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POINTWISE CONVERGENCE OF WAVELET EXPANSION OF ${\mathcal{K}^r_M}'(R)$

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Abstract. The expansion of a distribution of ${\mathcal{K}^r_M}'(R)$ in terms of regular orthogonal wavelets is considered. The expansion of a distribution of ${\mathcal{K}^r_M}'(R)$ is shown to converge pointwise to the value of the distribution where it exists.

1. Introduction

Both theories of wavelets and distributions began with an attempt to solve problems arising in technology by using new techniques. The two theories that were subsequently developed interact considerably. Many of the arguments in wavelet theory involve convergence in the sense of distributions and orthogonal wavelet expansions for certain classes of distributions [1]. Orthogonal wavelets have proved very useful in both applied and theoretical problems. Their usefulness in applications is largely based on their coefficient representation of signals; fewer coefficients are needed than with other methods. These are well-known results and have been extensively studied in [1] and [4]. In the theoretical domains, orthogonal wavelets have proved useful as unconditional bases for certain Banach spaces [1].

In this paper, we show that every tempered distributions of $e^{M(kx)}$-growth with derivative up to order r can be expanded by orthogonal wavelets of ordinary function and the wavelet expansion converges pointwise to the value of the distribution where it exists. Some of interactions between wavelets and tempered distributions were also presented in G.G.Walter's work [8] and [9]. G. G. Walter has found orthogonal wavelets expansion and the wavelet expansion's pointwise convergence of the tempered distributions of polynomial growth. These two G. G. Walter's results were

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extended by us to the case of the tempered distributions of exponential growth [6]. Our results in this paper are equivalent to the G. G. Walter's ones in [8] and [9] when $M(x) = \log(1+|x|)^p$ and to the previous ones of our's in [6] when $M(x) = |x|^p, p > 1$.

2. The generalized tempered distribution spaces $\mathcal{K}'_M(R)$

Let $\mu(\xi)$ $(0 \le \xi \le \infty)$ denote a continuous increasing function such that $\mu(0) = 0,\ \mu(\infty) = \infty$. For $x \ge 0$, we define

$$M(x) = \int_0^x \mu(\xi) d\xi.$$

The function $M(x)$ is an increasing, convex and continuous function with $M(0) = 0, M(\infty) = \infty$ and satisfies the fundamental convexity inequality $M(x_1) + M(x_2) \le M(x_1 + x_2)$. Further we define $M(x)$ for negative x by means of the equality $M(-x) = M(x)$. Note that since the derivative $\mu(x)$ of $M(x)$ is unbounded in R, the function $M(x)$ will grow faster than any linear function as $|x| \to \infty$. Now we list some properties of $M(x)$ which will be frequently used in this paper.

(1) $$M(x) + M(y) \le M(x+y) \text{ for all } x,\ y \ge 0.$$

(2) $$M(x+y) \le M(2x) + M(2y) \text{ for all } x,\ y \ge 0.$$

Using the function $M(x)$, we define the space $\mathcal{K}_M(R)$ as the space of all functions $\phi \in C^\infty(R)$ such that

(3) $$\nu_k(\phi) = \sup_{x \in R, \alpha \le k} e^{M(kx)} |D^\alpha \phi(x)| < \infty,\ k = 1, 2, ...,$$

where $D^\alpha = \frac{d^\alpha}{dx^\alpha}$. The topology in $\mathcal{K}_M(R)$ is defined by the family of the semi-norms ν_k. Then $\mathcal{K}_M(R)$ becomes a Fréchet space and the embeddings $\mathcal{D} \hookrightarrow \mathcal{K}_M \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{E}$ are continuous; here $\mathcal{E}$ denotes the space of all C^∞-functions, $\mathcal{S}$ the space of the tempered distributions of polynomial growth and $\mathcal{D}$ the space of C^∞-functions with compact supports. By $\mathcal{K}'_M(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_M(R)$. Pahk characterized the distributions in $\mathcal{K}'_M(R)$ by the growth at infinity [5, Theorem 2.3]; a distribution $T \in \mathcal{D}'$ is in $\mathcal{K}'_M(R)$ if and only if there exist positive integers α, k_0 and a bounded continuous function $f(x)$ on R such that

