SPECTRAL PROPERTIES OF
BIPARTITE TOURNAMENT MATRICES

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ABSTRACT. In this paper, we look at the spectral bounds of a bipartite tournament matrix \( M \) with arbitrary team size. Also we find the condition for the variance of the Perron vector of \( M \) to vanish.

1. Introduction

Let \( p \) and \( q \) be positive integers. A digraph obtained by orienting each edge of the complete bipartite graph \( K_{p,q} \) is called a bipartite tournament with team size \( p \) and \( q \), and the associated adjacency \((0,1)\)-matrix is called a bipartite tournament matrix. It is interpreted as the result of a round-robin competition between two teams in which each player in a team competes every player in the other team.

We assume that two teams respectively consist of players in the sets \( \{1, 2, \ldots, p\} \) and \( \{p+1, p+2, \ldots, p+q\} \). Let \( p + q = n \). Then a bipartite tournament matrix of order \( n \) with team size \( p \) and \( q \) is written \( M = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix} \), where \( O_p \) is the zero matrix of order \( p \), \( A \) is a \( p \times q \) \((0,1)\)-matrix, and \( B = J_{q,p} - A^t \), where \( J_{q,p} \) is the \( q \times p \) matrix with 1’s for all entries. The matrix \( M \) satisfies

\[
M + M^t = J_n - \begin{bmatrix} J_p & O_{p,q} \\ O_{q,p} & J_q \end{bmatrix} = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix},
\]

where \( O_{p,q} \) is the \( p \times q \) zero matrix and \( J_n = J_{n,n} \).

A matrix \( M \) is called reducible if \( P M P^t = \begin{bmatrix} M_1 & O \\ * & M_2 \end{bmatrix} \) for some permutation matrix \( P \), where \( M_1 \) and \( M_2 \) are nonvacuous square matrices, and irreducible otherwise. If a bipartite tournament matrix \( M \) is reducible,
then the submatrices $M_1$ and $M_2$ of $PMP^t$ are again bipartite tournament matrices. To study the spectral properties of $M$, it is enough to look at its irreducible components.

It is well known by Perron-Frobenius theorem [1] that a nonnegative irreducible matrix $M$ has its spectral radius $\rho$ as a positive eigenvalue, called the Perron value, and a corresponding eigenvector consists of all positive coordinates, and the eigenvector the sum of whose coordinates is 1 is called the Perron vector of $M$.

We find the spectral bounds of an irreducible bipartite tournament matrix $M$ with arbitrary team size $p$ and $q$. Especially, when $M$ is normal, $M$ has two nonzero real eigenvalues $\pm \sqrt{pq}/2$, and the variance of the Perron vector of $M$ vanishes if and only if $M1 = \frac{n}{q}1$, where $n = p + q$.

2. Spectral Properties

Let $M$ be a bipartite tournament matrix with team size $p \leq q$, $p+q = n$ and let $\lambda$ be an eigenvalue of $M$ and $v$ an eigenvector such that $Mv = \lambda v$.

Pre- and post-multiplying $v$ to equality (1) and applying Schwartz inequality, we obtain

$$
(2 \Re \lambda) v^* v = v^* (M + M^t) v
$$

$$
= v^* J_n v - \begin{bmatrix} \bar{v}_1, \ldots, \bar{v}_n \end{bmatrix} \begin{bmatrix} J_p & O_{p,q} \\ O_{q,p} & J_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}
$$

$$
= |v^* 1|^2 - \sum_{i=1}^{p} v_i \cdot \sum_{i=p+1}^{n} v_i - \sum_{i=1}^{n} \bar{v}_i \cdot \sum_{i=p+1}^{n} v_i
$$

$$
\geq |v^* 1|^2 - p (|v_1|^2 + \cdots + |v_p|^2) - q (|v_{p+1}|^2 + \cdots + |v_n|^2)
$$

$$
\geq |v^* 1|^2 - q v^* v,
$$

where $1 = (1, \ldots, 1)^t$.

