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THE MAXIMAL OPERATOR OF BOCHNER-RIESZ MEANS FOR RADIAL FUNCTIONS

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ABSTRACT. Author proves weak type estimates of the maximal function associated with the Bochner-Riesz means while it is claimed $p = 2n/(n+1+2\delta)$ and $0 < \delta \le (n-1)/2$ that the maximal function is bounded on L^p_{rad}.

1. Introduction

We consider the maximal operator associated with the Bochner-Riesz means defined on $\mathbb{R}^n$.

Let S^δ_ϵ, $\epsilon > 0$ and S^δ_* on $L^2(\mathbb{R}^n)$ by

$$S^\delta_\epsilon f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} (1 - |\epsilon\xi|^2)^\delta_+ \widehat{f}(\xi) e^{i<x,\xi>}\, d\xi,$$

and

$$S^\delta_* f(x) = \sup_{\epsilon>0} |S^\delta_\epsilon f(x)|.$$

respectively.

We set $S^\delta = S^\delta_1$. C. Herz [5] showed that, when restricted to radial functions on $L^p(\mathbb{R}^n)$, S^0 is bounded if and only if $2n/(n+1) < p < 2n/(n-1)$. This result can not be extended to arbitrary functions. This was proved by C. Fefferman [4]. Kenig and Tomas [8] showed that for $p = 2n/(n+1)$, S^0 is not of weak type on radial functions in L^p, and S. Chanillo [1] showed that S^0 is of restricted weak type on radial functions in L^p for $p = 2n/(n+1)$. Chanillo and Muckenhoupt [2] proved that S^δ is of weak type (p,p) on radial functions in L^p where $p = 2n/(n+1+2\delta)$ and $0 < \delta \le (n-1)/2$. Y. Kanjin [6, 7] showed that S^δ_* is bounded on radial functions in L^p when $2n/(n+1+2\delta) < p < 2n/(n-1-2\delta)$ and $0 \le \delta < (n-1)/2$.

The purpose of this paper is an extension of a result by Chanillo and Muckenhoupt [2] on Bochner-Riesz means.

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Let L^p_{rad} be the space of all measurable functions f of the form $f(x) = g(|x|)$, for which

$$||f||_{L^p_{rad}} = \left(\int_0^\infty |g(s)|^p s^{n-1}\, ds \right)^{1/p}$$

is finite.

THEOREM 1. *Let $f \in L^p_{rad}(\mathbb{R}^n)$. Then for $\alpha > 0$, $0 < \delta \le (n-1)/2$, and $n \ge 2$*

$$|\{x \in \mathbb{R}^n : |S^\delta_* f(x)| > \alpha\}| \le C \left(\frac{||f||_{L^p_{rad}(\mathbb{R}^n)}}{\alpha} \right)^p$$

with $p = 2n/(n+1+2\delta)$. The constant C does not depend on α or f.

COROLLARY 1. *If $p = 2n/(n+1+2\delta)$ and $0 < \delta \le (n-1)/2$, $n \ge 2$, then*

$$S^\delta_\epsilon f \to f$$

almost everywhere as $\epsilon \to 0$ when $f \in L^p_{rad}$.

A version of Theorem 1 for Jacobi expansions was obtained by Chanillo and Muckenhoupt [3].

2. The maximal operator

In order to prove the theorem, we will show that S^δ_* is of weak type (p,p) acting on radial functions in $L^p(\mathbb{R}^n)$ where p is the critical value $2n/(n+1+2\delta)$ and $0 < \delta \le (n-1)/2$.

