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SPECTRAL PROPERTIES OF BIPARTITE TOURNAMENT MATRICES

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ABSTRACT. In this paper, we look at the spectral bounds of a bipartite tournament matrix M with arbitrary team size. Also we find the condition for the variance of the Perron vector of M to vanish.

1. Introduction

Let p and q be positive integers. A digraph obtained by orienting each edge of the complete bipartite graph $K_{p,q}$ is called a *bipartite tournament* with team size p and q, and the associated adjacency $(0,1)$-matrix is called a *bipartite tournament matrix*. It is interpreted as the result of a round-robin competition between two teams in which each player in a team competes every player in the other team.

We assume that two teams respectively consist of players in the sets $\{1,2,\dots,p\}$ and $\{p+1,p+2,\dots,p+q\}$. Let $p+q=n$. Then a bipartite tournament matrix of order n with team size p and q is written $M = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix}$, where O_p is the zero matrix of order p, A is a $p\times q$ $(0,1)$-matrix, and $B = J_{q,p} - A^t$, where $J_{q,p}$ is the $q \times p$ matrix with 1's for all entries. The matrix M satisfies

$$M + M^t = J_n - \begin{bmatrix} J_p & O_{p,q} \\ O_{q,p} & J_q \end{bmatrix} = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix}, \tag{1}$$

where $O_{p,q}$ is the $p\times q$ zero matrix and $J_n = J_{n,n}$.

A matrix M is called *reducible* if $PMP^t = \begin{bmatrix} M_1 & O \\ * & M_2 \end{bmatrix}$ for some permutation matrix P, where M_1 and M_2 are nonvacuous square matrices, and *irreducible* otherwise. If a bipartite tournament matrix M is reducible,

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then the submatrices M_1 and M_2 of PMP^t are again bipartite tournament matrices. To study the spectral properties of M, it is enough to look at its irreducible components.

It is well known by Perron-Frobenius theorem [1] that a nonnegative irreducible matrix M has its spectral radius ρ as a positive eigenvalue, called the *Perron value*, and a corresponding eigenvector consists of all positive coordinates, and the eigenvector the sum of whose coordinates is 1 is called the *Perron vector* of M.

We find the spectral bounds of an irreducible bipartite tournament matrix M with arbitrary team size p and q. Especially, when M is normal, M has two nonzero real eigenvalues $\pm\sqrt{pq}/2$, and the variance of the Perron vector of M vanishes if and only if $M\mathbf{1} = \frac{n}{4}\,\mathbf{1}$, where $n = p + q$.

2. Spectral Properties

Let M be a bipartite tournament matrix with team size $p \leq q$, $p+q = n$ and let λ be an eigenvalue of M and v an eigenvector such that $Mv = \lambda v$.

Pre- and post-multiplying v to equality (1) and applying Schwartz inequality, we obtain

(2)

$$\begin{aligned}
(2\,\mathrm{Re}\,\lambda)v^*v &= v^*(M + M^t)v \\
&= v^*J_nv - [\bar{v}_1, \dots, \bar{v}_n] \begin{bmatrix} J_p & O_{p,q} \\ O_{q,p} & J_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
&= |v^*\mathbf{1}|^2 - \sum_{i=1}^{p} \bar{v}_i \cdot \sum_{i=1}^{p} v_i - \sum_{i=p+1}^{n} \bar{v}_i \cdot \sum_{i=p+1}^{n} v_i \\
&\geq |v^*\mathbf{1}|^2 - p\left(|v_1|^2 + \cdots + |v_p|^2\right) - q\left(|v_{p+1}|^2 + \cdots + |v_n|^2\right) \\
&\geq |v^*\mathbf{1}|^2 - q\,v^*v,
\end{aligned}$$

where $\mathbf{1} = (1, \dots, 1)^t$.

