WEYL SPECTRA OF THE $\chi$-CLASS OPERATORS

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Abstract. In this paper we introduce a notion of the $\chi$-class operators, which is a class including hyponormal operators and consider their spectral properties related to Weyl spectra.

Introduction

Throughout this paper let $\mathcal{H}$ denote an infinite dimensional separable Hilbert space. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$ and $\mathcal{K}(\mathcal{H})$ the closed ideal of compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$ write $N(T)$ and $R(T)$ for the null space and range of $T$; $\rho(T)$ for the resolvent set of $T$; $\sigma(T)$ for the spectrum of $T$; $\pi_0(T)$ for the set of eigenvalues of $T$; $\pi_{0f}(T)$ for the eigenvalues of finite multiplicity; $\pi_{0i}(T)$ for the eigenvalues of infinite multiplicity. Recall ([12],[13]) that $T \in \mathcal{L}(\mathcal{H})$ is called regular if there is an operator $T' \in \mathcal{L}(\mathcal{H})$ for which $T = TT'T$. It is familiar that if $T \in \mathcal{L}(\mathcal{H})$ then $T$ is regular if and only if $T$ has closed range. An operator $T \in \mathcal{L}(\mathcal{H})$ is called upper semi-Fredholm if it has closed range with finite-dimensional null space and lower semi-Fredholm if it has closed range with its range of finite co-dimension. If $T$ is either upper or lower semi-Fredholm, we call it semi-Fredholm and if $T$ is both upper and lower semi-Fredholm, we call it Fredholm. The index of a semi-Fredholm operator $T \in \mathcal{L}(\mathcal{H})$ is given by

$$\text{ind}(T) = \dim N(T) - \dim R(T)^\perp = (\dim N(T) - \dim N(T^*))$$

An operator $T \in \mathcal{L}(\mathcal{H})$ is called Weyl if it is Fredholm of index zero. An operator $T \in \mathcal{L}(\mathcal{H})$ is called Browder if it is Fredholm "of finite ascent and descent": equivalently ([13, Theorem 7.9.3]) if $T$ is Fredholm and $T - \lambda I$ is

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invertible for sufficiently small $\lambda \neq 0$ in $\mathbb{C}$. The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\omega(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in \mathcal{L}(\mathcal{H})$ are defined by

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm} \};$$

$$\omega(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl} \};$$

$$\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder} \};$$

then ([13])

(0.1) \quad \sigma_e(T) \subseteq \omega(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{acc } \sigma(T) \quad \text{and} \quad \omega(T) \subseteq \eta \sigma_e(T),

where we write acc $K$ and $\eta K$ for the accumulation points and the polynomially-convex hull, respectively, of $K \subseteq \mathbb{C}$. If we write iso $K = K \setminus \text{acc } K$, and $\partial K$ for the topological boundary of $K$, and

(0.2) \quad \pi_{00}(T) := \{ \lambda \in \text{iso } \sigma(T) : 0 < \dim (T - \lambda I)^{-1}(0) < \infty \}

for the isolated eigenvalues of finite multiplicity, and ([13])

(0.3) \quad p_{00}(T) := \sigma(T) \setminus \sigma_b(T)

for the Riesz points of $\sigma(T)$, then by the punctured neighborhood theorem, i.e., $\partial \sigma(T) \setminus \sigma_e(T) \subseteq \text{iso } \sigma(T)$ ([13], [14]),

(0.4) \quad \text{iso } \sigma(T) \setminus \sigma_e(T) = \text{iso } \sigma(T) \setminus \omega(T) = p_{00}(T) \subseteq \pi_{00}(T).

We say that Weyl's theorem holds for $T \in \mathcal{L}(\mathcal{H})$ if there is equality

(0.5) \quad \sigma(T) \setminus \omega(T) = \pi_{00}(T).

