A NOTE ON LATTICE IMPlication ALGEBRAS

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Abstract. In this paper, a simple axiom system of lattice implication algebras is presented, it is convenient for verifying whether an algebra of type $(2,2,2,1,0,0)$ becomes a lattice implication algebra.

1. Introduction

In order to do research on the logical system whose propositional value is given in a lattice, Y. Xu [6] proposed the concept of lattice implication algebras. Some of their fundamental properties were obtained in [1, 2, 4, 6, 7]. In this paper we will give an equivalent axiom system of lattice implication algebras, which simplifies Xu’s axioms in [6]. This is convenient for verifying whether an algebra of type $(2,2,2,1,0,0)$ becomes a lattice implication algebra.

2. Preliminaries

According to the notion of lattice implication algebras, originally given by Y. Xu [6], we can also describe it as follows:

Definition. An algebra $(X, V, \land, \rightarrow, ', 0, 1)$ of type $(2,2,2,1,0,0)$ is called
a lattice implication algebra if it satisfies the following axioms:

(L1) \( x \land x = x, \)
(L1') \( x \lor x = x, \)
(L2) \( x \land y = y \land x, \)
(L2') \( x \lor y = y \lor x, \)
(L3) \( (x \land y) \land z = x \land (y \land z), \)
(L3') \( (x \lor y) \lor z = x \lor (y \lor z), \)
(L4) \( x \land (x \lor y) = x, \)
(L4') \( x \lor (x \land y) = x, \)
(B1) \( x \land 0 = 0, \)
(B1') \( x \lor 1 = 1, \)
(L) \( (x \land y) \rightarrow z = (x \rightarrow z) \lor (y \rightarrow z), \)
(L') \( (x \lor y) \rightarrow z = (x \rightarrow z) \land (y \rightarrow z), \)
(C1) \( x \land y = x \Rightarrow x' \land y' = y', \)
(C2) \( (x')' = x, \)
(I1) \( x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z), \)
(I2) \( x \rightarrow x = 1, \)
(I3) \( x \rightarrow y = y' \rightarrow x', \)
(I4) \( x \rightarrow y = 1 = y \rightarrow x \Rightarrow x = y, \)
(I5) \( (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x, \)

for all \( x, y, z \in X. \)

**Lemma 1 ([6]).** In a lattice implication algebra \((X, \lor, \land, \rightarrow, ', 0, 1)\), the following hold (for all \( x, y, z \in X)\):

(1) \( 0 \rightarrow x = 1, \)
(2) \( x \rightarrow 0 = x', \)
(3) \( x \lor y = (x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x, \)
(4) \( x \land y = ((y \rightarrow x) \rightarrow y')' = ((x \rightarrow y) \rightarrow x')'. \)

**Lemma 2 ([5]).** An algebra \((X; *, 0)\) of type \((2, 0)\) is a commutative \(BCK\)-algebra if and only if the following conditions are satisfied (for all \( x, y, z \in X)\):

(1) \( 0 \ast x = 0, \)
(2) \( (x \ast y) \ast (x \ast z) = (z \ast y) \ast (y \ast x), \)
(3) \( x \ast 0 = x. \)

**Lemma 3 ([3]).** Suppose \((X; *, 0)\) is a bounded commutative \(BCK\)-algebra with unit \(1\). For any \( x, y, z \) in \( X \), let \( Nx = 1 \ast x, x \land y = y \ast (y \ast x), x \lor y = N(Nx \land Ny). \) Then \((X, \lor, \land)\) is a distributive lattice with a least element \( 0 \) and a greatest element \( 1. \)
3. Result

The following theorem gives a very simpler axiom system of lattice implication algebras.

**Theorem 1.** An algebra \((X, \lor, \land, \to, \to', 0, 1)\) of type \((2, 2, 2, 1, 0, 0)\) is a lattice implication algebra if and only if it satisfies the following conditions (for all \(x, y, z \in X\)):

\begin{align*}
(Z1) \quad & 0 \to x = 1, \\
(Z2) \quad & (x \to z) \to (x \to y) = (z \to x) \to (z \to y), \\
(Z3) \quad & x \to y = y' \to x', \\
(Z4) \quad & (x')' = x, \\
(Z5) \quad & x' = x \to 0, \\
(Z6) \quad & x \lor y = (x \to y) \to y, \\
(Z7) \quad & (x \land y)' = x' \lor y'.
\end{align*}