$$T = D^\alpha[e^{M(k_0 x)} f(x)].$$

DEFINITION 1. *For a natural number r, we denote by $\mathcal{K}_M^r(R)$ the space of all functions $\phi \in C^r(R)$ such that*

$$\nu_k^r(\phi) = \sup_{x \in R, \alpha \leq r} e^{M(kx)} |D^\alpha \phi(x)| < \infty, \ k = 1, 2, 3, \ldots .$$

The topology of $\mathcal{K}_M^r(R)$ is defined by the family of semi-norms $\{\nu_k^r\}_{k=1,2,\ldots}$. By $\mathcal{K}_M^r{}'(R)$, we mean the space of continuous linear functionals on $\mathcal{K}_M^r(R)$. Each $S \in \mathcal{K}_M^r{}'(R)$ is characterized by

$$S = D^r[e^{M(k_0 x)} f(x)], \tag{4}$$

where $f(x)$ is a bounded continuous function on R and $r, k_0 \in N$, the set of natural numbers, by the same method of the above $\mathcal{K}_M'$-case in [5, Theorem 2.3]. Similarly, we can define

$$\mathcal{S}_r(R) = \left\{\theta(t) \in C^r(R); |D^k\theta(t)| \leq C_{pk}(1 + |t|)^{-p}, \ p \in N, \ k = 0, 1, \ldots, r\right\}$$

and its dual $\mathcal{S}_r'(R)$. For further details, we refer to [5].

3. Multiresolution analysis of $L^2(R)$ associated with $\phi \in \mathcal{K}_M^r(R)$

Let $\phi \in \mathcal{K}_M^r(R)$. In order for it to qualify as a scaling function, there must be associated with ϕ a multiresolution analysis of $L^2(R)$, i.e., a nested sequence of closed subspaces $\{V_m\}_{m \in Z}$ for the set of integers Z such that

(i) $\{\phi(\cdot - n)\}$ is an orthonormal basis of V_0,

(ii) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(R)$,

(iii) $f(\cdot) \in V_m \Leftrightarrow f(2\cdot) \in V_{m+1}$,

(iv) $\cap_m V_m = \{0\}$, $\overline{\cup_m V_m} = L^2(R)$.

Then ϕ has an expansion

$$\phi(t) = \sum\nolimits_n c_n \sqrt{2}\phi(2t - n), \ \{c_n\} \in l^2, \ t \in R, \tag{5}$$

where $l^2 = \left\{\{c_n\}; \{\sum_n |c_n|^2\}^{\frac{1}{2}} < \infty\right\}$. Once we have the scaling function $\phi \in \mathcal{K}_M^r(R)$, we can obtain a mother wavelet ψ such that $\{\psi(t - n)\}$ is an orthogonal basis of the space W_0, given by the orthogonal complement of V_0 in V_1. Also, ψ has an expansion

$$\psi(t) = \sum\nolimits_n d_n \sqrt{2}\phi(2t-n),\ \{d_n\} \in l^2, \tag{6}$$

for d_n corresponding to c_n in (5). We will adopt the construction of a mother wavelet defined by $d_n = (-1)^n \overline{c_{1-n}}$. If such a $\psi(t)$ can be found, then $\psi_{mn}(t) = 2^{\frac{m}{2}}\psi(2^m t - n)$ is an orthogonal basis of W_m which is the orthogonal complement of V_m in V_{m+1}.

Example. In [1], Corollary 5.5.3 states that it is impossible that ψ has exponential decay and that $\psi \in C^\infty$, with all derivatives bounded, unless $\psi = 0$. Hence there is no mother wavelet $\psi \in \mathcal{K}_M(R)$. So we will restrict our attention to $\mathcal{K}_M^r(R)$. Daubechies' compactly supported wavelets are examples of $\mathcal{K}_M^r(R)$, but Battle-Lemarié's wavelets (in the page 152 of [1]) are not $\mathcal{K}_M^r(R)$ wavelets even if they have exponential decay and smoothness.

