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A NOTE ON THE ROOT SYSTEM OF AN AFFINE LIE ALGEBRA OF TYPE $A_{2l}^{(2)}$

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ABSTRACT. Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra of type $A_{2l}^{(2)} (l > 1)$ and W be its Weyl group. In this paper, we find $w \in W$ such that $\overset{\circ}{\Delta}_+ \cup \{\frac{1}{2}(\alpha_l + \delta), \frac{1}{2}(\alpha_l + 3\delta)\} \subset \Delta^+(w)$.

1. Notation and some basic facts about root systems of Kac-Moody algebras

We first recall some of the basic definitions in Kac-Moody theory. An $n \times n$ integral matrix $A = (a_{ij})_{i,j=1}^n$ is called a generalized Cartan matrix(GCM) if

$$(1.1)\qquad \begin{cases} a_{ii} = 2, & i = 1, 2, \dots, n, \\ a_{ij} \leq 0 & \text{if } \ i \neq j, \\ a_{ij} = 0 & \text{implies } \ a_{ji} = 0. \end{cases}$$

A realization of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is the complex vector space, $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ are indexed subsets in $\mathfrak{h}^*$and $\mathfrak{h}$, respectively, satisfying the following three conditions;

$$(1.2)\qquad \begin{cases} \Pi \text{ and } \Pi^\vee \text{ are linearly independent} \\ \alpha_j(\alpha_i^\vee) = a_{ij} \quad (i, j = 1, 2, \dots, n) \\ \dim \mathfrak{h} = 2n - l, \quad \text{where } l = \operatorname{rank} A. \end{cases}$$

The elements of Π (resp. $\Pi^\vee$) are called the simple roots (resp. simple coroots) of $\mathfrak{g}$. For each $i = 1, 2, \dots, n$, define

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$$r_i : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$$

by

$$r_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i,$$

for $\lambda \in \mathfrak{h}^*$. We have in particular,

$$r_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i.$$

The subgroup W of $GL(\mathfrak{h}^*)$ generated by the r_i's $(i = 1, 2, \ldots, n)$ is called the Weyl group of $\mathfrak{g}$. Let $Q = \sum_{i=1}^n \mathbb{Z}\alpha_i$, $Q_+ = \sum_{i=1}^n \mathbb{Z}_{\geq 0}\,\alpha_i$ and $Q_- = -Q_+$. We define a partial ordering $\geq$ on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ has the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$. An element $\alpha \in Q$ is called a root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. A root $\alpha > 0$ (resp. $\alpha < 0$) is called positive(resp. negative). For $\alpha = \sum_{i=1}^n k_i\alpha_i \in Q$, the number $ht(\alpha) = \sum_{i=1}^n k_i$ is called the height of α. A root $\alpha \in \Delta$ is called a real root if there exists $w \in W$ such that $w(\alpha)$ is a simple root. A root α which is not a real root is called an imaginary root.

A root of a finite root system Δ which satisfies above condition is called the highest root. It is well known that generalized Cartan matrix $A = (a_{ij})_{i,j=1}^n$ is of finite, affine or indefinite type [3].

Let $A = (a_{ij})_{i,j=0}^l$ be an indecomposable generalized Cartan Matrix of affine type of order $l+1$ (and rank l) and $\mathfrak{g} = \mathfrak{g}(A)$ be the associated affine Lie algebra. The Kac-Moody Lie algebra $\mathring{g} = g(\mathring{A})$ associated with the Cartan matrix $\mathring{A} = (a_{ij})_{i,j=1}^l$ is a finite dimensional simple Lie algebra. Let $\mathring{W}$ be the Weyl group of $\mathring{\mathfrak{g}}$ generated by simple reflections $r_1, \ldots, r_l$. Let Δ and $\mathring{\Delta}$ denote the set of roots for g and $\mathring{g}$ respectively. Then $\Delta = \Delta^{re} \cup \Delta^{im}$, where Δ^{re} and Δ^{im} denote the real and imaginary roots, respectively. It is known that the set of positive imaginary roots $\Delta_+^{\text{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$, where $\delta = \sum_{i=0}^l a_i\alpha_i$, $a_i \in \mathbb{Z}_{>0}$, $gcd(a_0, \ldots, a_n) = 1$ and $A(a_0, \ldots, a_n)^T = 0$. We have $a_0 = 1$ unless A is of type $A_{2l}^{(2)}$, in which case $a_0 = 2$. Note that $\Delta = \Delta_+ \cup \Delta_-$ $\mathring{\Delta} = \mathring{\Delta}_+ \cup \mathring{\Delta}_-$, $\Delta^{re} = \Delta_+^{re} \cup \Delta_-^{re}$ $\Delta^{im} = \Delta_+^{im} \cup \Delta_-^{im}$, where the subscript plus/minus denote the positive/negative roots. Let $\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_l\}$ and $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ denote the simple roots for $\mathring{g}$ and g respectively. We denote the the set of long(resp. short)roots for $\mathring{g}$ by $\mathring{\Delta}_l$(resp. $\mathring{\Delta}_s$).

