Bull. Korean Math. Soc. **38** (2001), No. 1, pp. 157–161

AFFINE MANIFOLD WITH MEASURE PRESERVING PROJECTIVE HOLONOMY GROUP

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ABSTRACT. In this paper, we prove that an affine manifold M is finitely covered by a manifold $\bar{M}$ where $\bar{M}$ is radiant or the tangent bundle of $\bar{M}$ has a conformally flat vector subbundle if the projective holonomy group of M admits an invariant probability Borel measure. This implies that $\chi(M)$ is zero.

1. Introduction

A smooth manifold M is said to be affinely(projectively resp.) flat if M is covered by coordinate charts in the affine(projective resp.) space and coordinate transitions are affine(projective resp.) transformations. An affinely flat manifold is called affine manifold, in short.

It was conjectured that the Euler characteristic of a closed affine manifold is zero in [7]. In [1] Benzecri proved that a closed orientable surface admits an affine structure if and only if it is a torus. The conjecture is affirmative in dimension 2, see also [7].

For a complete closed affine manifold, Kostant and Sullivan proved the conjecture [6]. For a general closed affine manifold M, Hirsh and Thurston showed the conjecture assuming that the fundamental group $\pi_1(M)$ is amenable [4].

In [5], Jo and Kim studied holonomy invariant measures of projectively flat manifolds. A holonomy invariant probability measure on $\mathbb{RP}^n$ induces a measure on a projectively flat manifold M. They proved that the Euler characteristic of M and the total measure induced on M are equal if M is even dimensional. Using this result, they proved that the Euler characteristic of an affine manifold vanishes if the holonomy group admits an invariant probability measure on $\mathbb{RP}^n$, the compactification of $\mathbb{R}^n$.

In this paper, we will prove that an affine manifold M is radiant or the tangent bundle of M contains a conformally flat vector subbundle if the

Received March 27, 2000. Revised July 28, 2000.
2000 Mathematics Subject Classification: 53C15, 57N16.
Key words and phrases: affine manifold, invariant measure, Euler characteristic.

projective holonomy group preserves a probability Borel measure. Hence such a manifold is of Euler characteristic zero. It is an affirmative result on the above conjecture and a generalization of the result in [4]. If the projective holonomy group leaves a probability Borel measure invariant then it can be represented to a projective general linear group $\mathrm{PGL}(V)$ for some vector space V so that its image has a compact closure(see Corollary 2.2). The holonomy group or its double covering group has a compact closure in $\mathrm{SL}(V)$. This is useful tool for this paper.

All measures mentioned below are Borel measures.

2. Measure preserving subgroup of $\mathrm{PGL}(n+1,\mathbb{R})$

Let X be a compact metric space. Let $M(X)$ be the space of probability measures on X and $C(X)$ continuous real-valued functions on X. By the Riesz representation theorem, $M(X)$ can be identified with a subset of the dual space $C(X)^*$. On $C(X)^*$, we endow the dual norm of the supremum norm on $C(X)$. Then $M(X)$ is a closed subset of the unit ball in $C(X)^*$. By the Banach-Alaoglu theorem the unit ball in $C(X)^*$ is weak*-compact and thus so is $M(X)$.

Suppose that a group G acts on X. This action induces an action of G on $M(X)$ as follows

$$(\mu \cdot g)(A) = \mu(gA)$$

for any $\mu \in M(X)$, $g \in G$ and any measurable set $A \subset X$.

The following lemma is proved by Furstenberg [3], [8].

LEMMA 2.1. *Suppose that* $g_m \in \mathrm{PGL}(n+1,\mathbb{R})$, μ, $\nu \in M(\mathbb{R}\mathrm{P}^n)$ *and that* $\mu \cdot g_m \to \nu$ *as* $m \to \infty$. *Then either*

(i) $\{g_m | m = 1, 2, \ldots\}$ *is bounded, i.e. has compact closure in* $\mathrm{PGL}(n+1,\mathbb{R})$ *or*

(ii) *there exist proper linear subspaces* V *and* W *of* $\mathbb{R}^{n+1}$ *such that* ν *is supported on* $[V] \cup [W]$.