The reproducing kernel of V_0 is given by

$$q(x,t) = \sum\nolimits_n \overline{\phi(x-n)}\phi(t-n),$$

where $\phi(x)$ is the scaling function. The series and its derivatives with respect to t of order $\leq r$ converge uniformly on $x \in R$ because of the regularity of $\phi \in \mathcal{K}_M^r(R)$, i.e.,

$$|\phi^{(\alpha)}(x)| \leq C_{\alpha k} e^{-M(kx)},\ \alpha = 0,1,...,r; k = 1,2,.... \tag{7}$$

We deduce the following properties [4, p. 33]:

(a) $q(x+k, t+k) = q(x,t)$ for all $k \in Z$.

(b)

$$\begin{aligned} |q(x,t)| &\leq \sum\nolimits_j |\phi(x-j)||\phi(t-j)| \\ &\leq \sum\nolimits_j c_{k+1}^2 e^{-M((k+1)(x-j))} e^{-M((k+1)(t-j))} \\ &\leq \sum\nolimits_j c_{k+1}^2 e^{-M(x-j)} e^{-M(t-j)} e^{-M(k(x-j))} e^{-M(k(t-j))} \\ &\leq \sum\nolimits_j c_{k+1}^2 e^{-M(x-j)} e^{-M(\frac{k}{2}((x-l)+(t-j)))} \\ &\leq c_k' e^{-M(\frac{k}{2}(x-t))}, k = 1,2,..., \end{aligned}$$

where we used the properties (1)and(2).

(c) $\int_{-\infty}^{\infty} q(x,t)t^{\alpha}dt = x^{\alpha},\ 0 \le \alpha \le r.$

The reproducing kernel for V_m is given by

$$q_m(x,t) = 2^m q(2^m x, 2^m t).$$

Similarly, we can define the reproducing kernel $r_m(x,t)$ for W_m by

$$r_m(x,t) = 2^m \sum_n \overline{\psi(2^m x - n)}\psi(2^m t - n),$$

where $\psi(t)$ is the mother wavelet.

Now, we will extend the expansion in orthogonal wavelets from $L^2(R)$ to $\mathcal{K}_M^r(R)$.

LEMMA 2. *If the scaling function ϕ is in $\mathcal{K}_M^r(R)$, then the mother wavelet ψ defined by* (6) *is also in $\mathcal{K}_M^r(R)$.*

Proof. By the orthogonality of $\phi(x)$ and $\phi(2x-n)$ and the regularity (7) of $\phi \in \mathcal{K}_M^r(R)$,

$$|c_n| = \left| \int \phi(x)\overline{\phi(2x-n)}dx \right| \le C' e^{-M(k'n)}, k' = 1, 2, ...$$

and

$$\begin{aligned} \left|\phi^{(\alpha)}(2t-n)\right| &\le C_{\alpha k''}\ e^{-M(k''(2t-n))} \\ &\le C_{\alpha k''}\ e^{-M((2k''(\frac{1}{2}(2t-n)))} \\ &\le C_{\alpha k''}\ e^{-M(2k''t)-M(k''(\frac{n}{2}))},\ \alpha = 0, 1, ..., r\ ; k'' = 1, 2, ..., \end{aligned}$$

where we used the property (1). Since $d_n = (-1)^n \overline{c_{1-n}}$, $|d_n| = |(-1)^n \overline{c_{1-n}}|$ $= |c_{1-n}| \le C' e^{-M(k'(1-n))}$. Hence, if we take k' which is sufficiently larger than $\frac{k''}{2}$,

$$\begin{aligned} \sum_n \left| d_n \sqrt{2} \phi^{(\alpha)}(2t-n) \right| &\le \sum_n C' C_{\alpha k''} e^{-M(k'(1-n))} e^{M(k'' \frac{n}{2})} e^{-M(2k''t)} \\ &\le \sum_n C'_{\alpha k''} e^{-M(n)} e^{-M(2k''t)} \\ &= C''_{\alpha k''} e^{-M(2k''t)},\ \alpha = 0, 1, 2, ..., r\ ;\ k'' = 1, 2, ... \end{aligned}$$

