CONFORMAL CHANGE OF THE TENSOR $S_{\omega \mu}^\nu$ IN 7-DIMENSIONAL $g$-UFT

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ABSTRACT. We investigate change of the torsion tensor $S_{\omega \mu}^\nu$ induced by the conformal change in 7-dimensional $g$-unified field theory. These topics will be studied for the second class with the first category in 7-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ [10]. CHUNG [8] also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case, for the second and third classes in 5-dimensional case, and for the first class in 5-dimensional $g$-UFT, and for the second class in 6-dimensional $g$-UFT were investigated by CHO [1-4].

In the present paper, we investigate change of the torsion tensor $S_{\omega \mu}^\nu$ induced by the conformal change in 7-dimensional $g$-unified field theory. These topics will be studied for the second class with the first category in 7-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to CHUNG [5-7], CHO [1-4].

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2.1. \( n \)-dimensional \( g \)-unified field theory

The \( n \)-dimensional \( g \)-unified field theory (\( n \)-\( g \)-UFT hereafter) was originally suggested by HLAVATÝ [10] and systematically introduced by CHUNG [9].

Let \( X_n \)\(^1\) be an \( n \)-dimensional generalized Riemannian manifold, referred to a real coordinate system \( x^\nu \) obeying coordinate transformations \( x^\nu \to x^\nu' \), for which

\[
\text{Det} \left( \frac{\partial x}{\partial x'} \right) \neq 0.
\]

In the usual Einstein’s \( n \)-dimensional unified field theory, the manifold \( X_n \) is endowed with a general real nonsymmetric tensor \( g_{\lambda\mu} \) which may be split into its symmetric part \( h_{\lambda\mu} \) and skew-symmetric part \( k_{\lambda\mu} \)\(^2\):

\[
g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}
\]

where

\[
\text{Det}((g_{\lambda\mu})) \neq 0, \quad \text{Det}((h_{\lambda\mu})) \neq 0.
\]

Therefore we may define a unique tensor \( h^{\lambda\nu} = h^{\nu\lambda} \) by

\[
h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu.
\]

In our \( n \)-\( g \)-UFT, the tensors \( h_{\lambda\mu} \) and \( h^{\lambda\nu} \) will serve for raising and/or lowering indices of the tensors in \( X_n \) in the usual manner.

The manifold \( X_n \) is connected by a general real connection \( \Gamma^\nu_{\omega\mu} \) with the following transformation rule:

\[
\Gamma^\nu_{\omega\mu'} = \frac{\partial x^\nu}{\partial x^\alpha} \left( \frac{\partial x^\beta}{\partial x^{\omega'}} \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma^\beta_{\gamma\mu} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}} \right)
\]

and satisfies the system of Einstein’s equations

\[
D_\omega g_{\lambda\mu} = 2S_{\omega\mu} \alpha g_{\lambda\alpha}
\]

where \( D_\omega \) denotes the covariant derivative with respect to \( \Gamma^\nu_{\omega\mu} \) and

\[
S_{\omega\mu} \nu = \Gamma^\nu_{[\omega\mu]}
\]

\(^1\)Throughout the present paper, we assumed that \( n \geq 2 \).

\(^2\)Throughout this paper, Greek indices are used for holonomic components of tensors. In \( X_n \) all indices take the values \( 1, \ldots, n \) and follow the summation convention.
is the torsion tensor of $\Gamma^\nu_{\omega\mu}$. The connection $\Gamma^\nu_{\omega\mu}$ satisfying (2.6) is called the Einstein's connection.

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used:

\[(2.8) a \quad g = \text{Det}(g_{\lambda\mu}) \neq 0, \quad h = \text{Det}(h_{\lambda\mu}) \neq 0, \quad t = \text{Det}(k_{\lambda\mu}), \]

\[(2.8) b \quad g = \frac{g}{h}, \quad k = \frac{t}{h}, \]

\[(2.8) c \quad K_p = k_{[\alpha_1}^{\alpha_1} \cdots k_{\alpha_p]}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots) \]

\[(2.8) d \quad (0) k^\nu_{\lambda} = \delta^\nu_{\lambda}, \quad (1) k^\nu_{\lambda} = k^\nu_{\lambda}, \quad (p) k^\nu_{\lambda} = (p-1) k^\nu_{\lambda} \alpha k_{\alpha}^{\nu}, \]

\[(2.8) e \quad K_{\omega\mu\nu} = \nabla_{\nu} k_{\omega\mu} + \nabla_{\omega} k_{\nu\mu} + \nabla_{\mu} k_{\omega\nu}, \]

\[(2.8) f \quad \sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}, \]

where $\nabla_{\omega}$ is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $\Gamma^\nu_{\lambda\mu}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

\[(2.9) a \quad K_0 = 1; K_n = k \text{ if } n \text{ is even}; \quad K_p = 0 \text{ if } p \text{ is odd}, \]

\[(2.9) b \quad g = 1 + K_2 + \cdots + K_{n-\sigma}, \]

\[(2.9) c \quad (p) k_{\lambda\mu} = (-1)^{p(p)} k_{\lambda\mu}, \quad (p) k^{\lambda\nu} = (-1)^{p(p)} k^{\nu\lambda}. \]