The projectivization of a linear subspace V of $\mathbb{R}^{n+1}$ is denoted by $[V]$. The above lemma implies that

COROLLARY 2.2. *Let* G *be a subgroup of* $\mathrm{PGL}(n+1,\mathbb{R})$ *leaving a probability measure invariant. Then either*

(i) G *has compact closure or*

(ii) *there are a proper linear subspace* $V \subset \mathbb{R}^{n+1}$ *and a finitely indexed subgroup* G' *of* G *such that* $[V]$ *is invariant under* G' *and* $G'|_{[V]}$ *has compact closure in* $\mathrm{PGL}(V)$.

Proof. Suppose that G leaves a probability measure μ on $\mathbb{RP}^n$ invariant. Suppose that the closure of G is not compact in $\mathrm{PGL}(n+1,\mathbb{R})$. Let $[V]$ be a subspace of minimal dimension such that $\mu[V] > 0$. By minimality, $\mu([V] \cap g[V]) = 0$ and $\mu([V] \cup g[V]) = 2\mu[V]$ if $[V] \neq g[V]$. Therefore $\{g[V] | g \in G\}$ has only finitely many elements. Let G' be the subgroup of G which leaves $[V]$ invariant. Since every projective subspace of $[V]$ is of measure zero, $G'|_{[V]}$ has compact closure. □

3. Structure of affine manifold

Let M be a compact orientable affine manifold of dimension $n > 1$. M can be considered as a projectively flat manifold using the standard inclusion from $\mathrm{Aff}(n)$ into $\mathrm{PGL}(n+1,\mathbb{R})$ given by

$$(a, A) \mapsto \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}.$$

Let $D : \tilde{M} \to \mathbb{R}^n$ be a developing map and $\rho : \pi_1(M) \to \mathrm{Aff}(n)$ be the corresponding holonomy homomorphism. We denote the holonomy group $\mathrm{im}\rho$ by Γ. Let D', ρ' and Γ' be the developing map, the holonomy homomorphism and the holonomy group of the induced projective structure, respectively. Here we identify $\mathbb{R}^n$ with the affine subspace $\mathbb{R}^n \times \{1\}$ of $\mathbb{R}^{n+1}$.

Suppose that μ is a Γ'-invariant probability measure on $\mathbb{RP}^n$. Then Γ' either has compact closure in $\mathrm{PGL}(n+1,\mathbb{R})$ or has a finitely indexed subgroup which has compact closure in $\mathrm{PGL}(V)$ for some proper linear subspace V of $\mathbb{R}^{n+1}$ by the Corollary 2.2.

First of all, we assume that Γ' has compact closure in $\mathrm{PGL}(n+1,\mathbb{R})$. Since the natural projection from $\mathrm{SL}(n+1,\mathbb{R})$ to $\mathrm{PGL}(n+1,\mathbb{R})$ is either a double covering or an isomorphism, the lifting Γ'' of Γ' has also compact closure in $\mathrm{SL}(n+1,\mathbb{R})$. Thus $\mathbb{R}^{n+1}$ admits a Γ''-invariant Riemannian metric. The action of Γ'' on the unit sphere with respect to this metric is orthogonal. Since the restriction of this metric on the sphere is invariant under the antipodal map, $\mathbb{RP}^n$ admits the induced Riemannian metric which is Γ'-invariant. The pull-back metric on $\tilde{M}$ is invariant under the deck transformation group $\pi_1(M)$ and M admits the induced Riemannian metric. Since M is compact, M is a complete Riemannian manifold and so is $\tilde{M}$. Since D' is an infinitesimal isometry, D' is a covering and hence $\tilde{M}$ is diffeomorphic to $\mathbb{S}^n$. This is absurd, because $\mathbb{S}^n$ admits no affine structure whenever $n > 1$.

