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OPERATOR DOMAINS ON FUZZY SUBGROUPS

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ABSTRACT. The various fuzzy subgroups of a group which are admissible under operator domains are studied. In particular, the classes of all inner automorphisms, automorphisms, and endomorphisms are applied on the fuzzy subgroups of a group. As results, several theorems and examples concerning the fuzzy subgroups following from these kinds of operator domains are obtained. Moreover, we prove that a necessary condition for a fuzzy subgroup to be characteristic is that the center of the fuzzy subgroup is characteristic.

1. Introduction

Since the concept of fuzzy sets was introduced by L. A. Zadeh in [7], a lot of mathematicians have been involved in extending the notions of groups, rings, ideals, fields, vector spaces, etc. in algebra to broader realms of the fuzzy set. Among others, A. Rosenfeld formulated the concept of a fuzzy subgroup and showed how some basic notions of group theory should be extended to develop the theory of fuzzy groups in his pioneering paper [6]. P. S. Das characterized fuzzy subgroups by their level subgroups in [1], since then many notions of fuzzy group theory can be equivalently characterized with the help of notion of level subgroups.

In this paper, we characterize the concept of a fuzzy normal subgroup (a fuzzy characteristic subgroup, a fuzzy fully invariant subgroup) in terms of notion of operator domain, which is the analogue of a normal subgroup (a characteristic subgroup, a fully invariant subgroup) and obtain several analogues of basic results of group theory.

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2. Preliminaries

This section presents basic definitions and results to be used in the sequel.

DEFINITION 1. Let X be a set. A map

$$\lambda : X \to [0,1]$$

is called a fuzzy subset of X.

DEFINITION 2. Let $\lambda : X \to [0,1]$ be a fuzzy subset of X and let $t \in [0,1]$. The set

$$\lambda_t = \{x \in X | \lambda(x) \geq t\}$$

is called a level subset of λ.

Let f be a map of X into Y, and λ and μ be fuzzy subsets of X and Y, respectively. The image $f(\lambda)$ of λ is the fuzzy subset of Y defined by, for $y \in Y$,

$$f(\lambda)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \lambda(x) & \text{if} \quad f^{-1}(y) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

The inverse image $f^{-1}(\mu)$ of μ under f is the fuzzy subset of X defined by, for $x \in X$,

$$f^{-1}(\mu)(x) = \mu\left(f(x)\right).$$

DEFINITION 3. Let G be a group. A fuzzy subset $\lambda : G \to [0,1]$ is called a fuzzy subgroup of G if

(1) $\lambda(xy) \geq \min\{\lambda(x), \lambda(y)\}$ for every $x, y \in G$.
(2) $\lambda(x) \geq \lambda(x^{-1})$ for every $x \in G$.

It is easy to see that if λ is a fuzzy subgroup of G, then for every $x \in G$, $\lambda(x) = \lambda(x^{-1})$ and $\lambda(e) \geq \lambda(x)$, where e is the identity of G. It is well known that a fuzzy subset λ of G is a fuzzy subgroup of G if and only if each level subset $\lambda_t, t \in \mathrm{Im}\lambda$, is a subgroup of G. See [1].

DEFINITION 4. If λ and μ are fuzzy subgroups of G and $\lambda \subseteq \mu$ then we say that λ is a fuzzy subgroup of μ.

The support of a fuzzy subgroup λ is a subgroup of G defined by $\mathrm{Supp}(\lambda) = \{x \in G | \lambda(x) > 0\}$.

For a group G, an operator domain on G is a nonempty class $\mathcal{D}$ of endomorphisms of G. In group theory we call a subset A of G admissible under $\mathcal{D}$, or simply $\mathcal{D}$-admissible if $f(A) \subseteq A$ for every $f \in \mathcal{D}$. A subgroup H of G is normal(resp, characteristic, fully invariant) if H is admissible under the class of all inner automorphisms(resp, all automorphisms, all endomorphisms) of G.

We can now define analogues of the concepts above point of a fuzzy subgroup.

LEMMA 1. *Let λ be a fuzzy subset of X and $f : X \to X$ be a map. Then the following two conditions are equivalent;*

(1) $f(\lambda) \subseteq \lambda$
(2) $\lambda \subseteq f^{-1}(\lambda)$.