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by $T$:

\[(2.10) a \quad T_{\omega\mu\nu} = T^p_{\alpha\beta\gamma} k^\alpha_{\omega} k^{\beta}_{\mu} k^{\gamma}_{\nu}, \]
(2.10)b \[ T = T_{\omega \mu \nu} = T_{\omega}^{000}, \]

(2.10)c \[ 2 T_{\omega|\lambda\mu|} = T_{\omega\lambda\mu}^{pq} - T_{\omega\mu\lambda}^{qr}, \]

(2.10)d \[ 2 T_{\omega\lambda\mu}^{(pq)r} = T_{\omega\lambda\mu}^{pq} + T_{\omega\lambda\mu}^{qr}. \]

We then have

(2.11) \[ T_{\omega\lambda\mu}^{pq} = - T_{\omega\lambda\mu}^{qr}. \]

If the system (2.6) admits $\Gamma_{\lambda\mu}^{\nu}$, using the above abbreviations it was shown that the connection is of the form

(2.12) \[ \Gamma_{\omega\mu}^{\nu} = \{\omega_{\mu}\} + S_{\omega\mu}^{\nu} + U_{\omega\mu}^{\nu}. \]

where

(2.13) \[ U_{\nu\omega\mu} = 2 S_{\nu(\omega\mu)}^{001}. \]

The above two relations show that our problem of determining $\Gamma_{\omega\mu}^{\nu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}^{\nu}$ satisfies

(2.14) \[ S = B - 3 S_{110}. \]

where

(2.15) \[ 2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta\kappa_{\omega}]}^{\alpha} k_{\nu}^{\beta}. \]

2.2. Some results for the second class in 7-g-UFT

In this section, we introduce some results of 7-g-UFT without proof, which are needed in our subsequent considerations.

They may be referred to CHUNG [5-7].

Definition 2.1. In 7-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class with the first category, if $K_2 \neq 0$, $K_4 = K_6 = 0$. 

Theorem 2.2. (Main recurrence relations) For the second class with the first category in 7-g-UFT, the following recurrence relation hold

\[ (p+3)k_\lambda^\nu = -K_2^{(p+1)}k_\lambda^\nu, \quad (p = 0, 1, 2, \cdots). \]

Theorem 2.3. (For the second class with the first category in 7-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is

\[ 1 - (K_2)^2 \neq 0. \]

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

\[ (1 - K_2^2)(S - B) = 1 - K_2^2 - (1 - K_2^2)2B - 2B + 2B - K_2B \]

3. Conformal change of the 7-dimensional torsion tensor for the second class

In this final chapter we investigate the change \( S_{\omega\mu}^{\nu} \to \overline{S}_{\omega\mu}^{\nu} \) of the torsion tensor induced by the conformal change of the tensor \( g_{\lambda\mu} \), using the recurrence relations and theorems introduced in the preceding chapter.

We say that \( X_n \) and \( \overline{X}_n \) are conformal if and only if

\[ \overline{g}_{\lambda\mu}(x) = e^\Omega g_{\lambda\mu}(x) \]

where \( \Omega = \Omega(x) \) is an at least twice differentiable function. This conformal change enforces a change of the torsion tensor \( S_{\omega\mu}^{\nu} \). An explicit representation of the change of 7-dimensional torsion tensor \( S_{\omega\mu}^{\nu} \) for the second class with the first category will be exhibited in this chapter.

Agreement 3.1. Throughout this section, we agree that, if \( T \) is a function of \( g_{\lambda\mu} \), then we denote \( \overline{T} \) the same function of \( \overline{g}_{\lambda\mu} \). In particular, if \( T \) is a tensor, so is \( \overline{T} \). Furthermore, the indices of \( T \) (\( \overline{T} \)) will be raised and/or lowered by means of \( h^{\lambda\nu}(\overline{h}^{\lambda\nu}) \) and/or \( h_{\lambda\mu}(\overline{h}_{\lambda\mu}) \).