If $T \in \mathcal{L}(\mathcal{H})$, write $r(T)$ for the spectral radius of $T$. It is familiar that $r(T) \leq ||T||$. An operator $T$ is called normaloid if $r(T) = ||T||$ and isoloid if $\sigma(T) \subseteq \pi_0(T)$. An operator $T$ is said to satisfy condition $(G_1)$ if $(T - \lambda I)^{-1}$ is normaloid for all $\lambda \notin \sigma(T)$. If $T \in \mathcal{L}(\mathcal{H})$, write $W(T)$ for the numerical range of $T$. It is also familiar that $W(T)$ is convex and $\sigma(T) \subseteq \text{cl } W(T)$. An operator $T$ is called convexoid if $\text{conv } \sigma(T) = \text{cl } W(T)$. Let $P$ be a property of operators. We say that an operator $T$ is restriction-$P$ if the restriction of $T$ to every invariant subspace has property $P$ and that $T$ is reduction-$P$ if every direct summand of $T$ has property $P$. Evidently, restriction-$P \implies$ reduction-$P$. It is known ([3]) that if $T \in \mathcal{L}(\mathcal{H})$ then we have:
(0.6) \((G_1) \implies\) convexoid and isoloid;
(0.7) seminormal \(\implies\) reduction-(\(G_1\)) \(\implies\) reduction-isoloid;
(0.8) hyponormal \(\implies\) restriction-convexoid \(\implies\) reduction-isoloid.

Note that seminormal operators are reduction-convexoid, but they may not be restriction-convexoid: for example consider the backward shift \(U^*\) on \(\ell_2\), where \(U\) is the unilateral shift ([4]). Thus the replacement of "reduction-
" by "restriction-" is very stringent. Now we shall say that an operator \(T \in \mathcal{L}(\mathcal{H})\) is in the \(\chi\)-class if \(T\) is restriction-convexoid and is reduced by each of its eigenspaces corresponding to isolated eigenvalues. Evidently, \(T\) hyponormal \(\implies\) \(T \in \chi\).

1. The \(\chi\)-class operators

An operator \(T \in \mathcal{L}(\mathcal{H})\) is called reguloid ([15]) if \(T - \lambda I\) is regular for each \(\lambda \in \text{iso} \sigma(T)\). We begin with:

**Lemma 1.1.** If \(T \in \mathcal{L}(\mathcal{H})\) then

\[(G_1) \implies \text{reguloid} \implies \text{isoloid} \]

and

\[(1.1.2) \text{restriction-convexoid} \implies \text{restriction-reguloid}.
\]

**Proof.** (1.1.1) is [15, Theorem 14]. For (1.1.2), suppose \(T\) is restriction-convexoid and \(\mathcal{M}\) is an invariant subspace of \(T\). Write \(S := T|\mathcal{M}\). Then \(S\) is also restriction-convexoid. Suppose \(\lambda \in \text{iso} \sigma(S)\). Observe that if \(T\) is convexoid then so is \(aT + bI\) for any \(a, b \in \mathbb{C}\). Thus we may write \(S\) in place of \(S - \lambda I\) and assume \(\lambda = 0\). Using the spectral projection at \(0 \in \mathbb{C}\) we can write \(S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}\), where \(\sigma(S_1) = \{0\}\) and \(\sigma(S_2) = \sigma(S) \setminus \{0\}\). Since by assumption, \(S_1\) is convexoid it follows that \(W(S_1) = \text{conv} \sigma(S_1) = \{0\}\), and hence \(S_1 = 0\). Thus we have

\[S = \begin{pmatrix} 0 & 0 \\ 0 & S_2 \end{pmatrix} = SS'S \text{ with } S' = \begin{pmatrix} 0 & 0 \\ 0 & S_2^{-1} \end{pmatrix},\]

which says that \(S\) is regular, and therefore \(T\) is restriction-reguloid. \(\square\)

It was shown in ([24]) that Weyl's theorem holds for restriction-convexoid operators. We can prove more:
Theorem 1.2. Let $T \in \mathcal{L}(\mathcal{H})$. If either $T$ or $T^*$ is restriction-convexoid then Weyl’s theorem holds for $T$.

