2b)
$$\Delta(\theta_0,\bar\theta_0)=\sum_{k=1}^{n+m}\sum_{i_1,i_2,\dots,i_k=1}^{n+m+1}\frac{1}{k!}\mathrm{Trace}(S_{\theta_{i_1}}(\theta_0)\tilde S_{\theta_{i_1},\dots,\theta_{i_k}}(\theta_0)\prod_{j=i_1}^{i_k}(\bar\theta_j-\theta_j).$$

Now, it is proved by contradiction that
(A.3)
$$\mathrm{Trace}(S_{\theta_{i_1}}(\theta_0)\tilde S_{\theta_{i_1},\dots,\theta_{i_k}}(\theta_0)=0 \quad \text{for all} \quad i_k\in\{1,\dots,n+m+1\},\ k=1,2,\dots,n+m$$

is impossible. Since (A.3) depends only on 'a priori' estimates of the plant parameters so that it is independent of the modification scheme. Now, assume that $|\mathrm{Det}S(\theta_0)|\neq\xi<\rho$ with $\xi>0$. Then, note from the definition of $S(\bar\theta_0)$ that $|\mathrm{Det}(S(\bar\theta_0))|=\xi$ with arbitrary nonzero ξ if the subsequent modification rule is used after 'a priori' estimation : $\delta a_i=-a_i$, $\delta b_j=-b_j$ and $\delta b_m=\pm\xi^{\frac{1}{n}}-b_m$ for $i=1,2,\dots,n$; $j=0,1,\dots,m$. Assume that (A.3) holds. Thus, one has the relationships $\xi=|\mathrm{Det}(S(\bar\theta_0))|=|\mathrm{Det}(S(\theta_0))|\neq\xi$ by using a Taylor series expansion in the parameter space of the modified estimates around the 'a priori' ones according to (A.2). Thus, (A.3) is false, since all the derivatives used in (A.2) are not dependent on the used modification scheme. Then, there is at least one parameter component θ_i of θ_0 for which and $\mathrm{Trace}(S(\theta_0)\tilde S_{\theta_{i_1},\dots,\theta_{i_k}}(\theta_0))\neq 0$ then $\bar C$ in (8.c)-(8.d) is nonzero. Direct calculation yields
(A.4)
$$|\mathrm{Det}(S(\bar\theta_0))|=|\mathrm{Det}(S(\theta_0))+\Delta(\theta_0,\bar\theta_0)|\geq\delta_\alpha|\bar C|-|\mathrm{Det}(S(\theta_0))|\geq\rho>0$$

with δ_α if $\delta_\alpha\geq 1$ (what implies that $\alpha^j\geq\alpha$ for $j\geq 1$) and $\delta_\alpha=\alpha^{n+m}$ if $\delta_\alpha<1$ (what implies that $\alpha^j<\alpha$ for $j\geq 1$) with δ_α and α being chosen according to (8.a)-(8.b). Such a constraint establishes the first inequality in (A.4) since $|\Delta(\theta_0,\bar\theta_0)|\geq\delta_\alpha\bar C$ from (A.2b). Thus, Proposition 1 has been proved. □

Proof of Proposition 2. One has from the definitions and properties of the l_2, l_1 and l_∞ matrix norms (see, for instance, [16-17])

$$(A.5)\quad \begin{aligned}(n+m+1)^{-\frac{1}{2}} &= ||S(\theta_0)||_2 = |\lambda_{\max}(S(\theta_0))| \\ &= \left|\frac{1}{\lambda_{\max}(S^{-1}(\theta_0))}\right| \\ &\leq (n+m+1)^{\frac{1}{2}}||S(\theta_0)||_1,\end{aligned}$$

where $|\lambda_{\max}(.)|$ and $|\lambda_{\min}(.)|$ denote the maximum and minimum moduli of the eigenvalues of the (.)-matrix, respectively. Thus, the two following inequalities follow directly from (A.5).

$$(A.6a)\quad \begin{aligned}|\lambda_{\min}(S^{-1}(\theta_0))| &= \frac{1}{|\lambda_{\max}(S(\theta_0))|} \\ &= \frac{1}{||(S(\theta_0))||_2} \\ &\geq \frac{1}{(n+m+1)^{\frac{1}{2}}||(S(\theta_0))||_1} \\ &= \frac{1}{(n+m+1)^{\frac{1}{2}}\mathrm{Max}\big(1+\sum_{i=1}^{n}|a_i|, \sum_{i=0}^{m}|b_i|\big)},\end{aligned}$$