PROOF. See [4]. □

DEFINITION 5. Let $\mathcal{D}$ be an operator domain on a group G. A fuzzy subset λ of G is admissible under $\mathcal{D}$ (or simply $\mathcal{D}$-admissible) if, for every $f \in \mathcal{D}$, one of two conditions of Lemma 1 is satisfied.

DEFINITION 6. Let Inn(G), Aut(G), and End(G) denote the classes of all inner automorphisms, all automorphisms, and all endomorphisms of a group G, respectively. A fuzzy subgroup λ of μ is a fuzzy normal (resp. fuzzy characteristic, fuzzy fully invariant) subgroup of μ if λ is admissible under $\mathrm{Inn}(\mathrm{Supp}(\mu))$ (resp. $\mathrm{Aut}(\mathrm{Supp}(\mu)), \mathrm{End}(\mathrm{Supp}(\mu))$). In particular, λ is a fuzzy normal (resp. fuzzy characteristic, fuzzy fully invariant) subgroup of G if λ is admissible under $\mathrm{Inn}(\mathrm{Supp}(\chi_G))$ $\big($resp. Aut $(\mathrm{Supp}(\chi_G))$, End $(\mathrm{Supp}\ (\chi_G))\big)$, where χ_G is the fuzzy mother subgroup of G.

Two examples will be shown in the sequel. One is a fuzzy characteristic subgroup which is not fuzzy fully invariant in Example 1, and the other is a fuzzy normal subgroup which is not fuzzy characteristic in Example 2.

EXAMPLE 1. Let S_3 be the symmetric group of order 3 whose elements are

$$\sigma_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and $t_i, 1 \leq i \leq 3$, be the numbers lying in the interval $[0, 1]$ such that $t_1 > t_2 > t_3$. The fuzzy subset $\lambda : S_3 \longrightarrow [0, 1]$ defined by $\lambda(\sigma_0) = t_1,\ \lambda(\sigma_1) =$

$\lambda(\sigma_2) = t_2$, $\lambda(\tau_1) = \lambda(\tau_2) = \lambda(\tau_3) = t_3$ is a fuzzy subgroup of S_3 because each of level subsets is a subgroup of S_3. Since the order of elements $\sigma_0, \sigma_1, \sigma_2, \tau_1, \tau_2, \tau_3$ is $1,3,3,2,2,2$ respectively, every automorphism f of S_3 is of the form $f(\sigma_0) = \sigma_0, f(\sigma_1) = \sigma_i, f(\sigma_2) = \sigma_i (i = 1,2)$, $f(\tau_1) = \tau_i, f(\tau_2) = \tau_i, f(\tau_3) = \tau_i (i = 1,2,3)$. This means that $\lambda\big(f(\alpha)\big) = \lambda(\alpha)$ for every $\alpha \in S_3$ and $f \in \mathrm{Aut}(S_3)$, and hence λ is a fuzzy characteristic subgroup of S_3. In fact, since $\mathrm{Inn}(S_3) = \mathrm{Aut}(S_3)$, the class of all fuzzy normal subgroups of S_3 is the same as the class of all fuzzy characteristic subgroups. On the other hand, the map $g : S_3 \longrightarrow S_3$ defined by $g(\sigma_i) = \sigma_0\,(i = 0,1,2)$, and $g(\tau_j) = \tau_1\,(j = 1,2,3)$ is an endomorphism of S_3, and

$$g^{-1}(\lambda)(\sigma_1) = \lambda\left(g(\sigma_1)\right) = \lambda(\sigma_0) = t_1 \neq t_2 = \lambda(\sigma_1).$$

Thus λ is not a fuzzy fully invariant subgroup of S_3.

3. Results

LEMMA 2. *Let λ be a fuzzy subgroup of G with* $\mathrm{Card}(\mathrm{Im}\lambda) < \infty$. *Then λ is a fuzzy characteristic subgroup of G if and only if each level subgroup of λ is a characteristic subgroup of G.*

PROOF. See [2]. □

DEFINITION 7. Let λ be a fuzzy subgroup of G. We denote by λ_x^* the fuzzy subgroup of G defined as

$$\lambda_x^*(g) = \lambda(x^{-1}gx) \quad \text{for every} \quad g \in G$$

and call the fuzzy conjugate subgroup of G determined by λ and x in G.