Let $\Delta^+(w) = \{\alpha \in \Delta_+ \mid w^{-1}(\alpha) < 0\}$ for $w \in W$. Since Δ_+^{im} is W-invariant, $\Delta^+(w) \subset \Delta_+^{\mathrm{re}}$.

We introduce the following important element:

$$\theta = \delta - a_0\alpha_0 = \sum_{i=1}^{l} a_i\alpha_i.$$

2. Properties of the root system of an Affine Lie algebra of type $A_{2l}^{(2)}$

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_{2l}^{(2)}$ and Δ be the root system of $\mathfrak{g} = \mathfrak{g}(A)$. We denote by Δ_s, Δ_m, and Δ_l the set of all short, medium, and long roots, respectively.

Since $\mathring{A}_{2l}^{(2)}$ is isomorphic to C_l, we have

$$\begin{aligned}\mathring{\Delta}_s &= \{\alpha_i + \cdots + \alpha_l + \cdots + \alpha_j \,|\, 1 \le i < j \le l\} \\ &\quad \cup \{\alpha_i + \cdots + \alpha_j \,|\, 1 \le i < j \le l-1\} \\ \mathring{\Delta}_l &= \{\alpha_i + \cdots + \alpha_l + \cdots + \alpha_i \,|\, 1 \le i \le l\}.\end{aligned}$$

It is left which is the description of the set of real roots. They are given by the following theorem.

PROPOSITION 2.1. [2] *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_{2l}^{(2)}$ and Δ be the root system of $\mathfrak{g} = \mathfrak{g}(A)$. Then we have*

(a) $\Delta_s^{\mathrm{re}} = \{\frac{1}{2}\,(\alpha + (2n-1)\delta) \mid \alpha \in \mathring{\Delta}_l,\ n \in \mathbb{Z}\}$,

(b) $\Delta_m^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}_s,\ n \in \mathbb{Z}\}$,

(c) $\Delta_l^{\mathrm{re}} = \{\alpha + 2n\delta \mid \alpha \in \mathring{\Delta}_l,\ n \in \mathbb{Z}\}$.

PROPOSITION 2.2. [2] *If A is of type $A_{2l}^{(2)}$, then $\theta \in (\mathring{\Delta}_+)_l$ and θ is the unique root in $\mathring{\Delta}$ of maximal height ($= h - a_0$).*

Let $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_t} \in \Pi$ (not necessarily distinct) and denote by r_i the simple reflection of W. When $w \in W$ is written as $w = r_{i_1} \cdots r_{i_t}$ ($\alpha_{i_j} \in \Pi$,

t minimal), we call the expression reduced. We call t the length of w and denote by $l(w)$.

PROPOSITION 2.3. [4] *Let $w = r_{i_1} \cdots r_{i_t} \in W$ be a reduced expression of w. Then*

$$\Delta^+(w) = \{\beta_1, \ldots, \beta_t\},$$

where $\beta_p = r_{i_1} \cdots r_{i_{p-1}}(\alpha_{i_p})\,(1 \le p \le t)$ and the β_p are all distinct. In particular, $l(w) = |\Delta^+(w)|$.