By (6), $\psi \in \mathcal{K}_M^r(R)$. □

Then expansion coefficients with respect to both $\{\phi(t-n)\}$ and $\{\psi(t-n)\}$ are well defined. Indeed, since $f \in \mathcal{K}_M^r{}'(R)$ is also characterized by

$$f = D^r\{e^{M(k_0x)}\mu\}, \ k_0 \in N$$

for a bounded measure μ on R, coefficients a_n may be found which satisfy

$$\begin{aligned} a_n &= (f, \phi(\cdot - n)) \\ &= (D^r(e^{M(k_0t)}\mu), \phi(\cdot - n)) \\ &= \int_{-\infty}^{\infty} e^{M(k_0t)}(-1)^r\phi^{(r)}(t-n)d\mu. \end{aligned}$$

Hence

$$\begin{aligned} |a_n| &\leq \int_{-\infty}^{\infty} e^{M(2k_0(t-n))}e^{M(2k_0n)}|\phi^{(r)}(t-n)|\,d|\mu| \\ &= \mathcal{O}(e^{M(k_1n)}). \end{aligned} \tag{8}$$

Similarly, $b_n = (f, \psi(t-n))$ satisfies the same kind of growth condition. Then, since

$$\begin{aligned} \sum_n |a_n\phi^{(j)}(t-n)| &\leq C\sum_n e^{M(k_1n)}e^{-M((2k_1+1)(t-n))} \\ &\leq C\sum_n e^{-M(t-n)}e^{M(2k_1t)} \\ &\leq C'e^{M(k_2t)}, \end{aligned}$$

$\sum_n a_n\phi(t-n)$ converges uniformly on bounded sets as do its first r derivatives. In fact, we have shown that the limiting function and its r derivatives are continuous functions of $e^{M(k_0x)}$ growth. These results enable us to imitate the multiresolution analysis of $L^2(R)$ in $\mathcal{K}_M^r{}'(R)$.

DEFINITION 3. *We define the spaces T_0 and U_0 by $T_0 = \{f; f(t) = \sum_n a_n\phi(t-n)$ for some sequence of complex numbers with $a_n = \mathcal{O}(e^{M(k_1n)})$ for some $k_1 \in N$ $\}$ and $U_0 = \{g; g(t) = \sum_n a_n\psi(t-n)$ for some sequence of complex numbers with $a_n = \mathcal{O}(e^{M(k_1n)})$ for some $k_1 \in N$ $\}$. We denote by*

T_m and U_m their corresponding dilation spaces, i.e., $f \in T_0 \Leftrightarrow f(2^m t) \in T_m$ and $g \in U_0 \Leftrightarrow g(2^m t) \in U_m$.

We may expect that a multiresolution analysis of $\mathcal{K}_p^{r\prime}(R)$ exists, namely,

$$(9) \qquad \cdots \subset T_{-m} \cdots \subset T_{-1} \subset T_0 \subset T_1 \cdots \subset T_m \subset \cdots \subset \mathcal{K}_M^r{}'(R)$$

and

$$\overline{\cup_m T_m} = \mathcal{K}_M^r{}'(R),$$

where the closure is in the topology of $\mathcal{K}_M^r{}'(R)$.

THEOREM 4. *Let the scaling function $\phi \in \mathcal{K}_M^r(R)$ satisfy the dilation equation (5) with $c_k = \mathcal{O}(e^{-M(lk)})$ for all $l \in N$, and have an associated multiresolution analysis in $L^2(R)$; let $\psi \in \mathcal{K}_M^r(R)$ be the mother wavelet given in (6). Then there exists a multiresolution analysis (9) of closed dilation subspaces $\{T_m\}$ whose union is dense in $\mathcal{K}_M^r{}'(R)$; the closed subspace U_m in Definition 2 is a complementary subspace of T_m in T_{m+1} and*

$$T_m = U_0 \oplus U_1 \oplus \cdots \oplus U_m \oplus T_0,$$

where $\oplus$ denotes the nonorthogonal direct sum.