THEOREM 1. *Let G be a group and λ be a fuzzy characteristic subgroup of G with* $\mathrm{Card}(\mathrm{Im}\lambda) < \infty$. *Then the fuzzy conjugate subgroup λ_x^* of G determined by λ and x, where $x \in G$, is also fuzzy characteristic subgroup of G.*

PROOF. Let $\mathrm{Im}\lambda = \{t_0, t_1, \cdots, t_n\}$ with $t_0 > t_1 > \cdots > t_n$ and the chain of level subgroups of λ be

$$\lambda_{t_0} \subseteq \lambda_{t_1} \subseteq \cdots \subseteq \lambda_{t_n} = G.$$

Define a map $i_x : G \to G$ by $i_x(g) = x^{-1}gx$ for all $g \in G$. Then $i_x \in \mathrm{Inn}(G)$. Since $\lambda_x^* = \lambda \circ i_x = i_x^{-1}(\lambda)$, it follows that $\mathrm{Im}\lambda = \mathrm{Im}\lambda_x^*$. Now

$$\begin{aligned} g \in (\lambda_x^*)_{t_k} &\iff \lambda_x^*(g) = \lambda(x^{-1}gx) \geq t_k \iff i_x(g) = x^{-1}gx \in \lambda_{t_k} \\ &\iff g \in i_x^{-1}(\lambda_{t_k}) = i_{x^{-1}}(\lambda_{t_k}) = x\lambda_{t_k}x^{-1} \end{aligned}$$

This means that $(\lambda_x^*)_{t_k} = x\lambda_{t_k}x^{-1}$ for all $k = 0, 1, \cdots, n$. Hence the chain of level subgroups of the fuzzy conjugate subgroup λ_x^* is given by

$$x\lambda_{t_0}x^{-1} \subseteq x\lambda_{t_1}x^{-1} \subseteq \cdots \subseteq x\lambda_{t_n}x^{-1} = G.$$

Since λ is a fuzzy characteristic subgroup of G, each $\lambda_{t_i} (i = 0, 1, \cdots, n)$ is a characteristic subgroup of G by Lemma 2, and so a normal subgroup of G. Hence we have $x\lambda_{t_i}x^{-1} = \lambda_{t_i}$ for all $i = 0, 1, \cdots, n$. Since λ_{t_i} is a characteristic subgroup of G, $f(x\lambda_{t_i}x^{-1}) = f(\lambda_{t_i}) \subseteq \lambda_{t_i} = x\lambda_{t_i}x^{-1}$ for every $f \in \mathrm{Aut}(G)$. Therefore, by Lemma 2, λ_x^* is a fuzzy characteristic subgroup of G. □

THEOREM 2. *If λ is a fuzzy characteristic subgroup of μ and μ is a fuzzy characteristic subgroup of η then λ is a fuzzy characteristic subgroup of η.*

In particular, if λ is a fuzzy characteristic subgroup of μ and μ is a fuzzy characteristic subgroup of G then λ is a fuzzy characteristic subgroup of G.

PROOF. Since μ is a fuzzy characteristic subgroup of η, $f^{-1}(\mu) = \mu$ for all$f \in \mathrm{Aut}(\mathrm{Supp}(\eta))$. Let $x \in \mathrm{Supp}(\mu)$ and $f \in \mathrm{Aut}(\mathrm{Supp}(\eta))$. Then we have $\mu(f(x)) = f^{-1}(\mu)(x) = \mu(x) > 0$, which implies that $f(\mathrm{Supp}(\mu)) \subseteq \mathrm{Supp}(\mu)$. Since $f^{-1} \in \mathrm{Aut}(\mathrm{Supp}(\eta))$, we have $f^{-1}(\mathrm{Supp}(\mu)) \subseteq (\mathrm{Supp}(\mu))$ in the same way. Hence the restriction map $g = f|_{\mathrm{Supp}(\mu)}$ of f to the $\mathrm{Supp}(\mu)$ is an automorphism of $\mathrm{Supp}(\mu)$. This means that $g^{-1}(\lambda) = \lambda$ because λ is a fuzzy characteristic subgroup of μ. For every $x \in \mathrm{Supp}(\mu)$, we have $f^{-1}(\lambda)(x) = \lambda(f(x)) = \lambda(g(x)) = g^{-1}(\lambda)(x) = \lambda(x)$. On the other hand, the fact $f^{-1}(\lambda)(x) = \lambda(x)$, for every $x \in \mathrm{Supp}(\eta) - \mathrm{Supp}(\mu)$, follows from