For the radial function $f(x)$, i.e., $f(x) = g(|x|)$, by Hankel transform we have

$$\widehat{f}(\xi) = \widetilde{g}(\rho) = \rho^{-(n-2)/2} \int_0^\infty g(s) J_{\frac{n-2}{2}}(\rho s) s^{n/2}\, ds.$$

Let $|x| = r$ and define $S^\delta_\epsilon f(x) = A^\delta_\epsilon g(r)$. Moreover,

$$A^\delta_\epsilon g(r) = \frac{1}{(2\pi)^n} r^{-(n-2)/2} \int_0^{1/\epsilon} (1 - \epsilon^2\rho^2)^\delta J_{\frac{n-2}{2}}(\rho r) \widetilde{g}(\rho) \rho^{n/2}\, d\rho.$$

Substituting the expression for $\widetilde{g}(\rho)$ into that for $A^\delta_\epsilon g(r)$, we see that

$$A^\delta_\epsilon g(r) = \frac{1}{(2\pi)^n} r^{-(n-2)/2} \int_0^\infty g(s) s^{n/2} K_\epsilon(r,s)\, ds$$

where

$$K_\epsilon(r,s) = \epsilon^{-2} \int_0^1 u(1-u^2)^\delta J_{\frac{n-2}{2}}(ur/\epsilon) J_{\frac{n-2}{2}}(us/\epsilon)\, du. \tag{2.1}$$

To proceed with the proof of the theorem, we state the following lemmas which are based on Chanillo and Muckenhaupt [2].

LEMMA 1. *Let $K_\epsilon(r,s)$ be defined as in (2.1). Then for $\delta > -1$ and $n \geq 2$, we have*

$$|K_\epsilon(r,s)| \leq \begin{cases} C\,\epsilon^{-2}\left((\frac{r}{\epsilon})(\frac{s}{\epsilon})\right)^{(n-2)/2} & \text{if } \frac{r}{\epsilon} \leq 2, \frac{s}{\epsilon} \leq 2, \\ C\,\epsilon^{-2}\left((1+\frac{r}{\epsilon})(1+\frac{s}{\epsilon})\right)^{-1/2} & \text{if } |\frac{r}{\epsilon} - \frac{s}{\epsilon}| \leq 2, \\ C\,\epsilon^{-2}\,|\frac{r}{\epsilon} - \frac{s}{\epsilon}|^{-(\delta+1)}\left((1+\frac{r}{\epsilon})(1+\frac{s}{\epsilon})\right)^{-1/2} & \text{if } |\frac{r}{\epsilon} - \frac{s}{\epsilon}| > 2. \end{cases}$$

LEMMA 2. *Let f be supported in $|x| \leq 2\epsilon$. Then for $|x| > 4\epsilon$, $p = 2n/(n+1+2\epsilon)$ and $0 < \epsilon \leq (n-1)/2$,*

$$|S^\delta_\epsilon f(x)| \leq C\,|x|^{-(n+1+2\delta)/2}\,||f||_{L^p_{rad}}.$$

Proof. From [9], p.171, it follows that

$$S^\delta_\epsilon f(x) = \frac{C}{(2\pi)^n}\int_{\mathbb{R}^n} f(y)\frac{\epsilon^{-n}}{(|x-y|/\epsilon)^{\delta+n/2}} J_{\delta+n/2}(|x-y|/\epsilon)\,dy.$$

Thus from the fact $|J_\mu(su)| \leq (1+su)^{-1/2}$ for $s,u > 0$ and $\mu \geq 0$ (see [9], p.158), we have

$$|S^\delta_\epsilon f(x)| \leq C\int_{|y|\leq 2\epsilon} f(y)\frac{\epsilon^{-n}}{(|x-y|/\epsilon)^{(n+1+2\delta)/2}}\,dy.$$

Suppose $p = 1$, that is $\delta = (n-1)/2$. If $|x| \geq 4\epsilon$, then $|x-y|/\epsilon \approx |x|/\epsilon$. Thus,

$$\begin{aligned} |S^{\frac{n-1}{2}}_\epsilon f(x)| &\leq C\,(|x|/\epsilon)^{-n}\int_{|y|\leq 2\epsilon}|f(y)|\,dy\,\epsilon^{-n} \\ &\leq C\,|x|^{-n}\,||f||_{L^1_{rad}}. \end{aligned}$$

Suppose $p > 1$, that is $0 < \delta < (n-1)/2$. With $1/p + 1/q = 1$, we have

$$\begin{aligned} |S^\delta_\epsilon f(x)| &\leq C\,\epsilon^{-n}\,(|x|/\epsilon)^{-(n+1+2\delta)/2}\int_{|y|\leq 2\epsilon}|f(y)|\,dy \\ &\leq C\,|x|^{-(n+1+2\delta)/2}\,||f||_{L^p_{rad}}. \end{aligned}$$ □

In the remaining part of our work we will prove Theorem 1.