Proof. We will only prove that T_0 is closed in the sense of $\mathcal{K}_M^r{}'(R)$ and $\overline{\cup_m T_m} = \mathcal{K}_M^r{}'(R)$. The other statements are the same as in the case of $\mathcal{S}_r'(R)$ [8]. It is clear that $f_m \to 0$ in $\mathcal{K}_M^r{}'(R)$ is equivalent to

$$f_m = D^r[e^{M(k_0 x)}\nu_m], \ r, k_0 \in N; \ \int d|\nu_m| \to 0,$$

where $\{\nu_m\}$ is a sequence of bounded measures on R. If $f_m \in T_0$, then we have that $a_{nm} = (f_m, \phi(\cdot - n)) \to 0$ as $m \to \infty$ and $|a_{nm}| \le ce^{M(k_1 n)}$ by (8). Hence if $f_m \to f$ in $\mathcal{K}_M^r{}'(R)$, the coefficient of f, a_n is of $\mathcal{O}(e^{M(k_1 n)})$ and hence its series $\sum_n a_n \phi(t-n) = f \in T_0$. Thus T_0 is closed in the sense of $\mathcal{K}_M^r{}'(R)$. Now by the facts that $\mathcal{S}_r'(R)$ is dense in $\mathcal{K}_M^r{}'(R)$, $L^2 = \overline{\cup_m V_m}$ is dense in $\mathcal{S}_r'(R)$ and $\cup_m V_m \subset \cup_m T_m$, $\cup_m T_m$ is dense in $\mathcal{K}_M^r{}'(R)$. □

REMARK. As in the case of $\mathcal{S}_r'(R)$ [8], the property $\cap_m V_m = \{0\}$ of the usual multiresolution analysis is lacking. By the moment property of the reproducing kernel $q_m(x,t)$, any polynomial of degree $\le r$ belongs to $\cap_m T_m$.

4. Convergence of the expansions of $\mathcal{K}_M^r{}'(R)$

A *quasi − positive delta sequence* is a sequence $\{\delta_m(\cdot, y)\}$ of functions in $L^1(R)$ with parameter $y \in R$ which satisfies the following:

(a) there is a $C > 0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x,y)|dx \le C, \ y \in R, \ m \in N;$$

(b) there is a $c > 0$ such that

$$\int_{y-c}^{y+c} \delta_m(x,y)dx \to 1$$

uniformly on compact subsets of R, as $m \to \infty$;

(c) for each $\gamma > 0$,

$$\sup_{|x-y|\le\gamma} |\delta_m(x,y)| \to 0 \text{ as } m \to \infty.$$

Then since $\mathcal{K}_M^r(R) \subset \mathcal{S}_r(R)$, we have following important two Lemmas as in [9]:

LEMMA 5. *Let $\{\delta_m(x,y)\}$ be a quasi-positive delta sequence and let $f \in L^1(R)$ be continuous on (a,b); then*

$$f_m(y) = \int_{-\infty}^{\infty} \delta_m(x,y)f(x)dx \to f(y) \text{ as } m \to \infty$$

uniformly on compact subsets of (a,b).

LEMMA 6. *If $q_m(x,y)$ is the reproducing kernel of V_m, $\phi \in \mathcal{K}_M^r(R)$, then $q_m(x,y)$ and $K_m(x,t) = \frac{(x-t)}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} q_m(x,t)$ for $\alpha \in N, 0 \le \alpha \le r$, are quasi-positive delta sequences on R.*

By the remark in section 3, the pure wavelet series of $f \in \mathcal{K}_M^r{}'(R)$, $\sum_{n,m} b_{mn}\psi_{mn}(t)$ does not necessarily converge to f. However, the mixed expansion

$$f = \sum_n a_n \phi(\cdot - n) + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} b_{mn}\psi_{mn},$$

converges to f in the sense of $\mathcal{K}_M^r{}'(R)$. This global convergence of the expansion of $\mathcal{K}_M^r{}'(R)$ is important for theoretical purposes. For computational purposes, we will study some sort of local convergence in the sense of S. Lojasiewicz [3].