Proof. If $T$ is restriction-convexoid then it follows from [24, Theorem 2.1] that Weyl’s theorem holds for $T$. Now suppose $T^*$ is restriction-convexoid. Let $\lambda \in \pi_{00}(T)$. Then $\bar{\lambda} \in \text{iso } \sigma(T^*)$. Since $T^*$ is restriction-convexoid, it follows from Lemma 1.1 that $T - \lambda$ has closed range. Therefore it follows from the punctured neighborhood theorem that $\lambda \in \sigma(T)\omega(T)$. Conversely, suppose $\lambda \in \sigma(T)\setminus \omega(T)$. Then $\bar{\lambda} \in \sigma(T^*)\setminus \omega(T^*)$. Since $T^*$ is restriction-convexoid, Browder’s theorem holds for $T^*$. Therefore $\bar{\lambda} \in p_{00}(T^*)$. It follows from the fact $p_{00}(T^*) = p_{00}(T)^*$ that $\lambda \in \pi_{00}(T)$. This completes the proof. \qed

In 1970, S. Berberian ([5]) raised the following question: if $T$ is restriction-convexoid and $\sigma(T)$ is countable, is $T$ normal? We now give a partial answer.

Theorem 1.3. Let $T \in \chi$. If $\sigma(T)$ is countable then $T$ is diagonal and normal.

Proof. Suppose $T \in \chi$ and $\sigma(T)$ is countable. Let $\delta$ be the set of all normal eigenvalues of $T$, i.e.,

$$\delta = \{ \lambda \in \pi_0(T) : N(T - \lambda I) = N(T^* - \bar{\lambda}I) \}.$$

We first claim that $\delta \neq \emptyset$. Since $\sigma(T)$ is countable, there exists a point $\lambda \in \text{iso } \sigma(T)$, so that $\lambda \in \pi_0(T)$ because by Lemma 1.1, $T$ is isoloid. Using the spectral projection at $\lambda \in \mathbb{C}$ we can represent $T$ as the direct sum

$$T = R \oplus S,$$

where $\sigma(R) = \pi_0(R) = \{ \lambda \}$ and $\sigma(S) = \sigma(T)\setminus \{ \lambda \}$.

Since by assumption $R$ is convexoid, we have that $W(R) = \text{conv } \{ \lambda \} = \{ \lambda \}$ and thus $\lambda \in \pi_0(R) \cap \partial W(R)$. Then an argument of Bouldin [6, Lemma 1] shows that $\lambda$ is a normal eigenvalue of $R$. By assumption we can write $T^* = R^* \oplus S^*$. But since $S^* - \bar{\lambda}I$ is invertible, it follows

$$N(T - \lambda I) = N(R - \lambda I) = N(R^* - \bar{\lambda}I) = N(T^* - \bar{\lambda}I),$$

which implies that $\delta \neq \emptyset$. Now if $\mathfrak{M}$ is the closed linear span of the eigenspaces $N(T - \lambda I)$ ($\lambda \in \delta$), then $\mathfrak{M}$ reduces $T$. Write

$$T_1 := T|\mathfrak{M} \quad \text{and} \quad T_2 := T|\mathfrak{M}^\perp.$$
Then an argument of Berberian [3, Proposition 4.1] shows that (i) $T_1$ is normal and diagonal; (ii) $\pi_0(T_1) = \delta$; (iii) $\sigma(T_1) = \text{cl} \delta$; (iv) $\pi_0(T_2) = \pi_0(T) \setminus \delta$. Thus it will suffice to show that $\mathfrak{M}^\perp = \{0\}$. Assume to the contrary that $\mathfrak{M}^\perp \neq \{0\}$. Then since $\sigma(T_2)$ is also countable, there exists a point $\mu \in \text{iso} \sigma(T_2)$. Since by assumption $T_2$ is restriction-convexoid and hence isoloid, it follows that $\mu \in \pi_0(T_2)$ and $\mu \notin \delta$. Thus using the spectral projection at $\mu \in \mathbb{C}$, we can decompose $T_2$ as the direct sum

