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COUNTING SELF-CONVERSE ORIENTED TREES

SUJEONG CHOI AND CHANGWOO LEE

ABSTRACT. We classify self-converse oriented trees into two types, namely, bicentral self-converse oriented trees and central ones, according to their centers and characterize these two types. Using characterizations and Pólya enumeration theorem, we derive the ordinary generating function for self-converse oriented trees.

1. Introduction

A *graph* G consists of a finite nonempty set $V = V(G)$ of p *vertices* together with a set $E = E(G)$ of q unordered pairs of distinct vertices of V. We say that the graph G has *order* p and *size* q. A pair $e = \{u, v\}$ of vertices in E is called an *edge* of the graph G. Perhaps the most important type of graph is a tree, because of its applications to many different fields. A *tree* is a connected graph with no cycles. An *oriented tree* is a tree in which each edge is assigned a unique direction, and a *self-converse oriented tree* is an oriented tree T whose *converse* T' obtained from T by reversing the direction of all edges is isomorphic to T. For definitions and notation not given here, see [1] and [2].

The object of this paper is to count the number of self-converse oriented trees. In particular, we show that the ordinary generating function $s(x)$ for self-converse oriented trees is

$$\begin{aligned} s(x) = x &+ x^2 + x^3 + 2x^4 + 3x^5 + 7x^6 + 10x^7 + 26x^8 \\ &+ 39x^9 + 107x^{10} + 160x^{11} + 458x^{12} + 702x^{13} \\ &+ 2058x^{14} + 3177x^{15} + 9498x^{16} + 14830x^{17} \\ &+ 44947x^{18} + 70678x^{19} + 216598x^{20} + \cdots . \end{aligned}$$

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2. Characterization

We want to characterize self-converse oriented trees. To do this, we need the following definitions. Let G be a connected graph and let v be a vertex of G. The *eccentricity* $e(v)$ of v is the distance from v to a vertex farthest from v. Thus $e(v) = max\{d(u,v) : u \in V\}$, where $d(u,v)$ is the length of a shortest path from u to v. The *radius* $r(G)$ of G is the minimum eccentricity of the vertices. Now v is a *central vertex* if $e(v) = r(G)$, and the *center* $C(G)$ of G is the set of all central vertices. Thus, the center consists of all vertices having minimum eccentricity.

THEOREM 1. [4, p. 55] *The center of a tree consists of either a single vertex or a pair of adjacent vertices.*

A tree is called *central* or *bicentral* depending on whether its center consists of a single vertex or two adjacent vertices. An oriented tree is called *central* or *bicentral* according as its underlying tree is central or bicentral.

Using the centers of oriented trees, we characterize self-converse oriented trees as follows:

THEOREM 2. *Let T be an oriented tree.*

(1) *If T is bicentral, then T is self-converse if and only if T corresponds to an ordered pair (A, A') of two rooted oriented trees A and A' such that A' is the converse of A. Moreover, the order of a bicentral self-converse oriented tree T must be even.*

(2) *If T is central, then T is self-converse if and only if T corresponds to a combination with repetition of n ordered pairs (A_1, A_1'), (A_2, A_2'), $\cdots$, (A_n, A_n') of rooted oriented trees A_i and A_i' such that A_i' is the converse of A_i for $i = 1, 2, \ldots, n$. Moreover, the order of a central self-converse oriented tree T must be odd.*

PROOF. (1) Let T be a bicentral self-converse oriented tree, and let both r and r' be the adjacent central vertices of T with the direction from r to r'. Remove the arc rr' from T and designate r and r' as roots. Then we have an ordered pair (A, A') of two rooted oriented trees A and A' with roots r and r' respectively. Clearly, A' is the converse of A.

Now, suppose that we have an ordered pair (A, A') of two oriented trees A and A' with roots r and r' respectively such that A' is the converse of A. To construct a bicentral self-converse oriented tree T from this pair, just add a new arc rr'.

It is straightforward to see that the order of T is even in this case.