The results in the following theorems are needed in our further considerations. They may be referred to CHUNG [8-9], CHO [1-3].
Theorem 3.2. In n-g-UFT, the conformal change (3.1) induces the following changes:

\begin{align*}
(3.2)a \\ (p)\bar{k}_{\lambda\mu} &= e^{\Omega(p)} k_{\lambda\mu}, & (p)\bar{k}_{\lambda}^{\nu} &= (p) k_{\lambda}^{\nu}, \\
(3.2)b \\ \bar{g} &= g, & \bar{K}_p &= K_p \\
(\text{for } p = 1, 2, \cdots).
\end{align*}

Theorem 3.3. (For all classes in 7-g-UFT). The change of the tensor $B_{\omega\mu\nu}$ induced by the conformal change (3.1) may be given by

\begin{align*}
(3.3) \quad \bar{B}_{\omega\mu\nu} &= e^{\Omega} (B_{\omega\mu\nu} + k_{\nu[\omega} \Omega_{\mu]} - k_{\omega\mu} \Omega_{\nu} \\
&\quad - h_{\nu[\omega} k_{\mu]} \delta \Omega_{\delta} + 2(2) k_{\nu[\omega} k_{\mu]} \delta \Omega_{\delta} + k_{\omega\mu} (2) k_{\nu} \delta \Omega_{\delta}).
\end{align*}

Now, we are ready to derive representations of the changes $S_{\omega\mu}^{\nu} \rightarrow \bar{S}_{\omega\mu}^{\nu}$ in 7-g-UFT for the second class with the first category induced by the conformal change (3.1).

Theorem 3.4. The conformal change (3.1) induces the following change:

\begin{align*}
(3.4) \quad \bar{B}_{\omega\mu\nu} &= e^{\Omega} \left[ 2 B_{\omega\mu\nu} + (-2(4) k_{\nu[\omega} k_{\mu]} \delta \\
&\quad + 2(2) k_{\nu[\omega} k_{\mu]} \delta - k_{\nu[\omega} (2) k_{\mu]} \delta \right] \Omega_{\delta} - (3) k_{\nu[\omega} \Omega_{\mu}].
\end{align*}

Theorem 3.5. The conformal change (3.1) induces the following change:

\begin{align*}
(3.5) \quad \bar{B}_{\omega\mu\nu} &= e^{\Omega} \left[ B_{\omega\mu\nu} + (-1)^{p} \left( 2(p+q+2) k_{\nu[\omega} (p+1) k_{\mu]} \delta \\
&\quad + (2(p+1)) k_{\omega\mu} \delta \right) k_{\nu} \delta \\
&\quad + (p+q+1) k_{\nu[\omega} (p) k_{\mu]} \delta - (p+q) k_{\nu[\omega} (p+1) k_{\mu]} \delta \right] \Omega_{\delta}. \\
(\text{for } p = 0, 1, 2, 3, 4, \cdots) \\
(\text{and } q = 0, 1, 2, 3, 4, \cdots)
\end{align*}
By the above relation (3.5), we obtain $\overline{B}, \overline{\overline{B}}, \overline{002}$.

**Theorem 3.6.** The change $S_{\omega \mu} \zeta \to \overline{S}_{\omega \mu} \zeta$ induced by conformal change (3.1) may be represented by

$$
\begin{align*}
\overline{S}_{\omega \mu} \zeta &= S_{\omega \mu} \zeta + 1 - k_{\omega k_{\mu}} \delta \Omega_0 \\
&\quad + (K_2 - 1) k_{\omega \mu} \Omega_0 + (1 - K_2) k_{\omega \mu} (2) k_{\mu \delta} \delta \Omega_0 \\
&\quad + \frac{1}{K_2^2 - 1} [(-1 + K_2) k_{\omega \mu} \Omega_0] \\
&\quad + (-1 + 2K_2 + K_2^2) (2) k_{\omega k_{\mu}} \delta \Omega_0 \\
&\quad + (K_2 + K_2^2 - 2K_2^3) k_{\omega (2) k_{\mu}} \delta \Omega_0.
\end{align*}
$$

(3.6)

**Proof.** In virtue of (2.18) and Agreement (3.1), we have

$$
(1 - K_2^2)(\overline{S} - \overline{B}) = 1 - \overline{K_2}^2 - (1 - K_2^2) \overline{B} - 2 \overline{B} + \overline{B} - 2 \overline{B}.
$$

(3.7) The relation (3.6) follows by substituting (3.2), (3.3), (3.4), (3.5), (2.16), Definition (2.1), into (3.7). \(\square\)

**References**


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