$$f^{-1}(\lambda)(x) = \lambda(f(x)) \leq \mu(f(x)) = 0 \text{and} \lambda(x) \leq \mu(x) = 0$$

because $x \notin \mathrm{Supp}(\mu)$ and $f(x) \notin \mathrm{Supp}(\mu)$. The foregoing results yield $f^{-1}(\lambda)(x) = \lambda(x)$ for every $x \in \mathrm{Supp}(\eta)$ and for every $f \in \mathrm{Aut}(\mathrm{Supp}(\eta))$. This completes the proof. □

THEOREM 3. *If λ is a fuzzy characteristic subgroup of μ and μ is a fuzzy normal subgroup of η then λ is a fuzzy normal subgroup η.*

PROOF. Let i be any inner automorphism of $\mathrm{Supp}(\eta)$. Since μ is a fuzzy normal subgroup of η, we have $\mu(i(x)) = \mu(x)$ for all $x \in \mathrm{Supp}(\eta)$. Since $\mu(i(x)) = \mu(x) > 0$ for all $x \in \mathrm{Supp}(\mu), i(\mathrm{Supp}(\mu)) \subseteq \mathrm{Supp}(\mu)$. This implies that the restriction map $g = i|_{\mathrm{Supp}(\mu)}$ of i to $\mathrm{Supp}(\mu)$ is the automorphism of $\mathrm{Supp}(\mu)$. Since λ is a fuzzy characteristic subgroup of μ, we have $g^{-1}(\lambda)(x) = \lambda(x)$ for all $x \in \text{Supp}(\mu)$. This induces $i^{-1}(\lambda)(x) = \lambda(i(x)) = \lambda(g(x)) = \lambda(x)$ for all $x \in \text{Supp}(\mu)$. On the other hand, for all $x \in \text{Supp}(\eta) - \text{Supp}(\mu)$, $i^{-1}(\lambda)(x) = \lambda(i(x)) \le \mu(i(x)) = 0$ and $\lambda(x) \le \mu(x) = 0$ because $x \notin \text{Supp}(\mu)$ and $i(x) \notin \text{Supp}(\mu)$. These yield $i^{-1}(\lambda)(x) = \lambda(x)$ for all $i \in \text{Inn}(\text{Supp}(\eta))$ and for all $x \in \text{Supp}(\eta)$. Therefore λ is a fuzzy normal subgroup of η. □

DEFINITION 8. Let λ be a fuzzy subgroup of G. The set

$$C(\lambda) = \{x \in G | \lambda(xg) = \lambda(gx) \text{ for all } g \in G\}$$

is called a center of λ.

It is easily verified that the center $C(\lambda)$ of a fuzzy subgroup λ of G is a subgroup of G.

THEOREM 4. *Let λ be a fuzzy subgroup of G. Then the followings are equivalent;*

(1) *λ is admissible under $\text{Inn}(G)$.*
That is, λ is a fuzzy normal subgroup of G.
(2) *λ assumes constant value on the conjugate class of G.*
(3) *$\lambda(xy) = \lambda(yx)$ for every $x, y \in G$.*
(4) *$C(\lambda) = G$.*

PROOF. See Theorem 4.2 of [3]. □

THEOREM 5. *If λ is a fuzzy characteristic subgroup of G, then the center $C(\lambda)$ of λ is a characteristic subgroup of G.*

PROOF. Let $x \in C(\lambda)$. The proof will be finished if we claim that $f(x) \in C(\lambda)$ for every $f \in \text{Aut}(G)$. For $g \in G$, there is $g' \in G$ such that $f(g') = g$. We have

$$\begin{aligned}
\lambda(f(x)g) &= \lambda\left(f(x)f(g')\right) = \lambda\left(f(xg')\right) \\
&= \lambda(xg') \text{ since } \lambda \text{ is Aut}(G)\text{-admissible} \\
&= \lambda(g'x) \text{because } x \in C(\lambda) \\
&= \lambda\left(f^{-1}(g)f^{-1}(f(x))\right) = \lambda\left(f^{-1}(gf(x))\right) \\
&= \lambda(gf(x)) \text{ since } \lambda \text{ is Aut}(G)\text{-admissible.}
\end{aligned}$$

That is, $f(x) \in C(\lambda)$. Therefore the center $C(\lambda)$ of λ is a charcteristic subgroup of G. □

The example that follows says the converse of Theorem 5 is not true.