LEMMA 2.4. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_{2l}^{(2)}$ and let r_i be the simple reflection. Then we have the following relations ;*

(a) $$r_{i-1}r_i\alpha_{i-1} = \alpha_i (1 < i < l),\ r_0r_1\alpha_0 = \alpha_0 + \alpha_1,$$
$$r_{l-1}r_l\alpha_{l-1} = \alpha_{l-1} + \alpha_l,$$

(b) $$r_ir_{i-1}\alpha_i = \alpha_{i-1}(1 < i < l),$$
$$r_lr_{l-1}\alpha_l = 2\alpha_{l-1} + \alpha_l,$$
$$r_1r_0\alpha_1 = 2\alpha_0 + \alpha_1,$$

(c) $$r_0r_1r_0\alpha_1 = \alpha_1,$$
$$r_{l-1}r_lr_{l-1}\alpha_l = \alpha_l,$$
$$r_{i-1}r_ir_{i-1}\alpha_i = -\alpha_{i-1}\ (1 < i < l).$$

PROOF. Let $A = (a_{ij})_{i,j=0}^{l}$ be a generalized Cartan matrix of $\mathfrak{g} = \mathfrak{g}(A)$. Since $a_{01} = -2$, $a_{10} = -1$, $a_{(l-1)l} = -2$, $a_{l(l-1)} = -1$, $a_{i(i+1)} = a_{(i-1)i} = -1$ for $1 \le i \le l-2$, and $a_{ij} = 0$ for $|i-j| > 1$, we have (a), (b), (c) immediately. □

LEMMA 2.5. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_{2l}^{(2)}$ and let $\theta = 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l$ be the highest long root of $A_{2l}^{(2)}$. Then the following three statements holds:*

(a) *There exists $w_0 \in \mathring{W}$ such that $\Delta^+(w_0) = \mathring{\Delta}_+$.*
(b) *There exists $w_1 \in W$ such that $w_1(\alpha_0) = \frac{1}{2}(\alpha_l + \delta)$.*
(c) *There exists $w_2 \in W$ such that $w_2(\alpha_0) = \frac{1}{2}(\theta + \delta)$.*

PROOF. (a) Let $w_0 = (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)\ (r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_{l-1} r_l r_{l-1}) r_l$. Using the relations in Proposition 2.6, we have ;

$$\begin{aligned}\Delta^+(w_0) = \{&\alpha_1,\ \alpha_1+\alpha_2,\ \dots, \alpha_1+\alpha_2+\dots+\alpha_{l-1},\ 2\alpha_1+\dots+2\alpha_{l-2}\\ &+2\alpha_{l-1}+\alpha_l, \alpha_1+\dots+\alpha_{l-2}+\alpha_{l-1}+\alpha_l,\ \alpha_1+\dots+\alpha_{l-2}\\ &+2\alpha_{l-1}+\alpha_l, \dots,\ \alpha_1+2\alpha_2+\dots+2\alpha_{l-1}+\alpha_l, \alpha_2,\ \alpha_2+\alpha_3,\\ &\dots,\ \alpha_2+\alpha_3+\dots+\alpha_{l-1},\ 2\alpha_2+\dots+2\alpha_{l-2}+2\alpha_{l-1}+\alpha_l,\\ &\alpha_2+\dots+\alpha_{l-2}+\alpha_{l-1}+\alpha_l,\ \alpha_2+\dots+\alpha_{l-2}+2\alpha_{l-1}+\alpha_l,\\ &\alpha_2+\dots+2\alpha_{l-2}+2\alpha_{l-1}+\alpha_l, \dots,\ \alpha_2+2\alpha_3+\dots+2\alpha_{l-1}\\ &+\alpha_l, \dots,\ \alpha_{l-2},\ \alpha_{l-2}+\alpha_{l-1},\ 2\alpha_{l-2}+2\alpha_{l-1}+\alpha_l,\ \alpha_{l-2}+\alpha_{l-1}\\ &+\alpha_l, \alpha_{l-1}, 2\alpha_{l-1}+\alpha_l,\ \alpha_{l-1}+\alpha_l,\ \alpha_l.\} = \mathring{\Delta}_+.\end{aligned}$$