**Proof.** Suppose \((X, \lor, \land, \to, \to', 0, 1)\) is a lattice implication algebra. By the definition of lattice implication algebras and Lemma 1, \((Z1)\) and \((Z3)\sim(Z6)\) hold. For \((Z2)\), from \((I1)\) and \((I3)\sim(I5)\) we obtain

\[
(x \to z) \to (x \to y) = x \to ((x \to z) \to y) = x \to (y' \to (x \to z)'),
\]

\[
= y' \to (x \to (x \to z')) = y' \to ((z' \to x') \to x'),
\]

\[
= y' \to ((x' \to z') \to z') = (x' \to z') \to (y' \to z'),
\]

\[
= (z \to x) \to (z \to y).
\]

For \((Z7)\), by \((Z3)\), \((Z4)\), \((Z6)\) and Lemma 1 \((4)\), we have

\[
x' \lor y' = (x' \to y') \to y' = (y \to x) \to y' = (x \land y)'.
\]

So far, the necessity is proved.

Conversely, suppose \((X, \lor, \land, \to, \to', 0, 1)\) satisfies the conditions \((Z1)\sim(Z7)\). For all \(x, y, z \in X\), let

\[
x \star y = (x \to y').
\]

We first prove \((X; \star, 0)\) is a bounded commutative \(BCK\)-algebra. In fact, by \((Z1)\) and \((Z5)\) we have \(0' = 1\), and so \(1' = 0\) by \((Z4)\). Therefore,
by (Z1) and (Z3)〜(Z5) we have

\[ 0 \ast x = (0 \to x)' = 1' = 0, \]
\[ x \ast 0 = (x \to 0)' = (x')' = x, \]
\[ x \ast 1 = (x \to 1)' = (0 \to x')' = 1' = 0. \]

Moreover, by (Z2)〜(Z5) we have

\[
(x \ast y) \ast (x \ast z) = ((x \to y)' \to (x \to z)')' = ((x \to z) \to (x \to y))'
\]
\[ = ((z \to x) \to (z \to y))' = ((z \to y)' \to (z \to x))'
\]
\[ = (z \ast y) \ast (z \ast x). \]

Thus, by Lemma 2 it is proved that \((X; \ast, 0)\) is a bounded commutative \(BCK\)-algebra with unit 1.

Next, by (Z3)〜(Z5) we have

\[ Nx = 1 \ast x = (1 \to x)' = (x' \to 0)' = (x'')' = x'. \]

By (Z3), (Z4), (Z6) and (Z7) we obtain

\[ x \land y = (x' \lor y')' = ((x' \to y') \to y')'
\]
\[ = (y \to (y \to x'))' = y \ast (y \ast x), \]
\[ x \lor y = (x \to y) \to (y' \to x')'
\]
\[ = (y' \ast (y' \ast x'))' = N(Nx \land Ny). \]

Summarizing the above discussions, by Lemma 3 it is proved that \((X, \lor, \land)\) is a distributive lattice with a least element 0 and a greatest element 1. Therefore (L1)〜(L4), (L' 1)〜(L' 4), (B1) and (B' 1) hold, and so (I5) holds by (L' 2) and (Z6). Since \(x \land y = x\) implies \(y = x \lor y\), by (Z4) and (Z7) we have \(y' = (x \lor y)' = x' \land y'\), i.e., (C1) holds.

Finally, from the fundamental properties [3] of bounded commutative
BCK-algebras we obtain

(I1) \( x \to (y \to z) = N(x \ast N(y \ast z)) = N((y \ast z) \ast Nx)\)
    \[= N((y \ast Nx) \ast z) = N((x \ast Ny) \ast z)\]
    \[= N((x \ast z) \ast Ny) = N(y \ast N(x \ast z))\]
    \[= y \to (x \to z).\]

(I2) \( x \to x = N(x \ast x) = N0 = 1.\)

(I4) \( x \to y = 1 = y \to x \iff N(x \to y) = N1 = N(y \to x)\)
    \[\iff x \ast y = 0 = y \ast x \iff x = y.\]

(L) \( (x \land y) \to z = N((x \land y) \ast z) = N((x \ast z) \land (y \ast z))\)
    \[= N(x \ast z) \lor N(y \ast z) = (x \to z) \lor (y \to z).\]

(L') \( (x \lor y) \to z = N((x \lor y) \ast z) = N((x \ast z) \lor (y \ast z))\)
    \[= N(x \ast z) \land N(y \ast z) = (x \to z) \land (y \to z).\]

Noticing that (I3) and (C2) hold in the case of the sufficiency, we obtain all the axioms of the definition are satisfied. So \((X, \lor, \land, \to, ^', 0, 1)\) is a lattice implication algebra. The proof is complete. \(\square\)

References


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