$$T_2 = T_3 \oplus T_4,$$

where $\sigma(T_3) = \pi_0(T_3) = \{\mu\}$ and $\sigma(T_4) = \sigma(T_2) \setminus \{\mu\}$. Since again $T_3$ is convexoid, the same argument as the above gives that $\mu$ is an isolated normal eigenvalue of $T_3$ and further by assumption $T_2 = T_3^* \oplus T_4^*$. But since $T_1 - \mu I$ and $T_4 - \mu I$ are both one-one we have

$$N(T - \mu I) = N(T_3 - \mu I) = N(T_3^* - \mu I).$$

Further since $\pi_0(T_4^*) = \delta$ and $\bar{\mu} \notin \sigma(T_3^*)$, it follows that $N(T^* - \bar{\mu} I) = N(T_3^* - \bar{\mu} I)$, and therefore $N(T - \mu I) = N(T^* - \bar{\mu} I)$, which implies that $\mu \in \delta$, giving a contradiction. This completes the proof. \hfill \Box

We have been unable to answer if restriction-convexoid operators are reduced by each of its eigenspaces corresponding to isolated eigenvalues. If the answer were affirmative then we would answer Berberian question affirmatively.

We recall that an operator $T \in \mathcal{L(H)}$ is called a Riesz operator if $\sigma_e(T) = \{0\}$. We then have:

**Corollary 1.4.** *If $T \in \chi$ is Riesz then $T$ is compact and normal.*

**Proof.** By Theorem 1.3, $T$ is normal with pure point spectrum. Note that the nonzero eigenvalues are Riesz points, so that they are either finite or form a null sequence, which implies that $T$ is compact. \hfill \Box

An operator $T \in \mathcal{L(H)}$ is said to be *polynomially compact* if there exists a nonzero complex polynomial $p$ such that $p(T)$ is compact. S. Berberian ([3]) considered a relationship between the polynomial compactness of the operator and the finiteness of its Weyl spectrum, and gave several sufficient conditions for the finiteness of the Weyl spectrum; for example, if $T$ is a
semisnormal operator then $T$ is polynomiably compact if and only if $\omega(T)$ is finite. Observe

(1.4.1) \hspace{1em} T \text{ is polynomiably compact} \implies \omega(T) \text{ is finite:}

indeed if $p(T)$ is compact then $p(\sigma_c(T)) = \sigma_c(p(T)) = \{0\}$, so that $\sigma_c(T)$ is finite, which together with (0.1) implies that $\sigma_c(T) = \omega(T)$. Recently, the finiteness of the Weyl spectrum was characterized in ([11]).

**Lemma 1.5 ([11, Lemma 3]).** If $\omega(T)$ is finite then $T \in \mathcal{L}(\mathcal{H})$ is decomposed into the finite direct sum

(1.5.1) \hspace{1em} T = \bigoplus_{i=1}^{n} (N_i + K_i + \lambda_i I),

where the $N_i$ are quasinilpotents, the $K_i$ are compact, and $\{\lambda_1, \cdots, \lambda_n\} = \omega(T)$.

The following corollary provides a structure theorem for polynomiably compact $\chi$-class operators (Compare with [5, Theorem 3]):

**Corollary 1.6.** If $T \in \chi$ then the following statements are equivalent:

(a) $T$ is polynomiably compact;
(b) $\omega(T)$ is finite;
(c) $T$ is the direct sum of finitely many thin normal operators, i.e.,

(1.6.1) \hspace{1em} T = \bigoplus_{i=1}^{n} (R_i + \lambda_i I),

where the $R_i$ are compact normal operators.