(2) Let T be a central self-converse oriented tree and consider the center r of T as a root. Delete this root r from T and introduce new roots for the resulting trees as components. These roots are vertices adjacent from or to

r. Then, the rooted branches come in pairs A_i and A'_i that are converses each other. Now we form ordered pairs (A_i, A'_i) in such a way that the root of A_i is adjacent to r and the root of A'_i is adjacent from r.

Suppose that we have ordered pairs (A_1, A'_1), (A_2, A'_2), $\cdots$, (A_n, A'_n) of rooted oriented trees A_i and A'_i such that A'_i is the converse of A_i for $i = 1, 2, \ldots, n$. We want to construct a central self-converse oriented tree T from these pairs as follows. Add one new vertex and make it adjacent from or to each of the roots of the n given rooted oriented trees A_i or A'_i, respectively.

It is straightforward to see that the order of T is odd in this case. □

3. Main Result

In this section, we count the number of self-converse oriented trees. Let $s(x)$ be the ordinary generating function for self-converse oriented trees. First of all, we find the first 7 coefficients of $s(x)$ by checking the diagrams of self-converse oriented trees. See Figure 1. Therefore, we know

$$s(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 7x^6 + 10x^7 + \cdots .$$

Let $R(x)$ be the ordinary generating function for rooted oriented trees. Then $R(x)$ satisfies

$$R(x) = x\left(exp\left\{\sum_{k=1}^{\infty} R(x^k)/k\right\}\right)^2$$

and

$$(3.1) \qquad R(x) = x + 2x^2 + 7x^3 + 26x^4 + 107x^5 + 458x^6 + 2058x^7 + 9498x^8 + 44947x^9 + 216598x^{10} + \cdots .$$

This, of course, is a well-known result [3, p. 139].

We shall first find the ordinary generating function for bicentral self-converse oriented trees. Let T be a bicentral self-converse oriented tree of order $2p$. We observed in Theorem 2 that T corresponds in a natural way to an ordered pair (A, A') of two rooted oriented trees A and A' such that A' is the converse of A. Note that two rooted oriented trees A and A' have the same order p. More specifically, given an ordered pair (A, A') of two rooted oriented trees of order p such that A' is the converse of A, a new bicentral self-converse oriented tree T of order $2p$ is formed by adding a new arc from the root of A to that of A'. Clearly all bicentral self-converse oriented trees of order $2p$ can be formed in this manner. Therefore the number of bicentral self-converse oriented trees of order $2p$ is the same as the number of ordered pairs (A, A') of two rooted oriented trees A and A'