(A.6b)

$$\begin{aligned}|\lambda_{\max}(S^{-1}(\theta_0))| &= \frac{1}{|\lambda_{\min}(S(\theta_0))|} \\ &\leq \frac{1}{|\lambda_{\max}(S(\theta_0))|} = \frac{1}{||(S(\theta_0))||_2} \\ &\leq \frac{1}{(n+m+1)^{-\frac{1}{2}}||(S(\theta_0))||_\infty} \\ &= \frac{1}{(n+m+1)^{-\frac{1}{2}}\Big(\sum_{i=1}^{n-1}|a_i|+\sum_{i=0}^{m-1}|b_i|\Big)+\mathrm{Max}\big(1, |a_n|+|b_m|\big)}\end{aligned}$$

which imply

$$(A.7)\qquad 0 < \frac{1}{\varepsilon_0'} \leq |\lambda_{\min}(S(\theta_0))| \leq |\lambda_{\max}(S(\theta_0))| \leq \frac{1}{\varepsilon_0} < \infty$$

if Condition 1 holds. Thus, Condition 1 guarantees that $S(\theta_0)$ is nonsingular (i.e., the 'a priori' estimated plant model is controllable) and a parameter modification is not performed in (10)-(11). If Condition 1 does not hold then $S(\theta_0)$ is not guaranteed to be nonsingular accordingly to the test of (A.7). Thus, the estimation modification procedure of (9.a), (10)-(11) when Condition 1 does not hold guarantees that

$$1 > \sum_{i=1}^{n} |a_i| + \sum_{i=0}^{m-1} |\bar{b}_i| \ ; \ |\bar{b}_m| > \sum_{i=1}^{n} |\bar{a}_i| + \sum_{i=1}^{m-1} |\bar{b}_i| + \gamma |\bar{b}_0|. \tag{A.8}$$

Now, note that if (A.8) holds then the modified $S(\bar{\theta}_0)$ is diagonally dominant what follows directly by inspection from its definition since for such a matrix structure, it suffices to guarantee diagonal dominance for the n-th and $(n+1)$-th rows. Since all diagonally dominant matrix is nonsingular, [16], $S(\bar{\theta}_0)$ is nonsingular and the modified estimated plant model is controllable. The proof has been completed. □

Proof of Theorem 1. The subsequent proof applies for both modification schemes (8)-(9) and (9.a), (10)-(11). (i)-(ii) Note that $\dot{P}^{-1} = P^{-1}\dot{P}P^{-1} = \varphi\varphi^T$ from (7.c). Define the Lyapunov function candidate $V = \tilde{\theta}^T P^{-1}\tilde{\theta}$ where $\tilde{\theta} = \hat{\theta} - \theta^*$ is the 'a priori' parametrical error. Thus, (7.a) can be rewritten as $e = -\tilde{\theta}^T\varphi$ and $\dot{V} = -(\tilde{\theta}^T\varphi)^2 = -e^2 \leq 0$ after straigthforward calculations with V and (7), [9]. Thus, $e \in L_2$ and $\infty > \tilde{\theta}^T P^{-1}\tilde{\theta} \geq \lambda_{\min}(P^{-1})\tilde{\theta}^T\tilde{\theta}$, with $\lambda_{\min}(P^{-1})$ being the minimum eigenvalue of P^{-1} so that $\tilde{\theta}$ is uniformly bounded for all $t \geq 0$ since the maximum eigenvalue of P, $\lambda_{\max}(P)$, is upper-bounded by a positive finite constant and then $\lambda_{\min}(P^{-1}) = \lambda_{\max}^{-1}(P) > 0$ for all $t \geq 0$. Thus, θ is uniformly bounded and $\bar{\delta}$ is uniformly bounded for all $t \geq 0$ from (9) since $\theta = (\theta_0^T, \varepsilon_0^T)^T$ and θ_0 and $\mathrm{Det}(S(\theta_0))$ are uniformly bounded for all $t \geq 0$. Thus, the 'a posteriori' modified parameter vector $\bar{\theta} = (\bar{\theta}_0^T, \varepsilon_0^T)^T$ is also uniformly bounded for all $t \geq 0$. The modified estimated plant model is controllable since $\infty > |\mathrm{Det}(S(\bar{\theta}_0))| \geq \rho > 0$ from (8)-(9) and the fact that $\bar{\theta}_0$ is uniformly bounded for all $t \geq 0$. On the other hand, $P\varphi \in L_2$ since $\mathrm{tr}(\dot{P}) = -||P\varphi||_2^2 \in L_1$ from (7.c) with $||.||_2$ denoting the spectral (or euclidean) vector norm. Thus, propositions (i)-(ii) have been proved.