The variance of a vector $v = (v_1, \ldots, v_n)^t$ is defined by

$$
\text{var } v = \sum_{1 \leq i < j \leq n} |v_i - v_j|^2.
$$

Let $M$ be an irreducible bipartite tournament matrix with team size $p \leq q$, $p+q = n$, $\rho$ the Perron value of the matrix, and $v = (v_1, \ldots, v_n)^t$
the corresponding eigenvector. Denote
\[ v^{(1)} = (v_1, \ldots, v_p)^t, \quad v^{(2)} = (v_{p+1}, \ldots, v_n)^t \]
\[ w = (w_1, w_2)^t, \quad w_1 = \sum_{i=1}^{p} v_i, \quad w_2 = \sum_{i=p+1}^{n} v_i. \]

Pre- and post-multiplying \( v \) to equality (1), we have
\[
v^*(M + M^t)v = [\bar{v}_1, \ldots, \bar{v}_n] \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = w^*w - |w_1 - w_2|^2.
\]

Since
\[
w^*w = |v_1 + \cdots + v_p|^2 + |v_{p+1} + \cdots + v_n|^2
\]
\[= p(|v_1|^2 + \cdots + |v_p|^2) - \sum_{1 \leq i < j \leq p} |v_i - v_j|^2
\]
\[+ q(|v_{p+1}|^2 + \cdots + |v_n|^2) - \sum_{p+1 \leq i < j \leq n} |v_i - v_j|^2
\]
\[= p v^{(1)*}v^{(1)} + q v^{(2)*}v^{(2)} - \text{var} v^{(1)} - \text{var} v^{(2)},
\]
we have
\[2\rho v^*v = p v^{(1)*}v^{(1)} + q v^{(2)*}v^{(2)} - \text{var} v^{(1)} - \text{var} v^{(2)} - \text{var} w
\]
or
\[0 \leq \text{var} v^{(1)} + \text{var} v^{(2)} + \text{var} w
\]
\[= (p - 2\rho) v^{(1)*}v^{(1)} + (q - 2\rho) v^{(2)*}v^{(2)}
\]
\[\leq (q - 2\rho) v^*v.
\]

**Theorem 1.** Let \( M \) be an irreducible bipartite tournament matrix with team size \( p \leq q, p + q = n \), and \( \rho \) the Perron value of \( M \). Then, for an eigenvalue \( \lambda \) of \( M \),

(i) \[-\frac{n}{2} \leq \text{Re} \lambda \leq \frac{n}{2}. \]

(ii) \( \text{Re} \lambda = -\frac{n}{2} \) if and only if \( p = q = \frac{n}{2}, \lambda = -\rho = -\frac{n}{4} \) and the corresponding eigenvector is \( v = (1, \ldots, 1, -1, \ldots, -1)^t \).

(iii) \( \text{Re} \lambda = \frac{n}{2} \) if and only if \( p = q = \frac{n}{2}, \lambda = \rho = \frac{n}{4} \) and the corresponding eigenvector is \( 1 = (1, \ldots, 1)^t \).
Since \( M = \begin{bmatrix} O_p & A \\ B & O_q \end{bmatrix} \), when \( p = q = \frac{n}{2} \), \( M1 = \frac{n}{4}1 \) if and only if \( Mv = -\frac{n}{4}v \), where \( v = (1, \ldots, 1, -1, \ldots, -1)^t \). In other words, \( M \) has either both eigenvalues \( \frac{n}{2} \) and \( -\frac{n}{2} \) or for any eigenvalue \( \lambda \), \( |\lambda| \leq \rho < \frac{n}{2} \). Note here that \( \frac{n}{4} \) is the row sum of \( M \) and so \( n \) is a multiple of 4.

**Proof.** From inequality (2), \((2 \Re \lambda + q) v^*v \geq 0 \) implies \( \Re \lambda \geq -\frac{q}{2} \), where the equality holds if and only if \( p = q = \frac{n}{2} \), \( v_1 = \cdots = v_p = v_{p+1} = \cdots = v_n \), and \( v^*1 = \sum_{i=1}^n v_i = 0 \). So \( \Re \lambda = -\frac{q}{2} \) if and only if \( p = q = \frac{n}{2} \) and the corresponding eigenvector is \( v = (1, \ldots, 1, -1, \ldots, -1)^t \).

On the other hand, using inequality (3), we have \( \Re \lambda \leq \rho \leq \frac{n}{2} \). And \( \Re \lambda = \rho = \frac{q}{2} \) if and only if \( p = q \) and \( \var v^{(1)} = \var v^{(2)} = \var w = 0 \), i.e., the corresponding eigenvector is 1.

**Corollary 2.** Let \( p = q = \frac{n}{2} \), and let \( u = (u_1, \ldots, u_n)^t \) be an eigenvector of \( M \), whose Perron value is \( \frac{n}{4} \), corresponding to an eigenvalue \( \mu \) with \( \Re \mu \neq -\frac{n}{4} \). Then \( u_1 + \cdots + u_p = u_{p+1} + \cdots + u_n \).