**Proof.** (a)\(\Rightarrow\)(b): This comes from (1.4.1).
(b)\(\Rightarrow\)(c): If $\omega(T)$ is finite then (1.5.1) holds with Riesz operators $R_i$. Thus if $T \in \chi$ then so is each $R_i$, and therefore it follows from Corollary 1.4 that each $R_i$ is a compact normal operator.
(c)\(\Rightarrow\)(a): Suppose $T$ satisfies (1.6.1). Then $p(T)$ is compact, where $p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$, with $\lambda_i$ as in (c).

In [10, Solution 178], it was shown that if $T$ is normal and if $T^n$ is compact for some $n \in \mathbb{N}$ then $T$ is compact. We can prove more:
**COROLLARY 1.7.** If $T \in \chi$ and if $T^n$ is compact for some $n \in \mathbb{N}$ then $T$ is a diagonal compact operator.

**Proof.** If $T \in \chi$ and if $T^n$ is compact for some $n \in \mathbb{N}$ then it follows from Corollary 1.6 that $\sigma(T)$ is countable. Thus by Theorem 1.3, $T$ is a diagonal normal operator with diagonal $\{\alpha_m\}_{m=1}^{\infty}$. But since $T^n$ is a diagonal compact operator with diagonal $\{\alpha_m^n\}_{m=1}^{\infty}$, we can see that $\alpha_m^n \to 0$, so that $\alpha_m \to 0$. Therefore $T$ is a diagonal compact operator. \hfill \Box

We consider here a relationship between convexoid operators and their spectral sets. Recall that a compact set $\sigma$ in $\mathbb{C}$ is called a *spectral set* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma(T) \subseteq \sigma$ and if $\|f(T)\| \leq \|f\|_\sigma := \max_{z \in \sigma} |f(z)|$ for every rational function $f$ with poles off $\sigma$. The following results are well-known:

(i) The closed unit disk $\mathbb{D}$ is a spectral set for every contraction operator ([26]).

(ii) The spectrum of a subnormal operator is a spectral set ([1], [18]).

(iii) There exists a hyponormal operator whose spectrum contains a disk and is not a spectral set ([27]).

We now have:

**THEOREM 1.8.** If $\text{conv} \sigma(T)$ is a spectral set for $T \in \mathcal{L}(\mathcal{H})$ then $T$ is convexoid.

**Proof.** Suppose $\text{conv} \sigma(T)$ is a spectral set for $T$. Thus $\|f(T)\| \leq \|f\|_{\text{conv} \sigma(T)}$ for every rational function $f$ with poles off $\text{conv} \sigma(T)$. If $K$ is a convex subset of $\mathbb{C}$, write $\text{Ext} K$ for the set of extreme points of $K$. Observe that if $K$ is a compact convex set in $\mathbb{C}$, then $\|z\|_K$ occurs on $\text{Ext} K$. But by the Krein-Milman theorem,

$$\text{conv} \sigma(T) = \overline{\text{conv}} \left( \text{Ext} \sigma(T) \right) \quad \text{and} \quad \text{Ext} \left( \text{conv} \sigma(T) \right) \subseteq \sigma(T),$$

where $\overline{\text{conv}}$ denotes the closed convex-hull. Thus for every $\lambda \in \mathbb{C},$

$$r(T - \lambda I) \leq \|T - \lambda I\| \leq \|z - \lambda\|_{\text{conv} \sigma(T)} = \|z\|_{\text{conv} \sigma(T - \lambda I)} = \|z\|_{\text{Ext} \text{conv} \sigma(T - \lambda I)} = \|z\|_{\sigma(T - \lambda I)} = r(T - \lambda I),$$

which implies that $r(T - \lambda I) = \|T - \lambda I\|$ for every $\lambda \in \mathbb{C}$. This says that $T - \lambda I$ is normaloid for every $\lambda \in \mathbb{C}$. It therefore follows that $T$ is convexoid ([4]). \hfill \Box
Note that $\sigma(T)$ need not be a spectral set for $T$ even though $\text{conv } \sigma(T)$ is. For example if $S$ is the bilateral shift on $L^2(\mathbb{T})$ of the unit circle $\mathbb{T}$, take

