ALMOST REGULAR OPERATORS ARE REGULAR

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ABSTRACT. We give a characterization of regular operators that allows us to prove that a bounded operator acting between Banach spaces is almost regular if and only if it is regular, solving an open problem in [5]. As an application, we show that some operators in the closure of the set of all regular operators are regular.

Recently, Lee and Choi [5] introduced a concept of almost regular operators, following a suggestion in [3, Preface]. They proved that if $X$ and $Y$ are Hilbert spaces, then $T \in L(X, Y)$ is almost regular if and only if $T$ is regular. However, for $X$ and $Y$ incomplete normed spaces, they gave an example of an almost regular operator which is not regular. In the case that $X$ and $Y$ are Banach spaces they propose as an open problem whether almost regular operators and regular operators coincide.

Here we give a positive answer to this problem.

Along the paper, $X$ and $Y$ denote real or complex Banach spaces and $L(X, Y)$ the set of all (bounded linear) operators acting from $X$ into $Y$. For every $T \in L(X, Y)$ we denote by $R(T)$ and $N(T)$ the range and the kernel of $T$, respectively.

**Definition 1.** An operator $T \in L(X, Y)$ is called *almost regular* if there exists a bounded sequence $\{A_n\} \subset L(Y, X)$ such that

$$\|TA_nT - T\| \to 0 \text{ as } n \to \infty.$$ 

The operator $T \in L(X, Y)$ is called *regular* if there exists $A \in L(Y, X)$ such that $TAT = T$.

It is clear that regular operators are almost regular and that regular operators have close range.

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Recall that a subspace $M$ of a Banach space $X$ is said to be complemented if there exists a projection $P \in L(X, X)$ such that $R(P) = M$. In particular, complemented subspaces are closed.

**Remark 1.** An operator $T \in L(X, Y)$ is regular if and only if $N(T)$ and $R(T)$ are complemented.

Indeed, if there is $A \in L(Y, X)$ such that $TAT = T$, then it is easy to check that $TA$ is a projection with range $R(TA) = R(T)$ and $AT$ is a projection with kernel $N(AT) = N(T)$.

On the other hand, if $R(T)$ and $N(T)$ are complemented, then $T$ has a continuous inverse $T|_M^{-1}$ on any closed complement $M$ of $N(T)$. We can define $A \in L(Y, X)$ equal to $T|_M^{-1}$ on $R(T)$ and equal to 0 in a fixed closed complement of $R(T)$, and we obtain $TAT = T$.

We recall some concepts about ultrapowers of Banach spaces and operators. See [4] for more information.

We fix a non-trivial ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ of all positive integers. For every Banach space $X$, we consider the Banach space $\ell_\infty(X)$ of all bounded sequences $(x_i)$ in $X$, endowed with the norm $\| (x_i) \|_\infty := \sup \{ \|x_i\| : i \in \mathbb{N} \}$. Let $N_{\mathcal{U}}(X)$ be the closed subspace of all sequences $(x_i) \in \ell_\infty(X)$ which converge to 0 following $\mathcal{U}$. The **ultrapower of $X$ following $\mathcal{U}$** is defined as the quotient

$$X_{\mathcal{U}} := \frac{\ell_\infty(X)}{N_{\mathcal{U}}(X)}.$$ 

The element of $X_{\mathcal{U}}$ including the sequence $(x_i) \in \ell_\infty(X)$ as a representative is denoted by $[x_i]$. Its norm in $X_{\mathcal{U}}$ is given by

$$\|[x_i]\| = \lim_{\mathcal{U}} \|x_i\|.$$ 

The constant sequences generate a subspace of $X_{\mathcal{U}}$ isometric to $X$. So we can consider the space $X$ embedded in $X_{\mathcal{U}}$. Moreover, every operator $T \in L(X, Y)$ admits an extension $T_{\mathcal{U}} \in L(X_{\mathcal{U}}, Y_{\mathcal{U}})$, defined by