The *variance* of a vector $v = (v_1, \dots, v_n)^t$ is defined by

$$\mathrm{var}\, v = \sum_{1 \leq i < j \leq n} |v_i - v_j|^2.$$

Let M be an irreducible bipartite tournament matrix with team size $p \leq q$, $p + q = n$, ρ the Perron value of the matrix, and $v = (v_1, \dots, v_n)^t$

the corresponding eigenvector. Denote

$$v^{(1)} = (v_1, \ldots, v_p)^t, \quad v^{(2)} = (v_{p+1}, \ldots, v_n)^t$$
$$w = (w_1, w_2)^t, \quad w_1 = \sum_{i=1}^{p} v_i, \quad w_2 = \sum_{i=p+1}^{n} v_i.$$

Pre- and post-multiplying v to equality (1), we have

$$\begin{aligned} v^*(M + M^t)v &= [\bar{v}_1, \ldots, \bar{v}_n] \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= w^*w - |w_1 - w_2|^2. \end{aligned}$$

Since

$$\begin{aligned} w^*w &= |v_1 + \cdots + v_p|^2 + |v_{p+1} + \cdots + v_n|^2 \\ &= p(|v_1|^2 + \cdots + |v_p|^2) - \sum_{1 \le i < j \le p} |v_i - v_j|^2 \\ &\quad + q(|v_{p+1}|^2 + \cdots + |v_n|^2) - \sum_{p+1 \le i < j \le n} |v_i - v_j|^2 \\ &= p\, v^{(1)*} v^{(1)} + q\, v^{(2)*} v^{(2)} - \operatorname{var} v^{(1)} - \operatorname{var} v^{(2)}, \end{aligned}$$

we have

$$2\rho\, v^* v = p\, v^{(1)*} v^{(1)} + q\, v^{(2)*} v^{(2)} - \operatorname{var} v^{(1)} - \operatorname{var} v^{(2)} - \operatorname{var} w$$

or

$$\begin{aligned} 0 &\le \operatorname{var} v^{(1)} + \operatorname{var} v^{(2)} + \operatorname{var} w \\ &= (p - 2\rho)\, v^{(1)*} v^{(1)} + (q - 2\rho)\, v^{(2)*} v^{(2)} \\ &\le (q - 2\rho)\, v^* v. \end{aligned} \tag{3}$$

THEOREM 1. *Let M be an irreducible bipartite tournament matrix with team size $p \le q$, $p + q = n$, and ρ the Perron value of M. Then, for an eigenvalue λ of M,*

(i) $-\frac{q}{2} \le \operatorname{Re}\lambda \le \frac{q}{2}$.
(ii) $\operatorname{Re}\lambda = -\frac{q}{2}$ *if and only if* $p = q = \frac{n}{2}$, $\lambda = -\rho = -\frac{n}{4}$ *and the corresponding eigenvector is* $v = (1, \ldots, 1, -1, \ldots, -1)^t$.
(iii) $\operatorname{Re}\lambda = \frac{q}{2}$ *if and only if* $p = q = \frac{n}{2}$, $\lambda = \rho = \frac{n}{4}$ *and the corresponding eigenvector is* $1 = (1, \ldots, 1)^t$.

Since $M = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix}$, when $p = q = \frac{n}{2}$, $M1 = \frac{n}{4}1$ if and only if $Mv = -\frac{n}{4}v$, where $v = (1, \dots, 1, -1, \dots, -1)^t$. In other words, M has either both eigenvalues $\frac{q}{2}$ and $-\frac{q}{2}$ or for any eigenvalue λ, $|\lambda| \le \rho < \frac{q}{2}$. Note here that $\frac{n}{4}$ is the row sum of M and so n is a multiple of 4.

Proof. From inequality (2), $(2\operatorname{Re}\lambda + q)\, v^*v \ge 0$ implies $\operatorname{Re}\lambda \ge -\frac{q}{2}$, where the equality holds if and only if $p = q = \frac{n}{2}$, $v_1 = \cdots = v_p$, $v_{p+1} = \cdots = v_n$, and $v^*1 = \sum_{i=1}^{n} \bar{v}_i = 0$. So $\operatorname{Re}\lambda = -\frac{q}{2}$ if and only if $p = q = \frac{n}{2}$ and the corresponding eigenvector is $v = (1, \dots, 1, -1, \dots, -1)^t$.

On the other hand, using inequality (3), we have $\operatorname{Re}\lambda \le \rho \le \frac{q}{2}$. And $\operatorname{Re}\lambda = \rho = \frac{q}{2}$ if and only if $p = q$ and $\operatorname{var} v^{(1)} = \operatorname{var} v^{(2)} = \operatorname{var} w = 0$, i.e., the corresponding eigenvector is 1. □

COROLLARY 2. *Let $p = q = \frac{n}{2}$, and let $u = (u_1, \dots, u_n)^t$ be an eigenvector of M, whose Perron value is $\frac{n}{4}$, corresponding to an eigenvalue μ with $\operatorname{Re}\mu \ne -\frac{n}{4}$. Then $u_1 + \cdots + u_p = u_{p+1} + \cdots + u_n$.*