$$(1.8.1) \quad T = S \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where the second summand is a two-dimensional operator. Then $\sigma(T) = \mathbb{T} \cup \{0\}$. Choose $f(z) = (z - \frac{1}{2})^{-1}$. Then $\|f\|_{\sigma(T)} = 2$, but $\|f(T)\| \geq \|\begin{pmatrix} -2 & -4 \\ 0 & -2 \end{pmatrix}\| > 2$, which implies that $\sigma(T)$ is not a spectral set for $T$. But since $\text{conv } \sigma(T) = \mathbb{D}$ and $T$ is a contraction operator it follows from the statement (i) in the remark above Theorem 1.8 that $\text{conv } \sigma(T)$ is a spectral set for $T$. Also note that the conclusion of Theorem 1.8 cannot be strengthened by "reduction-convexoid": for example consider the operator $T$ defined by (1.8.1).

2. Commutators

A commutator is an operator of the form $AB - BA$. Then Brown-Pearcy theorem [7, Theorem 3] says that $T \in \mathcal{L}(\mathcal{H})$ is a noncommutator if and only if $T$ is of the form $K + \lambda I$, where $\lambda \neq 0$ and $K$ is compact. Thus we have that

$$(2.0.1) \quad T \text{ is a noncommutator } \implies \omega(T) = \{\lambda\}, \; \lambda \neq 0.$$ 

But the converse of (2.0.1) is, in general, not true. We however have:

THEOREM 2.1. If $T \in \chi$ and $\omega(T) = \{\lambda\}, \; \lambda \neq 0$, then $T$ is a noncommutator.

Proof. Suppose $\omega(T) = \{\lambda\}, \; \lambda \neq 0$. Then $\sigma_{e}(T) = \{\lambda\}$, and hence $\sigma_{e}(T - \lambda I) = \{0\}$. Thus if $T \in \chi$ and hence so is $T - \lambda I$, then it follows from Corollary 1.4 that $T - \lambda I$ is a compact operator. Therefore by the Brown-Pearcy theorem, $T$ is a noncommutator. $\square$

In Theorem 2.1, "restriction-convexoid in the definition of $\chi$" cannot be replaced by "convexoid". To see this, let on $\ell_2$

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \left[ \left( \frac{1}{3} \right) 0 \right] \otimes 1_\infty \right].$$
Then we have that (i) \( \omega(T) = \{ \frac{1}{3} \} \); (ii) \( T \) is convexoid because \( \text{conv } \sigma(T) = W(T) \), which is the equilateral triangle whose vertices are the three cube roots of 1; (iii) \( T \) is a commutator because \( T \) has a "large" kernel (see [10, Problem 234]); (iv) \( T \) is not reduction-convexoid because \( \left( \begin{array}{ccc} 1 & 0 \\ \frac{1}{3} & \frac{1}{3} \end{array} \right) \) is not convexoid.

**Theorem 2.2.** If either \( \sigma(A) \) or \( \sigma(B) \) is not a singleton set then \( A \otimes B \) is a commutator. In particular if either \( A \) or \( B \) is a nonconstant convexoid operator then \( A \otimes B \) is a commutator.

**Proof.** Suppose either \( \sigma(A) \) or \( \sigma(B) \) is not a singleton set. Since [16, Theorem 4.2]

\[
\omega(A \otimes B) = \omega(A) \cdot \sigma(B) \cup \sigma(A) \cdot \omega(B),
\]

it follows that either \( \omega(A \otimes B) = \{0\} \) or \( \omega(A \otimes B) \) has at least two elements. Thus by (2.0.1), \( A \otimes B \) is a commutator. This proves the first assertion. For the second assertion we suppose that \( A \) is nonconstant and convexoid. In view of the first assertion it suffices to show that \( \sigma(A) \) is not a singleton set. Assume to the contrary that \( \sigma(A) = \{ \lambda \} \), \( \lambda \in \mathbb{C} \). Then \( A - \lambda I \) is convexoid and quasinilpotent. But since the only convexoid quasinilpotent is 0, it follows that \( A = \lambda I \), giving a contradiction. This completes the proof.