(b) Let $w_1 = (r_0 r_1)(r_0 r_2 r_1) \dots (r_0 r_{l-2} \dots r_2 r_1)(r_0 r_{l-1} r_{l-2} \dots r_2 r_1)$. Then

$$\begin{aligned}w_1(\alpha_0) &= (r_0 r_1)(r_0 r_2 r_1) \dots (r_0 r_{l-2} \dots r_2 r_1)(r_0 r_{l-1} r_{l-2} \dots r_2 r_1)(\alpha_0)\\ &= (r_0 r_1)(r_0 r_2 r_1) \dots (r_0 r_{l-2} \dots r_2 r_1) r_0 (\alpha_0+\alpha_1+\dots+\alpha_{l-1})\\ &= (r_0 r_1)(r_0 r_2 r_1) \dots (r_0 r_{l-2} \dots r_2 r_1) r_0 \frac{1}{2}(-\alpha_l+\delta)\\ &= \frac{1}{2}(-\alpha_l+\delta).\end{aligned}$$

(c) Let $w_2 = (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-1} (r_l r_{l-1} r_{l-2} \dots r_2 r_1)$. Using the relations in Proposition 2.5,

$$\begin{aligned}w_2(\alpha_0) &= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-1} (r_l r_{l-1} r_{l-2} \dots r_2 r_1)(\alpha_0)\\ &= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-1} (\alpha_0+\alpha_1+\dots+\alpha_{l-2}+\alpha_{l-1}+\alpha_l)\\ &= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-1} (\frac{1}{2}(\alpha_l+\delta)\\ &= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-2} (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)(\frac{1}{2}(\alpha_l+\delta))\\ &= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-2} (r_l r_{l-1} (\frac{1}{2}(\alpha_l+\delta)\\ &= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-2} (\frac{1}{2}(2\alpha_{l-1}+\alpha_l+\delta))\\ &\dots\dots\dots\dots\dots\dots\end{aligned}$$

$$= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)^{l-3}(\frac{1}{2}(2\alpha_{l-2} + 2\alpha_{l-1} + \alpha_l + \delta))$$

$$\dots\dots\dots\dots\dots\dots$$

$$= (r_l r_{l-1} r_{l-2} \dots r_2 r_1 r_0)(\frac{1}{2}(2\alpha_2 + \dots 2\alpha_{l-2} + 2\alpha_{l-1} + \alpha_l + \delta))$$

$$= (r_l r_{l-1} r_{l-2} \dots r_2)(\frac{1}{2}(2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + 2\alpha_{l-1} + \alpha_l + \delta))$$

$$= \frac{1}{2}(2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + 2\alpha_{l-1} + \alpha_l + \delta)$$

$$= \frac{1}{2}(\theta + \delta). \qquad \square$$

PROPOSITION 2.6. [4] *Let* $w = r_{i_1} \cdots r_{i_t} \in W$ *be a reduced expression of* w *and let* α_i *be simple root. Then the following three statements hold:*

(a) $l(wr_i) < l(w)$ *if and only if* $w(\alpha_i) < 0$.

(b) $w(\alpha_{i_t}) < 0$.

(c) *If* $l(wr_i) < l(w)$, *then there exists an* $s(1 \le s \le t)$ *such that* $r_{i_s} r_{i_{s+1}} \dots r_{i_t} = r_{i_{s+1}} \dots r_{i_t} r_i$.

THEOREM 2.7. *Let* $\mathfrak{g} = \mathfrak{g}(A)$ *be the affine Lie algebra of type* $A_{2l}^{(2)}$ *and* w_0 *be the element of* $\mathring{W}$ *as in Theorem 2.6 (a). Then* $l(w_0 r_i) < l(w_0)$ *for all* $i = 1, \dots, l$.

PROOF. We need to show that $w_0(\alpha_i) < 0$ for $i = 1, \dots, l$.

$$w_0(\alpha_1)$$
$$= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_{l-1} r_l r_{l-1}) r_l(\alpha_1)$$
$$= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2)(\alpha_1)$$
$$= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1})(\alpha_1 + \alpha_2 + \dots \alpha_{l-1} + \alpha_l)$$
$$= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l)$$
$$= (r_1 \dots r_{l-1} r_l)(\alpha_1 + \alpha_2 + \cdots + \alpha_{l-1} + \alpha_l)$$
$$= r_1(\alpha_1)$$
$$= -\alpha_1.$$

$$w_0(\alpha_l)$$
$$= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots (r_{l-1} r_l r_{l-1}) r_l(\alpha_l)$$
$$= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_{l-1} r_l r_{l-1})(-\alpha_l)$$
$$= (r_{l-1} r_l r_{l-1})^{l-1}(-\alpha_l)$$
$$= -\alpha_l$$