(iii) It is standard to prove that P and θ converge asymptotically from

(7.b) and the fact that

$$\lim_{t\to\infty}\Big(\int_0^t ||\dot{\theta}||d\tau \leq \frac{1}{2}\big[\lim_{t\to\infty}\Big(||P\varphi||^2 d\tau\Big) + \lim_{t\to\infty}\Big(\int_0^t e^2 d\tau\Big)\big] < \infty$$

since $P\varphi \in L_2$ and $e \in L_2$ what implies $\dot{\theta} \in L_1$ and the θ converges from (ii) (see [18]). Also, θ_0 converges since θ converges and thus the Sylvester and the modified Sylvester determinants converge to finite constant values as time tends to infinity. This implies that each controller parameter resulting in finite limit values for the coefficients of $R(D,t)$ and $S(D,t)$ from solving (15.a) [or, equivalently, (15.b) subject to (15.c)] and proposition (iii) has been proved.

(iv) Note that direct calculation from (12) yields for $m \leq n-1$:

$$D^n y_f = e + (\bar{\theta}^T - \bar{\delta}^T)\varphi = e + \sum_{i=0}^{m} \bar{b}_i D^{m-i} u_f - \sum_{i=1}^{n} \bar{a}_i D^{n-1} y_f - \delta_0^T \varphi_0$$

and the substitution $D^n u_f$ explicited from (13) into (12) yields for $m = n$:

$$\begin{aligned} D^n y_f &= e + (\bar{\theta}^T - \bar{\delta}^T)\varphi \\ &= e - \bar{b}_0\big[\sum_{i=1}^{n} s_i D^{n-i} u_f + \sum_{i=0}^{n-1} r_i D^{n-i-1} y_f\big] \\ &\quad + \big[\sum_{i=1}^{n} \bar{b}_i D^{n-i} u_f - \sum_{i=0}^{n-1} \bar{a}_i D^{n-i-1} y_f\big] - \delta_0^T \varphi_0. \end{aligned}$$

Thus, the substitution of the above identities together with (13) yield the following extended auxiliary dynamic system which describes the combination of the closed-loop dynamics and control law :

$$\dot{x} = Ax + w, \tag{A.9a}$$

$$\dot{z} = Ax + w_1 \tag{A.9b}$$

with

$$\begin{aligned} w &= [e + \varepsilon_0^T i_\varphi - \delta_0^T \varphi_0, 0]^T = \bar{w} + w_1; \\ \bar{w} &= [-\delta_0^T \varphi_0, 0]^T; \\ w_1 &= [e + \varepsilon_0^T i_\varphi, 0]^T, \end{aligned} \tag{A.10a}$$

$$A(t) = \begin{bmatrix} & \bar{P}^T & \\ I_{n-1} & \vdots & 0 \\ \dots & \vdots & \dots \\ & \bar{V}^T & \\ 0 & \vdots & I_{n-1} \end{bmatrix}, \tag{A.10b}$$

$$\bar{P} = \begin{cases} \bar{P}^{(1)} & \text{if} \quad m \le n-1 \\ \bar{P}^{(2)} & \text{if} \quad m = n, \end{cases} \tag{A.10c}$$

$$\bar{P}^{(1)T} = \left[-\bar{a}_1, -\bar{a}_2, \dots, -\bar{a}_n : 0, \dots, 0 : \bar{b}_0, \bar{b}_1, \dots, \bar{b}_m \right], \tag{A.10d}$$

$$\begin{aligned} \bar{P}^{(2)T} = \big[-(\bar{a}_1 + \bar{b}_0 r_0), -(\bar{a}_2 + \bar{b}_0 r_1), \dots, -(\bar{a}_n + \bar{b}_0 r_{n-1}) : \\ (\bar{b}_1 + \bar{b}_0 s_1), (\bar{b}_2 + \bar{b}_0 s_2), \dots, (\bar{b}_n + \bar{b}_0 s_n) \big], \end{aligned} \tag{A.10e}$$