Now suppose that there are a proper linear subspace $V \subset \mathbb{R}^{n+1}$ and a finite indexed subgroup G' of Γ' such that $[V]$ is invariant under G' and $G'|_{[V]}$ has compact closure in $\mathrm{PGL}(V)$. Taking a finite covering of M, we may assume that $[V]$ is Γ'-invariant and $\Gamma'|_{[V]}$ has compact closure. There is also a $\Gamma''|_V$-invariant Riemannian metric on V, as shown in the above case.

Suppose that V is not contained in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Let $W = V \cap (\mathbb{R}^n \times \{0\})$. Then W is of codimension 1 in V and Γ''-invariant. The orthogonal complement $W^\perp$ of W in V is also Γ''-invariant. $[W^\perp] \in \mathbb{RP}^n$ is a fixed point of Γ'. Thus Γ has a fixed point and M is a radiant manifold.

As the last case, suppose that $V \subset \mathbb{R}^n \times \{0\}$. Let $k = \dim V$. We choose an ordered basis of $\mathbb{R}^n$ such that first k vectors are orthonormal and generate V. Every element in Γ can be represented in the form:

$$\left(\begin{pmatrix} * \\ 0 \end{pmatrix}, \begin{pmatrix} A & * \\ 0 & * \end{pmatrix} \right)$$

where $A \in \mathbb{R}_+ \cdot \mathrm{O}(k)$. Therefore the bundle $\tilde{M} \times_{Lin \circ \rho} \mathbb{R}^n$ of M contains a conformally flat vector subbundle $\tilde{M} \times_{\varphi \circ Lin \circ \rho} V$ where $\varphi : Lin(\Gamma) \to \mathbb{R}_+ \cdot \mathrm{O}(k)$ takes the first $k \times k$ minor matrix. The homomorphism $Lin : \mathrm{Aff}(n) \to \mathrm{GL}(n, \mathbb{R})$ takes the linear part of an affine transformation. It is easy to show $\tilde{M} \times_{Lin \circ \rho} \mathbb{R}^n$ is the tangent bundle of M. Hence the tangent bundle contains a conformally flat vector subbundle.

We summarize the above discussion.

THEOREM 3.1. *Let M be a compact affine manifold of dimension $n > 1$. If the projective holonomy group leaves a probability measure invariant then there is a finite covering $\bar{M}$ of M such that $\bar{M}$ is a radiant manifold or the tangent bundle of $\bar{M}$ contains a conformally flat vector subbundle.*

The following Corollary is already proved by K. Jo and H. Kim [5]. Combining above theorem and lemma, we can prove it.

COROLLARY 3.2. *Suppose that the projective holonomy group of an affine manifold M admits an invariant probability measure. Then the Euler characteristic of M is zero.*

Proof. It is well known that a radiant affine manifold if of Euler characteristic zero. Hence we assume that M is not radiant. A space is of Euler characteristic zero if and only if so is its finite covering space. Thus we may assume that the tangent bundle of M contains a conformally flat vector subbundle ξ which is of Euler characteristic zero. The Euler characteristic of M, which is the same that the Euler number of the tangent bundle of M, vanishes. $\square$

The discussion written in the above of Theorem 3.1 can be applied to the case of a flat vector bundle.

THEOREM 3.3. *If E is a flat vector bundle of rank $n+1$ over a smooth manifold M and its projectivized linear holonomy group admits an invariant probability measure on $\mathbb{R}\mathrm{P}^n$, then E has a conformally flat vector subbundle and, in particular, the Euler number of E vanishes.*

It is natural to ask that which subgroup of $\mathrm{PGL}(n+1, \mathbb{R})$ can leave a probability measure on $\mathbb{R}\mathrm{P}^n$ invariant. A group G is said to be *amenable* if for every continuous G-action on a compact metrizable space X there is a G-invariant probability measure on X. See also [2], [8]. For example, abelian groups, solvable groups and homomorphic images of an amenable groups are amenable.

A famous result related to amenability was proved by M. Hirsh and W. Thurston. Our result gives another way of proving their theorem.

COROLLARY 3.4. *Let M be a closed affine manifold. If $\pi_1(M)$ is amenable then $\chi(M) = 0$.*

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