Proof. From theorem 1, $v = (1, \dots, 1, -1, \dots, -1)^t$ is the eigenvector of M corresponding to $-\frac{n}{4}$. Pre- and post-multiplying v and u to equality (1), we obtain

$$\begin{aligned}
(-\frac{n}{4} + \mu)v^*u &= v^*(M + M^t)u \\
&= v^*J_n u - v^* \begin{bmatrix} J_{\frac{n}{2}} & O_{\frac{n}{2}} \\ O_{\frac{n}{2}} & J_{\frac{n}{2}} \end{bmatrix} u \\
&= 0 - \frac{n}{2} v^*u
\end{aligned}$$

So we have $v^*u = 0$. □

3. Eigenvalues for normal bipartite tournament matrices

Now, we assume that M is an irreducible normal bipartite tournament matrix with team size $p \le q$, $p + q = n$. Then M satisfies $MM^t = M^tM$.

We have shown [5] that M is normal if and only if the row sums of $A =$ the column sums of $B = \frac{q}{2}$ and the row sums of $B =$ the column sums of $A = \frac{p}{2}$. A and B have the same number of 1's, in other words, in a normal bipartite tournament, the total numbers of winning games of the two teams are equal.

Since M, M^t, and $M + M^t$ all commute, they are simultaneously diagonalizable by a unitary matrix P. Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be the eigenvalues of M and $M + M^t = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix}$, respectively. Then we have

$$\begin{aligned} \begin{bmatrix} 2\operatorname{Re}\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 2\operatorname{Re}\lambda_n \end{bmatrix} &= P^*MP + (P^*MP)^* \\ &= P^*(M+M^t)P = \begin{bmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_n \end{bmatrix}. \end{aligned} \tag{4}$$

Since the eigenvalues of J_k are 0 (mult. $k-1$) and k, the eigenvalues of $(M + M^t)^2 = \begin{bmatrix} qJ_p & O_{p,q} \\ O_{q,p} & pJ_q \end{bmatrix}$ are 0 (mult. $n-2$) and pq (mult. 2). From $\operatorname{tr}(M + M^t) = 0$, we can see that the eigenvalues of $M + M^t$ should be 0 (mult. $n-2$), $\sqrt{pq}$, and $-\sqrt{pq}$. Hence, by (4), the eigenvalues of M are $\rho = \frac{1}{2}\sqrt{pq}$, $-\rho = -\frac{1}{2}\sqrt{pq}$ and $n-2$ purely imaginaries including 0.

Note that M can have 0 as an eigenvalue with multiplicity at most $n-4$, since $\operatorname{tr} M^2 = 0$. In fact, a bipartite irreducible tournament matrix M has at least 4 distinct eigenvalues [4], which means that M has at least two nonzero purely imaginary eigenvalues.

THEOREM 3. *An irreducible normal bipartite tournament matrix M has eigenvalues two nonzero real $\rho = \frac{\sqrt{pq}}{2}$, $-\rho = -\frac{\sqrt{pq}}{2}$, $2k$ purely imaginaries, and 0 of multiplicity $n-2k-2$, for some $k \geq 1$.*

Remark that in the above theorem when $p = q = \frac{n}{2}$, a normal tournament matrix is also a regular matrix where the row sums of M are all constant $\frac{n}{4}$, and vice versa [5]. So when team sizes are equal, the eigenvalues of a regular bipartite tournament matrix M are two nonzero integer $\rho = \frac{n}{4}$, $-\frac{n}{4}$, $2k$ purely imaginaries, and 0(mult. $n-2k-2$), for some $k \geq 1$.

4. The variance of the Perron vector

We have seen that $-\frac{q}{2} \leq \operatorname{Re}\lambda \leq \frac{q}{2}$ and $\operatorname{Re}\lambda = \frac{q}{2}$ is achieved when $p = q, \lambda = \rho$ for a regular bipartite tournament matrix M, that is, when M satisfies $M\mathbf{1} = \rho\mathbf{1}, \rho = \frac{n}{4}$. In this case, the Perron vector v satisfies $\operatorname{var} v^{(1)} = \operatorname{var} v^{(2)} = \operatorname{var} w = 0$, which implies that the players in the first and the second teams are evenly ranked and two teams get the same ranking according to Kendall-Wei scheme [3,7].