$$\bar{V}^T = [r_0, r_1, \dots, r_n \vdots s_1, s_2, \dots, s_n] \tag{A.10f}$$

with $x(0) = z(0) = x_0$, and $x = (D^{n-1}y_f, \dots, Dy_f, y_f, D^{n-1}u_f, \dots, Du_f, u_f)^T$ and $\varphi_0 = (D^{n-1}y_f, \dots, Dy_f, y_f, D^n u_f, D^{n-1}u_f, \dots, Du_f, u_f)^T$. The proof of boundedness and convergence to zero of the input, output, their filtered versions and the time- derivatives of those ones up till $(n-1)$-th order of the closed-loop system is immediate by first proving that (A.9b) is asymptotically stable in the large. Thus, by vector construction, $|D^n u_f| \le K'||x||$ from the controller equation (13) and, then, $||\varphi_0|| \le \text{Max}(|D^n u_f|, ||\chi||) \le K||x||$ with $K = 1 + K'$. Note from (A.9b) and (15.a) that both eigenvalues of $A(t)$ are less than or equal to $(-\sigma) < 0$, for some real constant $\sigma > 0$ which is less than or equal to the minimum absolute value of the roots of the strictly Hurwitz $C^*(D)$- polynomial for all $t \ge 0$ (equality applies when both roots are distinct, [17-18]). Thus, the common unforced version of both time-varying systems (A.9) is exponentially stable in the large. Now, direct calculus with the differential systems (A.9a) and (A.9b) yields that their solutions are related as follows:

$$x(t) = z(t) + \int_0^t \Psi(t, \tau) \bar{w}(\tau) \, d\tau \tag{A.11}$$

with $\Psi(t,\tau)$ being the fundamental matrix of the unforced system of (A.9a) and (A.9b), i.e., $x(t) = z(t) = \Psi(t,0)x_0$ for all $t \geq 0$ if $w \equiv w_1 \equiv 0$. Since such a system is exponentially stable in the large, one has for any matrix norm that $||\Psi(t,\tau)|| \leq Ke^{-\sigma(t-\tau)}$ for any t and τ fulfilling $t \geq \tau \geq 0$. In particular, one has $||\Psi(t,\tau)|| \leq e^{-\sigma(t-\tau)}$ (i.e., $K = 1$) for $t > \tau$ if the spectral matrix norm is used. Since A(t) is exponentially stable and, furthermore, $w_1 \in L_\infty \cap L_2$ from (i)-(ii) $z \in L_\infty \cap L_2$, $\dot{z} \in L_\infty \cap L_2$ and z converges exponentially to zero for any bounded initial condition (see [18]). Thus, by taking spectral vector and matrix norms in (A.11), one gets directly from the definition of $\bar{w}$ in (A. 9a):

$$\text{(A.12)} \qquad ||x(t)||_2 \leq ||z(t)||_2 + K\int_0^t e^{-\sigma(t-\tau)}||\delta_0||_2||x(\tau)||_2\, d\tau.$$

Now, define $\bar{z}_{t_{k,e}} = \sup_{t_k \leq t \leq T}(||z(t)||_2)$ and $\bar{z} = \sup_{t \in R_0^+}(||z_{t,e}||_2) = \sup_{t \in R_0^+}\left(\sup_{0 \leq t \leq T}(||z(t)||_2)\right)$ for all finite $t_k \in Z_0^+$ and $T \in R_0^+$ where Z_0^+ and R_0^+ are the sets of nonnegative integer and real number, respectively. It is obvious from the fact that $z \in L_\infty \cap L_2$ that there exists a sequence of time instants $T_s = \{t_k, k \geq 0\}$ with t_0 sufficiently large (but finite) such that $\bar{z}_{t_{k+1},e} < \bar{z}_{t_k,e} \leq \bar{z} < \infty$ and $\bar{z}_{t_k} \to 0$ as $k \to \infty$ since T_s is a monotonically increasing sequence and $z(t)$ converges to zero asymptotically since it is in $L_\infty \cap L_2$. Thus, one gets from (A.12)