Proof. We first consider the case $p > 1$, that is $0 < \delta < (n-1)/2$. We now consider the situation when $r \leq 4\epsilon$. We have

$$\begin{aligned} & |A_\epsilon^\delta g(r)| \\ \leq \;& C r^{-(n-2)/2}\left(\int_0^{8\epsilon} |g(s)| s^{n/2} \, |K_\epsilon(r,s)| \, ds + \int_{8\epsilon}^\infty |g(s)| s^{n/2} |K_\epsilon(r,s)| \, ds\right) \\ := \;& C r^{-(n-2)/2} \, (\mathcal{U} + \mathcal{V}). \end{aligned}$$

By Lemma 1 when $r \leq 4\epsilon$ and $s < 8\epsilon$, $|K_\epsilon(r,s)| \leq C \ \epsilon^{-2}\left((r/\epsilon)\right.$ $\left.(s/\epsilon)\right)^{(n-2)/2}$. Thus,

$$\begin{aligned} r^{-(n-2)/2}\, \mathcal{U} &\leq C\, \epsilon^{-n}\left(\int_0^{8\epsilon} |g(s)| s^{n-1} \, ds\right) \\ &\leq C\, \epsilon^{-n}\left(\int_0^{8\epsilon} |g(s)|^p s^{n-1} \, ds\right)^{1/p}\left(\int_0^{8\epsilon} s^{[(n-1)-\frac{(n-1)}{p}]q} \, ds\right)^{1/q} \\ &\leq C\, \epsilon^{-n/p} \, ||f||_{L^p_{rad}}. \end{aligned}$$

Since $r \leq 4\epsilon$,

$$r^{-(n-2)/2}\, \mathcal{U} \leq C\,(r/\epsilon)^{-n/p}\, ||f||_{L^p_{rad}}\, \epsilon^{-n/p} = C\, r^{-(n+1+2\delta)/2}\, ||f||_{L^p_{rad}}.$$

Now consider $r^{-(n-2)/2}\, \mathcal{V}$. Since $r \leq 4\epsilon$ and $s > 8\epsilon$, we have $s > 2r$, and thus by Lemma 1, with $1/p + 1/q = 1$, we have

$$\begin{aligned} r^{-(n-2)/2}\, \mathcal{V} &\leq C\, r^{-(n-2)/2} \int_{8\epsilon}^\infty |g(s)| s^{n/2}\, (s/\epsilon)^{-(\delta+\frac{3}{2})}\, \epsilon^{-2}\, ds \\ &\leq C\, \epsilon^{(-\frac{1}{2}+\delta)}\, r^{-(n-2)/2}\left(\int_{8\epsilon}^\infty |g(s)|^p s^{n-1} \, ds\right)^{1/p} \\ &\quad \times \left(\int_{8\epsilon}^\infty s^{[\frac{n}{2}-(\delta+\frac{3}{2})-\frac{(n-1)}{p}]q} \, ds\right)^{1/q} \\ &\leq C\,(r/\epsilon)^{-(n-2)/2}\, \epsilon^{-(n+1+2\delta)/2} ||f||_{L^p_{rad}}. \end{aligned}$$

Since $(r/\epsilon)^{-(n-2)/2} \leq (r/\epsilon)^{-(n+1+2\delta)/2}$,

$$r^{-(n-2)/2}\, \mathcal{V} \leq C\, r^{-(n+1+2\delta)/2}\, ||f||_{L^p_{rad}}.$$

Thus for $r \leq 4\epsilon$,

$$|A_\epsilon^\delta g(r)| \leq C\, r^{-(n+1+2\delta)/2}\, ||f||_{L^p_{rad}}. \tag{2.2}$$