$$T_{\mathcal{U}}([x_i]) := [Tx_i], \quad [x_i] \in X_{\mathcal{U}}.$$ 

**Proposition 1.** Let $T \in L(X, Y)$ be an almost regular operator. Then $R(T)$ is closed.
Proof. We take a bounded sequence \( \{A_n\} \subset L(Y, X) \) such that \( \|TA_nT - T\| \to 0 \) as \( n \to \infty \), and define \( A \in L(Y, X) \) by
\[
A([y_i]) := [A_i y_i], \quad [y_i] \in Y\).
\]
Clearly \( A \) is well-defined and satisfies \( T_i A T_i = T_i \). Indeed,
\[
\|T_i A T_i - T_i\| = \lim_{i \to \infty} \|T A_i T - T\| = 0.
\]
Therefore \( T_i \) is regular; in particular \( R(T_i) \) is closed; hence \( R(T) \) is closed [2, Proposition 16].

**Remark 2.** (1) If \( X \) and \( Y \) are Hilbert spaces, then Proposition 1 implies that almost regular operators are regular.

Since every closed subspace of a Hilbert space is complemented, in this case the regular operators are precisely the operators with closed range.

(2) If \( X \) is reflexive, then we can give a direct proof of the fact that every almost regular \( T \in L(X, Y) \) is regular, using ultrafilter techniques.

Indeed, take a bounded sequence \( \{A_n\} \subset L(Y, X) \) such that \( \|TA_nT - T\| \to 0 \) as \( n \to \infty \). Since every bounded sequence in \( X \) is relatively weakly compact, for every \( y \in Y \) the sequence \( \{A_i y\} \) is weakly convergent following \( U \) [4]. Hence, we can define \( A \in L(Y, X) \) by \( A y := \text{weak-lim}_i A_i y \), and it is not difficult to check that \( T A T = T \); hence \( T \) is regular.

The following characterization of regular operators was known from [3, Theorem 3.8.2], however we give a different proof.

**Theorem 1.** An operator \( T \in L(X, Y) \) is regular if and only if there exists \( A \in L(Y, X) \) so that \( R(TAT) = R(T) \) and \( N(TAT) = N(T) \). In this case,
\[
X = N(T) \oplus R(AT) \quad \text{and} \quad Y = N(TA) \oplus R(T).
\]

**Proof.** The direct implication is clear. So we only have to prove the converse one. Moreover, by [3, Theorem 4.8.2] and the closed graph theorem, it is enough to prove that \( X = N(T) \oplus R(AT) \) and \( Y = N(TA) \oplus R(T) \) algebraically.

Suppose that \( x \in N(T) \cap R(AT) \). Then \( x = ATz \) for some \( z \in X \); hence \( 0 = Tx = TATz \). Thus \( z \in N(TAT) = N(T) \) and we conclude that \( x = ATz = 0 \).
Moreover, for every \( x \in X \) we can find \( z \in X \) so that \( Tx = TATz \). So we can write \( x = (x - ATz) + ATz \), and we have proved that \( X = N(T) \oplus R(AT) \).

For the remaining equality, suppose that \( y \in N(TA) \cap R(T) \). Then \( y = Tx \) for some \( x \in X \); hence \( 0 = TAx = TATx \). Thus \( x \in N(TAT) = N(T) \) and we conclude that \( y = Tx = 0 \).

Moreover, observe that \( R(TA) = R(TAT) \). Therefore, for every \( y \in Y \) we can find \( z \in X \) so that \( TAy = TATz \). So we can write \( y = (y - Tz) + Tz \), and we have proved that \( Y = N(TA) \oplus R(T) \).

For \( T \in L(X, Y) \), we consider the \textit{minimum modulus} \( \gamma(T) \), defined by

\[
\gamma(T) := \inf\{|\|Tx\|| : x \in X, \text{dist}(x, N(T)) = 1\}.
\]

It is well-known that \( R(T) \) is closed if and only if \( \gamma(T) > 0 \) [1, Theorem IV.1.6].