$$\text{(A.13)} \qquad \begin{aligned} &||x(t_k)||_2 \\ &\leq ||z(t_k)||_2 + K\int_0^{t_k} e^{-\sigma(t-\tau)}||\delta_0||_2||x(\tau)||_2\, d\tau \\ &\leq ||\bar{z}(t_k,e)||_2 + K\int_0^{t_k} e^{-\sigma(t-\tau)}||\delta_0||_2||x(\tau)||_2\, d\tau \end{aligned}$$

for all $t_k \in T_s$. Now, it follows from

$$\text{(A.14)} \qquad \begin{aligned} &||x(t_k+\tau)||_2 \\ &\leq ||\bar{z}_{t_k,e}||_2 e^{\frac{\delta_0}{\sigma}(1-e^{-\sigma(t_k+\tau)})}||x(t_k)||_2 \\ &\leq e^{\frac{\delta_0}{\sigma}}||\bar{z}_{t_k,e}||_2 \\ &< \infty \quad \text{for all} \quad t_k \in T_k, \quad \text{and} \quad \tau \geq t_k, \end{aligned}$$

where $\bar{\delta}_0 = K \sup_{T\in R_0^+} \sup_{0\le t\le T} ||\bar{\delta}(t)||_2)$ by applying Bellman-Gronwall's Lemma (see [17]) to (A.13). Thus, $||x(t_k+\tau)||_2 < \infty$ and $||\dot{x}(t_k+\tau)||_2 < \infty$ are uniformly bounded from (A.14), boundedness of both the estimation error and $\bar{\delta}$ and (A.9)-(A.10). One has, in addition, from (A.14) that $x(t_k+\tau) \to 0$ as $k \to \infty$ for all $\tau \in [t_k, \infty)$ since $\bar{\delta}$ is bounded from (i). As a result, $x \in L_\infty$, $\dot{x} \in L_\infty$ and $x \to 0$ and $\dot{x} \to 0$ as $t \to \infty$. Thus, the proof of (iv) follows from the definition of the x-vector. □

Appendix B

Outline of Proof of Theorem 2. Define the positive function $V := \tilde{\theta}^T P^{-1}\tilde{\theta}$, all $t \ge 0$ as in Theorem 1. Now, from (18.c) $\dot{P}^{-1} = b\varphi\varphi^T$ which yields by using (18.a)-(18.c)

$$\text{(A.15)} \qquad \begin{aligned} \dot{V} :&= b[2(\eta_f - e)\eta_f - (\eta_f - e)^2] \\ &\le b[\bar{\eta}_f^2 - e^2] \\ &\le \begin{cases} -\chi \le 0 & \text{if} \quad |e| > \mu\bar{\eta}_f \\ 0 & \text{if} \quad |e| \le \mu\bar{\eta}_f \end{cases} \end{aligned}$$

for all $t \ge 0$ with $\chi := \frac{g(\mu^2-1)f^2}{\mu^2(1+\gamma\phi^T P\phi)} \le \frac{gs(\mu^2-1)e^2}{\mu^2(1+\gamma\phi^T P\phi)} = \frac{\mu^2-1}{\mu^2} be^2$ since $|e| \ge |f|$ $\Rightarrow se^2 = fe \ge f^2$ and $(\mu^2-1)s^2/\mu^2 \le 1$. Now $V(t_2) \le V(t_1) \le V(0) < \infty$, all t_1, t_2 with $t_2 \ge t_1 \ge 0$ and $||\tilde{\theta}|| \in L_\infty$ and $||\theta|| \in L_\infty$. $P \in L_\infty$ from (18.c) since b is not positive so that $\dot{P} \le 0$ and when using any of the estimation schemes, $||\bar{\delta}|| \in L_\infty$. Also, $b \in L_\infty$ from (18.d)-(18.f) since $gs \le g_2$. It is direct as in Theorem 1 to prove the convergence of θ and $\bar{\delta}$, thus, that of $\tilde{\theta}$ while the remaining of the proof follows as the corresponding parallel items of Theorem 1. The main variation compared to Theorem 1 is that the state z of the auxiliary system (A.9a) is now only proved to be Lyapunov stable in the large, by using a similar technique as that used in Theorem1, but not exponentially stable, in general. □

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Instituto de Investigación y Desarrollo de Procesos IIDP, Departamento de Ingenieria de Sistemas y Automática, Facultad de Ciencias. Universidad del Pais Vasco, Leioa (Bizkaia) - Aptdo. 644 de Bilbao. SPAIN
E-mail: msen@we.lc.ehu.es