EXAMPLE 2. Let D_4 be the dihedral group of order 8, that is,

$$\begin{aligned} D_4 &= \left\langle \left\{\sigma, \tau | \sigma^4 = \tau^2 = 1,\ \tau\sigma = \sigma^{-1}\tau\right\} \right\rangle \\ &= \left\{1, \sigma, \sigma^2, \sigma^3, \tau, \sigma\tau, \sigma^2\tau, \sigma^3\tau\right\} \end{aligned}$$

and $t_i, 1 \leq t_i \leq 3$, be the numbers lying in the interval $[0, 1]$ such that $t_1 > t_2 > t_3$. The fuzzy subset $\lambda : D_4 \longrightarrow [0, 1]$ defined by

$$\begin{aligned} &\lambda(1) = t_1, \\ &\lambda(\tau) = \lambda(\sigma^2\tau) = \lambda(\sigma^2) = t_2 \\ &\lambda(\sigma) = \lambda(\sigma^3) = \lambda(\sigma\tau) = \lambda(\sigma^3\tau) = t_3 \end{aligned}$$

is a fuzzy subgroup of D_4 because each of level subsets is a subgroup of D_4. It can be easily checked by the multiplication table which follows that $C(\lambda) = D_4$, and hence $C(\lambda)$ is a characteristic subgroup of D_4. Now define the map $\alpha : D_4 \longrightarrow D_4$ as follows:

$$\begin{array}{llll} \alpha(1) = 1, & \alpha(\sigma) = \sigma, & \alpha(\sigma^2) = \sigma^2, & \alpha(\sigma^3) = \sigma^3 \\ \alpha(\tau) = \sigma\tau, & \alpha(\sigma\tau) = \sigma^2\tau, & \alpha(\sigma^2\tau) = \sigma^3\tau, & \alpha(\sigma^3\tau) = \tau. \end{array}$$

Then it follows that λ is not a fuzzy characteristic subgroup of D_4 because we get

$$\alpha^{-1}(\lambda)(\tau) = \lambda\left(\alpha(\tau)\right) = \lambda(\sigma\tau) = t_3 \neq t_2 = \lambda(\tau)$$

for the automorphism α of D_4. Therefore the converse of Theorem 5 is not true. On the other hand, λ is a fuzzy normal subgroup of D_4 from Theorem 4.

	1	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
1	1	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
σ	σ	σ^2	σ^3	1	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$
σ^2	σ^2	σ^3	1	σ	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$
σ^3	σ^3	1	σ	σ^2	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ
τ	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	1	σ	σ^2	σ^3
$\sigma\tau$	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ	σ^3	1	σ	σ^2
$\sigma^2\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$	σ^2	σ^3	1	σ
$\sigma^3\tau$	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$	σ	σ^2	σ^3	1

THEOREM 6. *If λ and μ are fuzzy fully invariant subgroups of G, so is $\lambda \cap \mu$. The theorem is true in case we replace fuzzy fully invariant subgroups with either fuzzy normal subgroups or fuzzy characteristic subgroups.*

PROOF. Let $f \in \mathrm{End}(G)$ and $y \in G$. In case $f^{-1}(y) = \emptyset$, it follows from the definition $0 = f(\lambda \cap \mu)(y) \le (f(\lambda) \cap f(\mu))(y) \le (\lambda \cap \mu)(y)$. In the other case $f^{-1}(y) \neq \emptyset$, we have

$$\begin{aligned} f(\lambda \cap \mu)(y) &= \sup_{f(x)=y} (\lambda \cap \mu)(x) \\ &\le \sup_{f(x)=y} \lambda(x) = f(\lambda)(y). \end{aligned}$$

In the same way, we have $f(\lambda \cap \mu)(y) \le f(\mu)(y)$. Putting these facts together, we have

$$f(\lambda \cap \mu) \subseteq f(\lambda) \cap f(\mu) \subseteq \lambda \cap \mu.$$

Therefore $\lambda \cap \mu$ is a fuzzy fully invariant subgroup of G. □

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