For i with $1 < i < l$,

$$
\begin{aligned}
&w_0(\alpha_i)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_{l-1} r_l r_{l-1}) r_l(\alpha_i)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_{i+1} \dots r_l \dots r_{i+1})(\alpha_i)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_{i+1} \dots r_{l-1})\\
&\quad (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1} + \alpha_l)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_i \dots r_{l-1} r_l r_{l-1} \dots r_i)\\
&\quad (\alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_{l-1} + \alpha_l)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2) \dots (r_i \dots r_l)\\
&\quad (\alpha_i + \alpha_{i+1} + \cdots + \alpha_{l-1} + \alpha_l)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1) \dots (r_{i-1} \dots r_l r_{l-1} \dots r_{i-1}) r_i(\alpha_i)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1) \dots (r_{i-1} \dots r_l r_{l-1} \dots r_{i-1})(-\alpha_i)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1) \dots (r_{i-1} \dots r_l r_{l-1} \dots r_{i+1})(-\alpha_{i-1})\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1) \dots (r_{i-2} \dots r_l r_{l-1} \dots r_{i-2})(r_{i-1} r_i)(-\alpha_{i-1})\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1) \dots (r_{i-2} \dots r_l r_{l-1} \dots r_{i-2})(-\alpha_i)\\
&= -\alpha_i.
\end{aligned}
$$

$$
\begin{aligned}
&w_0(\alpha_l)\\
&= (r_1 \dots r_{l-1} r_l r_{l-1} \dots r_1)(r_2 \dots r_{l-1} r_l r_{l-1} \dots r_2 \dots (r_{l-1} r_l r_{l-1}) r_l(\alpha_l)\\
&= (r_{l-1} r_l r_{l-1})^{l-1}(-\alpha_l).\\
&= -\alpha_l.
\end{aligned}
$$

Thus by Proposition 2.6(a), $l(w_0 r_i) < l(w_0)$ for $i = 1, \dots, l$. □

THEOREM 2.8. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_{2l}^{(2)}$. Then there exists $w \in W$ such that $\mathring{\Delta}_+ \cup \{\frac{1}{2}(\alpha_l + \delta),\ \frac{1}{2}(\alpha_l + \delta)\} \subset \Delta^+(w)$.*

PROOF. Let $w = w_0 w_1 r_0 w_2 r_0$ where w_0, w_1, w_2 are elements in W which are constructed in Theorem 2.6. We know $\mathring{\Delta}_+ = \Delta^+(w_0) \subset \Delta^+(w)$.

$$\begin{aligned}
&w_0w_1(\alpha_0)\\
&= w_0(\frac{1}{2}(-\alpha_l+\delta))\\
&= (r_1\dots r_{l-1}r_lr_{l-1}\dots r_1)(r_2\dots r_{l-1}r_lr_{l-1}\dots r_2)\dots(r_{l-1}r_lr_{l-1})\\
&\quad r_l(\frac{1}{2}(-\alpha_l+\delta))\\
&= (r_1\dots r_{l-1}r_lr_{l-1}\dots r_1)(r_2\dots r_{l-1}r_lr_{l-1}\dots r_2)\dots(r_{l-1}r_lr_{l-1})\\
&\quad (\frac{1}{2}(\alpha_l+\delta))\\
&= (r_{l-1}r_lr_{l-1})^{(l-1)}\frac{1}{2}(\alpha_l+\delta))\\
&= \frac{1}{2}(\alpha_l+\delta) \ \in \Delta^+(w).
\end{aligned}$$