We now pass to the case $r > 4\epsilon$. We make a preliminary reduction. Let $g(s) = g_1(s) + g_2(s)$, where $g_1(s) = g(s) \cdot \chi_{(s<2\epsilon)}$. Then by Lemma 2,

$$|A_\epsilon^\delta g(r)| \;\leq\; C\, r^{-(n+1+2\delta)/2}\, \left(\int |g_1(s)|^p s^{n-1} ds\right)^{1/p}. \tag{2.3}$$

We now estimate $S^\delta_\epsilon g_2(r)$. We then have

$$\begin{aligned} |A^\delta_\epsilon g_2(r)| &\leq C\, r^{-(n-2)/2}\Big(\int_{2\epsilon}^{r/2} + \int_{r/2}^{2r} + \int_{2r}^{\infty} |g_2(s)| s^{n/2}\, |K_\epsilon(r,s)|\, ds \Big) \\ &:= C\, r^{-(n-2)/2}\, (\mathcal{X} + \mathcal{Y} + \mathcal{Z}). \end{aligned}$$

By Lemma 1, if $2\epsilon < s < r/2$, then $|K_\epsilon(r,s)| \leq C\,\epsilon^{-2}(r/\epsilon)^{-(\delta+3/2)}(s/\epsilon)^{-1/2}$. Thus with $1/p + 1/q = 1$, we have

$$\begin{aligned} r^{-(n-2)/2}\, \mathcal{X} &\leq C\, r^{-(n-2)/2} \int_{2\epsilon}^{r/2} |g(s)| s^{n/2}\, (r/\epsilon)^{-(\delta+3/2)}\, (s/\epsilon)^{-1/2}\, \epsilon^{-2}\, ds \\ &\leq C\, (r/\epsilon)^{-(n+1+2\delta)/2}\, \epsilon^{-(n+1)/2} \left(\int_0^\infty |g(s)|^p s^{n-1}\, ds \right)^{1/p} \qquad (2.4) \\ &\quad \times \left(\int_{2\epsilon}^{\infty} s^{[\frac{(n-1)}{2} - \frac{(n-1)}{p}]q}\, ds \right)^{1/q} \\ &\leq C\, r^{-(n+1+2\delta)/2} ||f||_{L^p_{rad}}. \end{aligned}$$

For the integral $\mathcal{Z}$, since $s > 2r$,

$$|K_\epsilon(r,s)| \leq C\epsilon^{-2}(s/\epsilon)^{-(\delta+1)}\big((r/\epsilon)(s/\epsilon)\big)^{-1/2}.$$

Thus with $1/p + 1/q = 1$, we have

$$\begin{aligned} r^{-(n-2)/2} \mathcal{Z} &\leq C\, r^{-(n-2)/2} \int_{2r}^{\infty} |g(s)| s^{n/2}\, (s/\epsilon)^{-(\delta+\frac{3}{2})}\, (r/\epsilon)^{-1/2}\, \epsilon^{-2}\, ds \\ &\leq C\, r^{-(n-1)/2}\, \epsilon^{\delta} \left(\int_0^\infty |g(s)|^p s^{n-1}\, ds \right)^{1/p} \qquad (2.5) \\ &\quad \times \left(\int_{2r}^{\infty} s^{[\frac{n}{2} - (\delta+\frac{3}{2}) - \frac{(n-1)}{p}]q}\, ds \right)^{1/q} \\ &\leq C\, (r/\epsilon)^{-\delta}\, r^{-(n+1+2\delta)/2}\, ||f||_{L^p_{rad}}. \end{aligned}$$

Since $r > 4\epsilon$,

$$r^{-(n-2)/2}\ \mathcal{Z} \leq C\, r^{-(n+1+2\delta)/2} ||f||_{L^p_{rad}}.$$

We now consider $r^{-(n-2)/2}\mathcal{Y}$. We first break up the range of the integration for the integral $\mathcal{Y}$ as follows:

$$\begin{aligned} r^{-(n-2)/2}\, \mathcal{Y} &\leq r^{-(n-2)/2} \int_{\{r/2<s<2r\}\cap\{|r-s|\leq 2\epsilon\}} |g(s)| s^{n/2}\, |K_\epsilon(r,s)|\, ds \\ &\quad + r^{-(n-2)/2} \int_{\{r/2<s<2r\}\cap\{|r-s|>2\epsilon\}} |g(s)| s^{n/2}\, |K_\epsilon(r,s)|\, ds. \end{aligned}$$