DEFINITION 7. Let $f \in \mathcal{K}'_M(R)$. f is said to have a value γ of order α at x_0 if there exists a continuous function $F(x)$ of exponential growth of $e^{M(k_0x)}$ for some $k_0 \in N$ such that $D^\alpha F = f$ in some neighborhood of x_0 and

$$\lim_{x\to x_0}\frac{F(x)}{(x-x_0)^\alpha}=\frac{\gamma}{\alpha!}.$$

Example. The δ distribution has value 0 everywhere except at 0.

THEOREM 8. *Let $f \in \mathcal{K}^r_M{}'(R)$ and let have a value γ of order $\alpha \le r$ at $x = x_0$. Then the function f_m given by $f_m(x) = (f(\cdot),\ q_m(x,\cdot))$ satisfies*

$$f_m(x_0) \to \gamma \text{ as } m \to \infty.$$

Proof. Although each $f \in \mathcal{K}^r_M(R)$ is a derivative of order $\beta \le r$ of a continuous function G of $e^{M(kx)}$-growth, we may obtain that $G = F$ and $\alpha = \beta$, where F is in the definition. In fact, if we take $G(x) = (x-x_0)^{\beta-\alpha}F(x)$ when $\beta \ge \alpha$ and $G(x) = (x-x_0)^{\alpha-\beta}F(x)$ when $\alpha > \beta$, then $\lim_{x\to x_0}\frac{F(x)}{(x-x_0)^\alpha} = \lim_{x\to x_0}\frac{G(x)}{(x-x_0)^\beta}$ and $\lim_{x\to x_0}\frac{F(x)}{(x-x_0)^\beta} = \lim_{x\to x_0}\frac{G(x)}{(x-x_0)^\alpha}$, respectively. We may assume $\alpha > \beta$.

Using integration by parts, for some $A > 0$,

$$\begin{aligned} f_m(x) &= \int_{-\infty}^{\infty}(-1)^\alpha \partial_y^\alpha q_m(x,y)F(y)dy \\ &= \int_{-\infty}^{\infty}\frac{(x-y)^\alpha}{\alpha!}\partial_y^\alpha q_m(x,y)\frac{F(y)\alpha!}{(y-x)^\alpha}dy \\ &= \int_{x-A}^{x+A} + \int_{x+A}^{\infty} + \int_{-\infty}^{x-A} \end{aligned}$$

Now, we claim that

$$\int_{x+A}^{\infty} + \int_{-\infty}^{x-A} \longrightarrow 0 \quad \text{as} \quad m\to\infty$$

First,

$$\begin{aligned} &\int_{x+A}^{\infty}\frac{(x-y)^\alpha}{\alpha!}\partial_y^\alpha q_m(x,y)\frac{F(y)\alpha!}{(y-x)^\alpha}dy \\ &\qquad = \sum_n\sum_{k=0}^{\alpha}(2^m x-n)^k\binom{\alpha}{k}\overline{\phi(2^m x-n)} \\ &\qquad\quad \times\int_{2^m(x+A)}^{\infty}\frac{\phi^{(\alpha)}(y-n)}{\alpha!}(n-y)^{\alpha-k}\frac{2^{m\alpha}F(\frac{y}{2^m})\alpha!}{(y-2^m x)^\alpha}\,dy. \end{aligned}$$