A **self-commutator** is an operator of the form \( A^* A - AA^* \). Then Radjavi's theorem ([25]) says that a self-adjoint operator \( T \in \mathcal{L}(\mathcal{H}) \) is a self-commutator if and only if \( 0 \in \text{conv } \omega(T) \). Thus the Radjavi's theorem gives the following:

**Theorem 2.3.** If \( T \in \mathcal{L}(\mathcal{H}) \) is a self-adjoint operator whose direct summands are nonconstant then \( T \) is a self-commutator if and only if either \( 0 \in \omega(T) \) or \( T \) is not semi-definite.

**Proof.** If \( 0 \in \omega(T) \), then evidently \( T \) is a self-commutator. If instead \( T \) is not semi-definite, then there exist \( \lambda, \mu \in \sigma(T) \) such that \( \lambda > 0 \) and \( \mu < 0 \). We now claim that \( \lambda, \mu \in \omega(T) \). Assume to the contrary that \( \lambda \notin \omega(T) \). Then it follows from Weyl's theorem that \( \lambda \in \text{iso } \sigma(T) \). Thus \( T \) should be of the form \( T = \lambda I \oplus S \), which contradicts to our assumption. Therefore we have that \( 0 \in \text{conv } \omega(T) \), and hence by the Radjavi's theorem, \( T \) is a self-commutator. The converse is evident.

\( \square \)
Theorem 2.3 is readily applicable for self-adjoint operators with no eigenvalues (e.g., Toeplitz operators with real-valued symbols).

An invertible operator $T \in \mathcal{L}(\mathcal{H})$ is called a multiplicative commutator if it is of the form $ABA^{-1}B^{-1}$. By contrast, a commutator $AB - BA$ is often called an additive commutator. It is known that if $T$ is a multiplicative commutator of the form $K + \lambda I$, where $K$ is compact and $\lambda \in \mathbb{C}$, then $|\lambda| = 1$ ([8, Theorem 1], [10, Problem 238]). It remains open whether a multiplicative commutator is not of the form $K + \lambda I$, where $|\lambda| \neq 1$ and $K$ is compact. But another argument of Brown and Pearcy [8, Theorem 5] shows that an invertible normal operator $T \in \mathcal{L}(\mathcal{H})$ is a multiplicative commutator if and only if $T$ is not of the form $K + \lambda I$, where $|\lambda| \neq 1$ and $K$ is compact. Thus if $T \in \mathcal{L}(\mathcal{H})$ is invertible and normal then

\begin{equation}
(2.3.1) \quad T \text{ is not a multiplicative commutator } \implies \omega(T) = \{\lambda\}, \quad |\lambda| \neq 1.
\end{equation}

The following theorem shows that the converse of (2.3.1) is also true with a weaker condition:

**Theorem 2.4.** If $T \in \chi$ is invertible and $\omega(T) = \{\lambda\}$, then

$$T \text{ is a multiplicative commutator } \iff |\lambda| = 1.$$  

**Proof.** If $\omega(T) = \{\lambda\}$, then $\sigma_\omega(T - \lambda I) = \{0\}$. By Corollary 1.4, $T - \lambda I$ is a compact normal operator, say $K$. But then $T = K + \lambda I$ is invertible and normal. Therefore by the Brown-Pearcy characterization [8, Theorem 5], $T$ is a multiplicative commutator if and only if $|\lambda| = 1$. \qed

Theorems 2.1 and 2.4 show that if $T$ is an invertible $\chi$-class operator and $\omega(T) = \{\lambda\}$, it is impossible that $T$ is both a multiplicative and an additive commutator: for if $|\lambda| = 1$, then $T$ is a multiplicative commutator but not an additive commutator, and if $|\lambda| \neq 1$, then $T$ is neither a multiplicative nor an additive commutator.

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