By Lemma 1, when $|r-s| \le 2\epsilon$, $|K_\epsilon(r,s)| \le C\,\epsilon^{-2}((r/\epsilon)(s/\epsilon))^{-1/2}$. Thus,

$$\begin{aligned} & r^{-(n-2)/2} \int_{\{r/2<s<2r\}\cap\{|r-s|\le 2\epsilon\}} |g(s)| s^{n/2}\, |K_\epsilon(r,s)|\, ds \\ \le\ & C\,\epsilon^{-1} \int_{|r-s|\le 2\epsilon} |g(s)|\, \chi_{\{r/2<s<2r\}}\, ds. \end{aligned} \tag{2.6}$$

Moreover, by Lemma 1, when $|r-s| > 2\epsilon$, we thus have

$$\begin{aligned} & r^{-(n-2)/2} \int_{\{r/2<s<2r\}\cap\{|r-s|>2\epsilon\}} |g(s)| s^{n/2}\, |K_\epsilon(r,s)|\, ds \\ \le\ & C\,\epsilon^{-1}\, (r/\epsilon)^{-(n-1)/2} \\ & \quad \int_{|r-s|>2\epsilon} |g(s)| (s/\epsilon)^{(n-1)/2}\, |r/\epsilon - s/\epsilon|^{-(\delta+1)}\, \chi_{\{r/2<s<2r\}}\, ds \\ \le\ & C\,\epsilon^{-1} \int_{|r-s|>2\epsilon} |g(s)|\, \chi_{\{r/2<s<2r\}}\, |r/\epsilon - s/\epsilon|^{-(\delta+1)}\, ds \\ \le\ & C \sum_{k\ge 0} 2^{-k\delta} \Big(2^{-k}\, \epsilon^{-1} \int_{|r-s|\sim 2^k\epsilon} |g(s)|\, \chi_{\{r/2<s<2r\}}\, ds \Big). \end{aligned} \tag{2.7}$$

Hence from (2.6) and (2.7),

$$\begin{aligned} r^{-(n-2)/2}\, \mathcal{Y}\ \le\ & C\,\epsilon^{-1} \int_{|r-s|\le 2\epsilon} |g(s)|\, \chi_{\{r/2<s<2r\}}\, ds \\ & + C \sum_{k\ge 0} 2^{-k\delta} \Big(2^{-k}\, \epsilon^{-1} \int_{|r-s|\sim 2^k\epsilon} |g(s)|\, \chi_{\{r/2<s<2r\}}\, ds \Big). \end{aligned} \tag{2.8}$$

Together with (2.2) – (2.5) and (2.8), we obtain

$$\begin{aligned} A_*^\delta g(r) &= \sup_{\epsilon>0} |A_\epsilon^\delta g(r)| \\ &\le\ C\, r^{-(n+1+2\delta)/2}\, \|f\|_{L^p_{rad}} + C\,\mathcal{M}[g\chi_{D_r}](r), \end{aligned}$$

where $\mathcal{M}$ is the Hardy-Littlewood operator and $D_r = \{s : r/2 < s < 2r\}$.

Thus, letting $I_k = \{r : 2^{k-1} \le r < 2^k\}$ and $D_k = \{s : 2^{k-2} \le s < 2^{k+1}\}$, we have

$$
\begin{aligned}
& \int_{\{r:\, |A_*^\delta g(r)|>\alpha\}} r^{n-1}\,dr \\
\le\ & \int_{\{r:\, r^{-(n+1+2\delta)/2}\,||f||_{L^p_{rad}} > \alpha/2C\}} r^{n-1}\,dr + \int_{\{r:\, \mathcal{M}[g\chi_{D_r}](r)>\alpha/2C\}} r^{n-1}\,dr \\
\le\ & C\,\alpha^{-p}||f||^p_{L^p_{rad}} + \sum_{k=-\infty}^{\infty} \int_{\{r\in I_k:\, \mathcal{M}[g\chi_{D_k}](r)>\alpha/2C\}} r^{n-1}\,dr \\
\le\ & C\,\alpha^{-p}||f||^p_{L^p_{rad}} + \sum_{k=-\infty}^{\infty} 2^{k(n-1)} \int_{\{r\in I_k:\, \mathcal{M}[g\chi_{D_k}](r)>\alpha/2C\}} dr.
\end{aligned}
$$