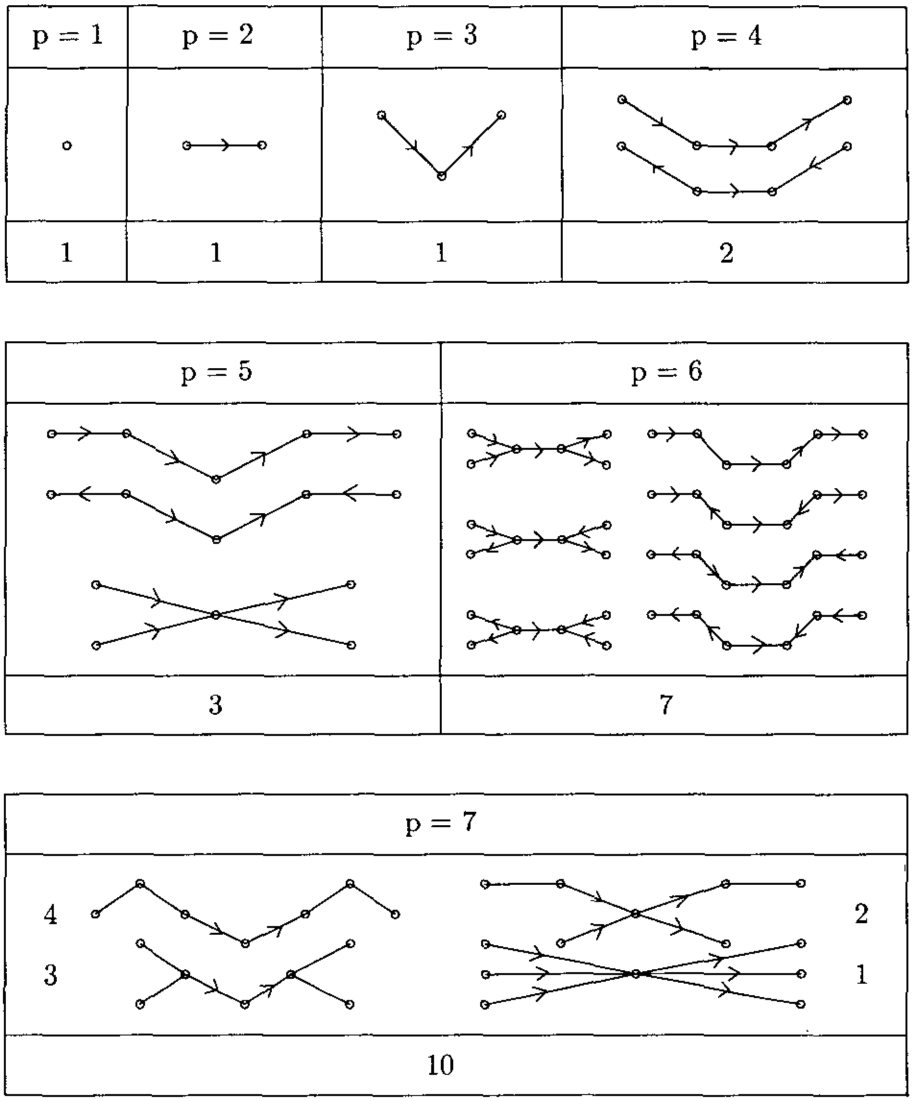

FIGURE 1. Self-converse oriented trees up to order $p \leq 7$

such that A' is the converse of A, which is exactly the number of rooted oriented trees of order p. Hence, from the counting series (3.1) for rooted oriented trees, the ordinary generating function for bicentral self-converse oriented trees is

$$R(x^2) = x^2 + 2x^4 + 7x^6 + 26x^8 + 107x^{10} + 458x^{12} + 2058x^{14} + 9498x^{16} + 44947x^{18} + 216598x^{20} + \cdots . \tag{3.2}$$

To find the generating function for central self-converse oriented trees, we need some preliminaries. Let A be a permutation group with object set $X = \{1, 2, \ldots, n\}$. It is well known that each permutation α in A can be written uniquely as a product of disjoint cycles and so for each integer k from 1 to n we let $j_k(\alpha)$ be the number of cycles of length k in the disjoint cycle decomposition of α. Then the *cycle index* of A, denoted by $Z(A) = Z(A; s_1, s_2, \ldots, s_n)$, is the polynomial in the variables $s_1, s_2, \ldots, s_n$ defined by

$$Z(A) = |A|^{-1} \sum_{\alpha \in A} \prod_{k=1}^{n} s_k^{j_k(\alpha)}.$$

Let S_n be the symmetric group on X. Define $Z(S_0) = 1$. Then we have a recursive formula

$$Z(S_n) = \frac{1}{n} \sum_{k=1}^{n} s_k Z(S_{n-k}).$$

Let $Z(S_n, R(x^2))$ denote the cycle index of S_n in which the variable s_k is replaced by $R(x^{2k})$ for k from 1 to n. For details, see [2].