Now, we assume that

$$\text{var}\, v^{(1)} = \text{var}\, v^{(2)} = \text{var}\, w = 0, \tag{5}$$

for an eigenvector v corresponding to an eigenvalue λ of an irreducible bipartite tournament matrix M with team size $p \leq q$, $p + q = n$.

Equation (5) holds if and only if $v_1 = \cdots = v_p$, $v_{p+1} = \cdots = v_n$, and $v_1 + \cdots + v_p = v_{p+1} + \cdots + v_n$, equivalently, if and only if $v = (\underbrace{q, \ldots, q}_{p \text{ times}}, \underbrace{p, \ldots, p}_{q \text{ times}})^t$ is an eigenvector corresponding to λ.

From $Mv = \lambda v$, we obtain

$$\begin{bmatrix} ps_1 \\ \vdots \\ ps_p \\ qt_1 \\ \vdots \\ qt_q \end{bmatrix} = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix} \begin{bmatrix} q \\ \vdots \\ q \\ p \\ \vdots \\ p \end{bmatrix} = \lambda \begin{bmatrix} q \\ \vdots \\ q \\ p \\ \vdots \\ p \end{bmatrix},$$

where $(s_1, \ldots, s_p)^t$ and $(t_1, \ldots, t_q)^t$ are the row sum vectors of A and B, respectively. So we have $s_1 = \cdots = s_p = s$, $t_1 = \cdots = t_q = t$, and

$$ps = \lambda q, \quad qt = \lambda p. \tag{6}$$

Since M satisfies (1), the number of 1's in M is pq, and so s and t should satisfy $ps + qt = pq$. Then using (6), we obtain $\lambda = \frac{ps+qt}{p+q} = \frac{pq}{p+q} = \frac{pq}{n}$, and the row sums of A and B are $s = \frac{q^2}{n}$ and $t = \frac{p^2}{n}$, respectively.

Here we see that $n = p+q$ is of the form $n = a^2 b$ for an integer $a \geq 2$ and a square free integer $b \geq 1$, and p and q have ab as a common divisor. For, if n is not divisible by a square, n can be written $n = \prod_{i=1}^{m} p_i$, for some distinct primes $p_i, i = 1, \ldots, m$. Then $n|p^2$ implies each $p_i|p$ and so $n|p$, which is a contradiction. Now, the fact that $n = a^2 b$ divides both p^2 and q^2 implies $a|p$, $a|q$ and $b|p$, $b|q$. Hence we have $p = abk$ and $q = ab(a-k)$, where $1 \leq k \leq [\frac{a}{2}]$, and $\lambda = \frac{pq}{n} = bk(a-k)$ is a positive integer. We summarize these results in the following theorem.

THEOREM 4. *Let M be an irreducible bipartite tournament matrix with team size $p \leq q, p + q = n$. Suppose an eigenvector v of M satisfies (5).*

Then there exist an integer $a \geq 2$, a square free integer $b \geq 1$, and an integer k with $1 \leq k \leq [\frac{a}{2}]$ such that the team sizes of this tournament are $p = abk$ and $q = ab(a-k)$; the corresponding eigenvalue is $\lambda = bk(a-k)$, which is a positive integer; the row sums of A and B are constants $s = b(a-k)^2$ and $t = bk^2$, respectively.

In particular, when a is even and $k = \frac{a}{2}$, we have a regular bipartite tournament matrix M with team size $p = q = \frac{n}{2}$, that is, row sums of M are all constant $\frac{n}{4}$.

COROLLARY 5. *Let M in theorem 4 be normal. Equation* (5) *holds for the Perron vector v if and only if M is regular, that is, $M\mathbf{1} = \frac{n}{4}\mathbf{1}$.*

Proof. It suffices to prove the necessity. Since M is normal, the row sums of $A =$ the column sums of $B = s = \frac{q}{2}$ and the row sums of $B =$ the column sums of $A = t = \frac{p}{2}$ [5]. The Perron value is $\rho = \frac{\sqrt{pq}}{2}$ by theorem 3. On the other hand, from equation (6) with $\lambda = \rho$, we have $\rho = \frac{p}{q}\frac{q}{2} = \frac{p}{2}$. Hence, we obtain $p = q = \frac{n}{2}$ and $\rho = \frac{n}{4}$, which means $M\mathbf{1} = \frac{n}{4}\mathbf{1}$, by theorem 1. $\square$

Note that we can rewrite corollary 5 as equation (5) holds for the Perron vector v if and only if $p = q = \frac{n}{2}$; for when $p = q$, M is normal if and only if it is regular [5].

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