$$\begin{aligned}
&w_0w_1r_0w_2\alpha_0\\
&= w_0w_1r_0(\frac{1}{2}(\theta+\delta))\\
&= w_0w_1r_0(\frac{1}{2}(2\alpha_1+2\alpha_2+\dots+2\alpha_{l-1}+\alpha_l+\delta))\\
&= w_0w_1(\frac{1}{2}(4\alpha_0+2\alpha_1+2\alpha_2+\dots+2\alpha_{l-1}+\alpha_l+\delta))\\
&= w_0w_1(\frac{1}{2}(-2\alpha_1-2\alpha_2-\dots-2\alpha_{l-1}-\alpha_l+3\delta))\\
&= w_0(r_0r_1)(r_0r_2r_1)\dots(r_0r_{l-2}\dots r_2r_1)(r_0r_{l-1}r_{l-2}\dots r_2r_1)\\
&\quad (\frac{1}{2}(-2\alpha_1-2\alpha_2-\dots-2\alpha_{l-1}-\alpha_l+3\delta))\\
&= w_0(r_0r_1)(r_0r_2r_1)\dots(r_0r_{l-2}\dots r_2r_1)(\frac{1}{2}(-\alpha_l+3\delta))\\
&= w_0(\frac{1}{2}(-\alpha_l+3\delta)\\
&= (r_1\dots r_{l-1}r_lr_{l-1}\dots r_1)(r_2\dots r_{l-1}r_lr_{l-1}\dots r_2)\dots(r_{l-1}r_lr_{l-1})\\
&\quad r_l(\frac{1}{2}(-\alpha_l+3\delta))\\
&= (r_1\dots r_{l-1}r_lr_{l-1}\dots r_1)(r_2\dots r_{l-1}r_lr_{l-1}\dots r_2)\dots(r_{l-1}r_lr_{l-1})\\
&\quad (\frac{1}{2}(\alpha_l+3\delta))
\end{aligned}$$

$$= (r_{l-1}r_lr_{l-1})^{(l-1)}(\frac{1}{2}(\alpha_l + 3\delta))$$
$$= \frac{1}{2}(\alpha_l + 3\delta)$$
$$\subset \Delta^+(w).$$
□

THEOREM 2.9. *Let $\mathfrak{g} = \mathfrak{g}(A)$ be the affine Lie algebra of type $A_{2l}^{(2)}$. Then for some $w \in W$ we can construct a sequence $\{\gamma_n\} \in \Delta^+(w)$ with $h - 1(= 2l)$ terms such that each partial sum is a root.*

PROOF. We can construct a sequence $\{\gamma_n\} \in \Delta^+(w)$ as follows;

$$\gamma_1 = \frac{1}{2}(\alpha_1 + \delta),\ \gamma_2 = \frac{1}{2}(\alpha_1 + 3\delta),\ \gamma_3 = \gamma_4 = \alpha_{l-1},\ \ldots, \gamma_{h-2} = \gamma_{h-1} = \alpha_1.$$

Then $\sum_{i=1}^{s} \gamma_i$ is a root for all s with $1 \le s \le h-1$. □

EXAMPLE. In $A_4^{(2)}$, let $w = r_1r_2r_1r_2r_0r_1r_0r_2r_1r_0r_2r_1r_0$. Then we have

$$\Delta^+(w) = \Big\{\alpha_1,\ 2\alpha_1 + \alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_2,\ \frac{1}{2}(2\alpha_1 + \alpha_2 + \delta),\ \alpha_1 + \alpha_2 + \delta,$$
$$\frac{1}{2}(\alpha_2 + \delta),\ 2\alpha_1 + \alpha_2 + 2\delta,\ \alpha_1 + \alpha_2 + 2\delta,\ \frac{1}{2}(2\alpha_1 + \alpha_2 + 3\delta),$$
$$\alpha_2 + 2\delta, \alpha_1 + \alpha_2 + 3\delta,\ \frac{1}{2}(\alpha_2 + 3\delta)\Big\}.$$

If we choose

$$\gamma_1 = \frac{1}{2}(\alpha_2 + \delta),$$
$$\gamma_2 = \frac{1}{2}(\alpha_2 + 3\delta),$$
$$\gamma_3 = \alpha_1, \quad \gamma_4 = \alpha_1,$$

then

$$\gamma_1 = \frac{1}{2}(\alpha_2 + \delta),$$
$$\gamma_1 + \gamma_2 = \alpha_2 + 2\delta,$$
$$\gamma_1 + \gamma_2 + \gamma_3 = \alpha_1 + \alpha_2 + 2\delta,$$
$$\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 2\alpha_1 + \alpha_2 + 2\delta$$

are roots.

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