By the weak type (p,p) estimates for the Hardy-Littlewood operator, we can majorize the expression above by

$$
\begin{aligned}
& C\,\alpha^{-p}\left(||f||^p_{L^p_{rad}} + \sum_{k=-\infty}^{\infty} 2^{k(n-1)} \int_{D_k} |g(s)|^p\,ds\right) \\
\le\ & C\,\alpha^{-p}||f||^p_{L^p_{rad}}.
\end{aligned}
$$

We now prove the case $p=1$, that is $\delta = (n-1)/2$. The only expressions we treat differently for this case are the integrals $\mathcal{V}$, $\mathcal{X}$, and $\mathcal{Z}$. The other parts are similar to the case $p>1$. Thus if $p=1$, since $r \le 4\epsilon$ and $s > 8\epsilon$,

$$
\begin{aligned}
r^{-(n-2)/2}\,\mathcal{V} &\le C\,r^{-(n-2)/2} \int_{8\epsilon}^{\infty} |g(s)| s^{n/2}\,(s/\epsilon)^{-(\delta+3/2)}\,\epsilon^{-2}\,ds \\
&\le C\,r^{-(n-2)/2}\,\epsilon^{(n-2)/2} \int_{8\epsilon}^{\infty} |g(s)| s^{-1}\,ds \\
&\le C\,(r/\epsilon)^{-n} \int_{8\epsilon}^{\infty} |g(s)| (s/\epsilon)^n\,s^{-1}\,ds \\
&\le C\,r^{-n} \int_{8\epsilon}^{\infty} |g(s)| s^{n-1}\,ds = C\,r^{-n}\,||f||_{L^1_{rad}}.
\end{aligned}
$$

Now, since $r > 4\epsilon$ and $s < r/2$,

$$
\begin{aligned}
r^{-(n-2)/2}\,\mathcal{X} &\le C\,r^{-(n-2)/2} \int_{2\epsilon}^{r/2} |g(s)| s^{n/2}\,(r/\epsilon)^{-(\delta+3/2)}\,(s/\epsilon)^{-1/2}\,\epsilon^{-2} ds \\
&\le C\,(r/\epsilon)^{-n} \int_{2\epsilon}^{r/2} |g(s)| (s/\epsilon)^{(n-1)/2}\,\epsilon^{-1}\,ds \\
&\le C\,(r/\epsilon)^{-n} \int_{2\epsilon}^{\infty} |g(s)| (s/\epsilon)^{(n-1)}\,\epsilon^{-1}\,ds \le C\,r^{-n}\,||f||_{L^1_{rad}}.
\end{aligned}
$$

Likewise, since $r > 4\epsilon$ and $s > 2r$,

$$\begin{aligned} r^{-(n-2)/2}\,\mathcal{Z} &\leq C\,r^{-(n-2)/2}\int_{2r}^{\infty}|g(s)|s^{n/2}\,(s/\epsilon)^{-(\delta+3/2)}\,(r/\epsilon)^{-1/2}\,\epsilon^{-2}\,ds \\ &\leq C\,(r/\epsilon)^{-(n-1)/2}\int_{2r}^{\infty}|g(s)|(s/\epsilon)^{-1}\,\epsilon^{-1}\,ds \\ &\leq C\,(r/\epsilon)^{-(n-1)/2}\left(\int_{2r}^{\infty}|g(s)|(s/\epsilon)^{n-1}\,ds\right)(r/\epsilon)^{-n}\,\epsilon^{-1} \\ &\leq C\,r^{-n}\,\|f\|_{L^1_{rad}}. \end{aligned}$$

We may proceed as in the case $p > 1$. This proves Theorem 1. □

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