**Proof.** From theorem 1, \( v = (1, \ldots, 1, -1, \ldots, -1)^t \) is the eigenvector of \( M \) corresponding to \( -\frac{n}{4} \). Pre- and post-multiplying \( v \) and \( u \) to equality (1), we obtain

\[
\left(-\frac{n}{4} + \mu\right)v^*u = v^*(M + M^t)u \\
= v^*J_nu - v^* \begin{bmatrix} \frac{n}{2} & O_{\frac{n}{2}} \\ O_{\frac{n}{2}} & \frac{n}{2} \end{bmatrix} u \\
= 0 - \frac{n}{2} v^*u
\]

So we have \( v^*u = 0 \). \( \square \)

3. Eigenvalues for normal bipartite tournament matrices

Now, we assume that \( M \) is an irreducible normal bipartite tournament matrix with team size \( p \leq q, p + q = n \). Then \( M \) satisfies \( MM^t = M^tM \).

We have shown [5] that \( M \) is normal if and only if the row sums of \( A \) = the column sums of \( B = \frac{n}{2} \) and the row sums of \( B \) = the column sums of \( A = \frac{n}{2} \). \( A \) and \( B \) have the same number of 1's, in other words, in a normal bipartite tournament, the total numbers of winning games of the two teams are equal.
Since $M, M^t$, and $M + M^t$ all commute, they are simultaneously diagonalizable by a unitary matrix $P$. Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be the eigenvalues of $M$ and $M + M^t = \begin{bmatrix} O_p & J_{p,q} \\ J_{q,p} & O_q \end{bmatrix}$, respectively. Then we have

$$\begin{bmatrix} 2 \text{Re} \lambda_1 & 0 \\ \vdots & \ddots \\ 0 & 2 \text{Re} \lambda_n \end{bmatrix} = P^*MP + (P^*MP)^*$$

(4)

$$= P^*(M + M^t)P = \begin{bmatrix} \mu_1 & 0 \\ \vdots & \ddots \\ 0 & \mu_n \end{bmatrix}.$$  

Since the eigenvalues of $J_k$ are 0 (mult. $k - 1$) and $k$, the eigenvalues of $(M + M^t)^2 = \begin{bmatrix} qJ_p & O_{p,q} \\ O_{q,p} & pJ_q \end{bmatrix}$ are 0 (mult. $n - 2$) and $pq$ (mult. 2). From tr$(M + M^t) = 0$, we can see that the eigenvalues of $M + M^t$ should be 0 (mult. $n - 2$), $\sqrt{pq}$, and $-\sqrt{pq}$. Hence, by (4), the eigenvalues of $M$ are $\rho = \frac{1}{2} \sqrt{pq}$, $-\rho = -\frac{1}{2} \sqrt{pq}$ and $n - 2$ purely imaginaries including 0.

Note that $M$ can have 0 as an eigenvalue with multiplicity at most $n - 4$, since tr$M^2 = 0$. In fact, a bipartite irreducible tournament matrix $M$ has at least 4 distinct eigenvalues [4], which means that $M$ has at least two nonzero purely imaginary eigenvalues.

**Theorem 3.** An irreducible normal bipartite tournament matrix $M$ has eigenvalues two nonzero real $\rho = \frac{\sqrt{pq}}{2}$, $-\rho = -\frac{\sqrt{pq}}{2}$, 2k purely imaginaries, and 0 of multiplicity $n - 2k - 2$, for some $k \geq 1$.

Remark that in the above theorem when $p = q = \frac{n}{2}$, a normal tournament matrix is also a regular matrix where the row sums of $M$ are all constant $\frac{n}{2}$, and vice versa [5]. So when team sizes are equal, the eigenvalues of a regular bipartite tournament matrix $M$ are two nonzero integer $\rho = \frac{n}{4}, -\frac{n}{4}$, 2k purely imaginaries, and 0(mult. $n - 2k - 2$), for some $k \geq 1$.

**4. The variance of the Perron vector**

We have seen that $-\frac{q}{2} \leq \text{Re} \lambda \leq \frac{q}{2}$ and $\text{Re} \lambda = \frac{q}{2}$ is achieved when $p = q, \lambda = \rho$ for a regular bipartite tournament matrix $M$, that is, when $M$ satisfies $M1 = \rho1, \rho = \frac{n}{4}$. In this case, the Perron vector $v$ satisfies $\text{var} v^{(1)} = \text{var} v^{(2)} = \text{var} w = 0$, which implies that the players in the first
and the second teams are evenly ranked and two teams get the same ranking according to Kendall-Wei scheme [3,7].

Now, we assume that

\[
\text{var } v^{(1)} = \text{var } v^{(2)} = \text{var } w = 0,
\]

for an eigenvector \( v \) corresponding to an eigenvalue \( \lambda \) of an irreducible bipartite tournament matrix \( M \) with team size \( p \leq q, p + q = n \).