Now, let T be a central self-converse oriented tree and consider the center r of T as a root. Note that the root r is incident with even number of arcs. By Theorem 2, central self-converse oriented trees in which the root is incident with $2n$ arcs correspond to combinations with repetition of n ordered pairs (A_1, A_1'), (A_2, A_2'), $\cdots$, (A_n, A_n') of rooted oriented trees A_i and A_i' such that A_i' is the converse of A_i for $i = 1, 2, \ldots, n$. On applying Pólya's Enumeration Theorem to the symmetric group S_n with $R(x^2)$ as the figure counting series, we have $Z(S_n, R(x^2))$ as the function counting series, and the coefficient of x^p in $Z(S_n, R(x^2))$ is the number of central self-converse oriented trees of order $p+1$ whose centers (or roots) are incident with $2n$ arcs. Multiplication of $Z(S_n, R(x^2))$ by x corrects the weights so that the coefficient of x^p in $xZ(S_n, R(x^2))$ is the number of these trees of order p. Then summing over all possible values of n, we obtain the ordinary generating function for central self-converse oriented trees:

$$x \sum_{n=0}^{\infty} Z(S_n, R(x^2)). \tag{3.3}$$

We want to find the first few terms of this generating function. To do this, we evaluate (3.3) for small n;

$$
\begin{aligned}
xZ(S_0, R(x^2)) &= x, \\
xZ(S_1, R(x^2)) &= xR(x^2) \\
&= x^3 + 2x^5 + 7x^7 + 26x^9 + 107x^{11} + 458x^{13} \\
&\quad + 2058x^{15} + 9498x^{17} + 44947x^{19} + \cdots, \\
xZ(S_2, R(x^2)) &= \frac{x}{2!}\{R(x^2)^2 + R(x^4)\} \\
&= x^5 + 2x^7 + 10x^9 + 40x^{11} + 187x^{13} + 854x^{15} \\
&\quad + 4074x^{17} + 19602x^{19} + \cdots, \\
xZ(S_3, R(x^2)) &= \frac{x}{3!}\{R(x^2)^3 + 3R(x^2)R(x^4) + 2R(x^6)\} \\
&= x^7 + 2x^9 + 10x^{11} + 44x^{13} + 208x^{15} + 988x^{17} \\
&\quad + 4843x^{19} + \cdots, \\
xZ(S_4, R(x^2)) &= \frac{x}{4!}\{R(x^2)^4 + 6R(x^2)^2R(x^4) + 8R(x^2)R(x^6) \\
&\quad + 3R(x^4)^2 + 6R(x^8)\} \\
&= x^9 + 2x^{11} + 10x^{13} + 44x^{15} + 213x^{17} \\
&\quad + 1016x^{19} + \cdots, \\
xZ(S_5, R(x^2)) &= x^{11} + 2x^{13} + 10x^{15} + 44x^{17} + 213x^{19} + \cdots, \\
xZ(S_6, R(x^2)) &= x^{13} + 2x^{15} + 10x^{17} + 44x^{19} + \cdots, \\
xZ(S_7, R(x^2)) &= x^{15} + 2x^{17} + 10x^{19} + \cdots, \\
xZ(S_8, R(x^2)) &= x^{17} + 2x^{19} + \cdots, \\
xZ(S_9, R(x^2)) &= x^{19} + \cdots, \\
&\vdots
\end{aligned}
$$

Summing these up, we have

$$
\begin{aligned}
(3.4) \qquad x\sum_{n=0}^{\infty} Z(S_n, R(x^2)) &= x + x^3 + 3x^5 + 10x^7 + 39x^9 \\
&\quad + 160x^{11} + 702x^{13} + 3177x^{15} \\
&\quad + 14830x^{17} + 70678x^{19} + \cdots.
\end{aligned}
$$

We have checked this computation using the program *Mathematica.*

Combining (3.2) and (3.4), we obtain the generating function desired:

THEOREM 3. *The ordinary generating function $s(x)$ for self-converse oriented trees is given by*

$$\begin{aligned} s(x) = {} & x + x^2 + x^3 + 2x^4 + 3x^5 + 7x^6 + 10x^7 + 26x^8 \\ & + 39x^9 + 107x^{10} + 160x^{11} + 458x^{12} + 702x^{13} \\ & + 2058x^{14} + 3177x^{15} + 9498x^{16} + 14830x^{17} \\ & + 44947x^{18} + 70678x^{19} + 216598x^{20} + \cdots . \end{aligned} \qquad \square$$

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Department of Mathematics
University of Seoul
Seoul 130–743, Korea
E-mail: chlee@uoscc.uos.ac.kr