If we denote by I the last integral above, then

$$
\begin{aligned}
I &= \left| \int_{2^m(x+A)}^{\infty} \phi^{(\alpha)}(y-n)(y-n)^{(\alpha-k)} \frac{2^{m\alpha} F(\frac{y}{2^m})}{(y-2^m x)^{\alpha}} dy \right| \\
&\leq \int_{2^m(x+A)}^{\infty} C_{\alpha,j} e^{-j|y-n|^p} |y-n|^{(\alpha-k)} e^{k_0 M(\frac{y}{2^m})} dy \\
&\leq \int_{2^m(x+A)}^{\infty} C_{\alpha,j} e^{-jM(y-n)} |y-n|^{(\alpha-k)} e^{k_0 M(2y-2n)} e^{k_0 M(2n)} dy \\
&\leq \int_{2^m(x+A)}^{\infty} C_{\alpha,j} e^{-jM(y-n)} e^{c_{\alpha,k} M(y-n)} e^{2k_0 M(y-n)} e^{2k_0 M(n)} e^{M(y)-M(y)} dy \\
&\leq C_{\alpha,j} e^{(2k_0+2)M(n)} e^{(-j+c_{\alpha,k}+2k_0+2)M(2^m(x+A)-n)} \int_{2^m(x+A)}^{\infty} e^{-M(y)} dy \\
&= C'_{\alpha,j} e^{(2k_0+2)M(n)} e^{(-j+c_{\alpha,k}+2k_0+2)M(2^m(x+A)-n)}
\end{aligned}
$$

for $j > c_{\alpha,k} + 2k_0 + 2$.

Hence

$$
\left| \int_{x+A}^{\infty} \frac{(x-y)^{\alpha}}{\alpha!} \partial_y^{\alpha} q_m(x,y) \frac{F(y)\alpha!}{(y-x)^{\alpha}} dy \right|
$$

$$
\leq \frac{C'_{\alpha,l,l'}}{2^m} \sum_n \frac{e^{k_3 M(n)}}{e^{lM(2^m x - n)} e^{l' M(2^m(x+A)-n)}}, \tag{10}
$$

for $l = 1, 2, 3, \ldots$ and sufficiently large l'.

Let us estimate the exponent of the term of the summation in (10). Assume that $l - l' > 0$. By the properties (1) and (2), we have

$$
\begin{aligned}
k_3 M(n) &- lM(2^m x - n) - l'M(2^m(A+x) - n) \\
&\leq k_3 M(n) - (l-l')M(2^m x - n) - l'M(2^m A) \\
&\leq (k_3 - (l-l'))M(n) + (l-l')M(2^m x) - l'M(2^m A) \\
&\leq (k_3 - (l-l'))M(n) + 2^m(l-l')M(\frac{x}{2}) - 2^m l' M(A).
\end{aligned}
$$

Take l satisfying $k_3 - (l-l') = -1$. Then $l - l' = k_3 + 1$, whose right-hand side is a constant. Take l' such that $l'M(A) > (k_3+1)M(\frac{x}{2})$. Then the right-hand side of (10) is estimated by

$$
\leq C_{\alpha,l,l'} e^{-(l'M(A)-(k_3+1)M(\frac{x}{2}))2^m} \sum_n e^{-M(n)}
$$

$$= C'_{\alpha,l,l'} e^{-(l'M(A)-(k_3+1)M(\frac{x}{2}))2^m}.$$

The same method for the estimation of $|\int_{-\infty}^{x-A}|$ induces

$$\left|\int_{x+A}^{\infty}\right| + \left|\int_{-\infty}^{x-A}\right| \leq C''_{\alpha,l,l'} e^{-(l'M(A)-(k_3+1)M(x))2^m}.$$

Hence

$$\int_{x+A}^{\infty} + \int_{-\infty}^{x-A} \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

Thus we can express f_m as

$$f_m(x) = \int_{-\infty}^{\infty} K_m(x,y) F_A(x,y)dy + o(1),$$

where $K_m(x,y) = \frac{(x-y)^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial y^\alpha} q_m(x,y)$ and $F_A(x,y)$ is continuous for all y except for $y = x \pm A$ and has a compact support. Hence F_A is bounded. Since $K_m(x,y)$ is a quasi-positive delta sequence by Lemma 6, Lemma 5 implies that

$$f_m(x_0) \longrightarrow F_A(x_0,x_0) = \gamma \quad \text{as} \quad m \to \infty. \qquad \square$$

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