Equation (5) holds if and only if \( v_1 = \cdots = v_p, v_{p+1} = \cdots = v_n, \) and \( v_1 + \cdots + v_p = v_{p+1} + \cdots + v_n \), equivalently, if and only if \( v = (q, q, \ldots, q, p, \ldots, p)^t \)

\[
p \text{ times } q \text{ times}
\]

is an eigenvector corresponding to \( \lambda \).

From \( Mv = \lambda v \), we obtain

\[
\begin{bmatrix}
ps_1 \\
\vdots \\
ps_p \\
qt_1 \\
\vdots \\
qt_q
\end{bmatrix} = \begin{bmatrix}
O_p & A \\
B & O_q
\end{bmatrix} \begin{bmatrix}
q \\
\vdots \\
p
\end{bmatrix} = \lambda \begin{bmatrix}
q \\
\vdots \\
p
\end{bmatrix},
\]

where \((s_1, \ldots, s_p)^t \) and \((t_1, \ldots, t_q)^t \) are the row sum vectors of \( A \) and \( B \), respectively. So we have \( s_1 = \cdots = s_p = s, t_1 = \cdots = t_q = t \), and

\[
\text{(6) } ps = \lambda q, \quad qt = \lambda p.
\]

Since \( M \) satisfies (1), the number of 1's in \( M \) is \( pq \), and so \( s \) and \( t \) should satisfy \( ps + qt = pq \). Then using (6), we obtain \( \lambda = \frac{ps + qt}{p + q} = \frac{pq}{p + q} = \frac{pq}{n} \), and the row sums of \( A \) and \( B \) are \( s = \frac{q^2}{n} \) and \( t = \frac{p^2}{n} \), respectively.

Here we see that \( n = p + q \) is of the form \( n = a^2b \) for an integer \( a \geq 2 \) and a square free integer \( b \geq 1 \), and \( p \) and \( q \) have \( ab \) as a common divisor. For, if \( n \) is not divisible by a square, \( n \) can be written \( n = \prod_{i=1}^{m} p_i \), for some distinct primes \( p_i, i = 1, \ldots, m \). Then \( n|p^2 \) implies each \( p_i|p \) and so \( n|p \), which is a contradiction. Now, the fact that \( n = a^2b \) divides both \( p^2 \) and \( q^2 \) implies \( a|p, a|q \) and \( b|p, b|q \). Hence we have \( p = abk \) and \( q = ab(a - k) \), where \( 1 \leq k \leq \lfloor \frac{q}{p} \rfloor \), and \( \lambda = \frac{pq}{n} = bk(a - k) \) is a positive integer. We summarize these results in the following theorem.

**Theorem 4.** Let \( M \) be an irreducible bipartite tournament matrix with team size \( p \leq q, p + q = n \). Suppose an eigenvector \( v \) of \( M \) satisfies (5).
Then there exist an integer \( a \geq 2 \), a square free integer \( b \geq 1 \), and an integer \( k \) with \( 1 \leq k \leq \left[ \frac{a}{2} \right] \) such that the team sizes of this tournament are \( p = abk \) and \( q = ab(a - k) \); the corresponding eigenvalue is \( \lambda = bk(a - k) \), which is a positive integer; the row sums of \( A \) and \( B \) are constants \( s = b(a - k)^2 \) and \( t = bk^2 \), respectively.

In particular, when \( a \) is even and \( k = \frac{a}{2} \), we have a regular bipartite tournament matrix \( M \) with team size \( p = q = \frac{a}{2} \), that is, row sums of \( M \) are all constant \( \frac{n}{4} \).

**Corollary 5.** Let \( M \) in theorem 4 be normal. Equation (5) holds for the Perron vector \( v \) if and only if \( M \) is regular, that is, \( M1 = \frac{n}{4}1 \).

**Proof.** It suffices to prove the necessity. Since \( M \) is normal, the row sums of \( A \) = the column sums of \( B = s = \frac{n}{2} \) and the row sums of \( B = \) the column sums of \( A = t = \frac{n}{2} \) [5]. The Perron value is \( \rho = \frac{\sqrt{pt}}{2} \) by theorem 3. On the other hand, from equation (6) with \( \lambda = \rho \), we have \( \rho = \frac{p}{q} \cdot \frac{n}{2} = \frac{n}{2} \). Hence, we obtain \( p = q = \frac{n}{2} \) and \( \rho = \frac{n}{4} \), which means \( M1 = \frac{n}{4}1 \), by theorem 1. \( \square \)

Note that we can rewrite corollary 5 as equation (5) holds for the Perron vector \( v \) if and only if \( p = q = \frac{n}{2} \); for when \( p = q \), \( M \) is normal if and only if it is regular [5].

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**References**


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