**Theorem 2.** Every almost regular operator \( T \in L(X, Y) \) is regular.

**Proof.** Let \( \{A_n\} \subset L(Y, X) \) be a sequence such that \( \|TA_nT - T\| \to 0 \) as \( n \to \infty \). Since \( R(T) \) is closed by Proposition 1, we can consider \( T \) and \( TA_nT \) as operators in \( L(X, R(T)) \). Thus the conjugate \( T^* \) is bounded below (as an operator in \( L(R(T)^*, X^*) \)). Moreover, \( \|T^*A_nT^* - T^*\| \to 0 \) as \( n \to \infty \). Therefore, there exists an integer \( n_1 \) so that \( T^*A_nT^* \) is bounded below (and hence \( TA_nT \) is surjective) for \( n > n_1 \). In this way, \( R(T) = R(TA_nT) \) for \( n > n_1 \). Moreover, \( N(T) \neq N(TA_nT) \) implies \( \|T - TA_nT\| \geq \gamma(T) > 0 \). Hence there exists an integer \( n_2 \) so that \( N(T) = N(TA_nT) \) for \( n > n_2 \). Taking \( n > \max\{n_1, n_2\} \) we obtain \( R(T) = R(TA_nT) \) and \( N(T) = N(TA_nT) \). Thus, the result follows from Theorem 1. \( \square \)

**Remark 3.** (1) In the definition of almost regular operator, the condition \( \{A_n\} \) bounded is not superfluous.

For instance, the operator \( T : \ell_2 \to \ell_2 \) given by \( T(x_n) := (x_n/n) \) is not regular. However, the operators \( A_n : \ell_2 \to \ell_2 \), given by

\[
A_n(x_1, x_2, \ldots) := (x_1, x_2, \ldots, nx_n, 0, 0, \ldots)
\]

satisfy \( \|TA_nT - T\| \to 0 \) and \( \{A_n\} \) is not bounded.

(2) An operator \( T \in L(X, Y) \) is regular if and only if it has closed range and there exists a (not necessarily bounded) sequence \( \{A_n\} \) in \( L(Y, X) \) so that \( \|TA_nT - T\| \to 0 \).
Almost regular operators are regular

It is enough to check the proof of Theorem 2.

(3) If in the definition of almost regular operator $T$ the operators $A_n$ can be taken to be bijective, then $\dim N(T) = \dim Y/R(T)$. This can be seen as a “zero index” condition, although sometimes $\dim N(T) = \dim Y/R(T) = \infty$.

Observe that, from the time that $N(T) = N(TA_nT)$ and $R(T) = R(TA_nT)$, the operators $A^{-1}_n$ apply $N(T)$ onto a complement of $R(T)$, which is isomorphic to $Y/R(T)$.

Finally, we give a result for operators in the closure of the set of all regular operators.

**Theorem 3.** Let $\{T_n\} \subset L(X,Y)$ be a sequence of regular operators. Assume that $T_n \to T$ as $n \to \infty$ and there exists a bounded sequence $\{U_n\} \subset L(Y,X)$ such that $T_nU_nT_n = T_n$ for all $n \in \mathbb{N}$. Then $T$ is regular.

**Proof.** From the equality $T_nU_nT_n = T_n$, it follows that

$$
\|TU_nT - T\| = \|(T - T_n)U_nT + T_nU_n(T - T_n) + T_n - T\|
$$

$$
\leq \|T - T_n\||U_nT\| + \|T_nU_n\||T - T_n\| + \|T_n - T\|.
$$

Since $\{U_n\}$ is bounded, we obtain that $\|TU_nT - T\| \to 0$ as $n \to \infty$. Hence $T$ is regular by Theorem 2.

**Remark 4.** (1) The condition of existence of a bounded sequence $\{U_n\}$ in Theorem 3 is not necessary in order that the limit of a sequence of regular operators be regular.

Given a Banach space $X$, the sequence of operators $\{T_n\} \subset L(X \times X, X \times X)$ defined by $T_n(x, y) := (x, y/n)$, converges to a regular operator, but there is no bounded sequence $\{T_n\} \subset L(X \times X, X \times X)$ so that $T_nU_nT_n = T_n$ for every $n$.

(2) If the sequence $\{U_n\}$ in Theorem 3 is unbounded, then there exists a sequence $\{Z_n\} \subset L(Y,X)$ of norm one operators such that $TZ_nT \to 0$ as $n \to \infty$.

Without lost of generality, we assume that $\|U_n\| \to \infty$. We define $Z_n := \frac{U_n}{\|U_n\|}$ and, proceeding as in the proof of Theorem 3, we get $\|TZ_nT\| \to 0$ as $n \to \infty$. 
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References


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