Comm. Korean Math. Soc. **16** (2001), No. 1, pp. 103–111

ON THE STABILITY OF N-DIMENSIONAL QUADRATIC FUNCTIONAL EQUATION

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ABSTRACT. In this paper, we investigate a generalization of the stability of a new quadratic functional equation $f(\sum_{i=1}^{n} x_i) + \sum_{1\leq i<j\leq n} f(x_i - x_j) = n\sum_{i=1}^{n} f(x_i)$ $(n \geq 2)$ in the spirits of Hyers, Ulam, Rassias and Gavruta.

1. Introduction

The problem of the stability of functional equations has originally been stated by S. M. Ulam [12]. In paper [5], D. H. Hyers has proved the stability of the linear functional equation for the case when G_1 and G_2 are Banach spaces, and the result of Hyers has been further generalized by Th. M. Rassias (see [10]). Since then, the stability problems of functional equations have been extensively investigated by a number of mathematicians (ref. [4], [6], [7], and [8]).

The quadratic function $f(x) = cx^2$ $(x \in \mathbb{R})$ satisfies the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.1}$$

Hence, the above equation is called the quadratic functional equation, and every solution of the quadratic equation (1.1) is called a quadratic function.

A Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved by F. Skof for functions $f : E_1 \to E_2$ where E_1 is a normed space and E_2 a Banach space (see [11]). P. W. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an abelian group. In the paper [3], S. Czerwik proved the Hyers-Ulam-Rassias stability of the quadratic functional equations, and this result was generalized by J. M. Rassias (see [9]).

Received May 2, 2000. Revised October 6, 2000.
2000 Mathematics Subject Classification: 39B52, 39B72.
Key words and phrases: stability, quadratic equation.

Consider the following functional equations:

$$(1.2)\quad f(x+y+z)+f(x-y)+f(y-z)+f(z-x) = 3f(x)+3f(y)+3f(z).$$

Recently, the author investigated in his paper [1] the Hyers-Ulam- Rassias stability of the equation (1.2) and the Hyers-Ulam stability of the equation (1.2) on restricted (unbounded) domains. In this paper, we prove the stability of the quadratic functional equations:

$$(1.3)\qquad f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) = n\sum_{i=1}^{n} f(x_i) \quad (n \ge 2).$$

2. Main results

Throughout this section, let $(E_1, \|\cdot\|)$ be a real normed space and $(E_2, \|\cdot\|)$ a Banach space. By $\mathbb{N}$ and $\mathbb{R}$ we denote the set of positive integers and of real numbers, respectively. We denote by $\varphi : E_1^n \to [0, \infty)$ a function such that either

$$(2.1)\qquad \psi_k(x_1, x_2, \cdots, x_n) := \sum_{i=0}^{\infty} k^{-2i}\varphi(k^i x_1, k^i x_2, \cdots, k^i x_n) < \infty$$

for all $x_1, x_2, \cdots, x_n \in E_1$, or

$$(2.2)\quad \tilde{\psi}_k(x_1, x_2, \cdots, x_n) := \sum_{i=0}^{\infty} k^{2(i+1)}\varphi\left(\frac{x_1}{k^{i+1}}, \frac{x_2}{k^{i+1}}, \cdots, \frac{x_n}{k^{i+1}}\right) < \infty$$

for all $x_1, x_2, \cdots, x_n \in E_1$.

For convenience, we use the following abbreviations:

$$Df(x_1, x_2, \cdots, x_n) = f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) - n\sum_{i=1}^{n} f(x_i).$$

We assume that the function $f : E_1 \to E_2$ satisfies the inequality

$$(2.3)\qquad \|Df(x_1, x_2, \cdots, x_n)\| \le \varphi(x_1, x_2, \cdots, x_n)$$

for all $x_1, x_2, \cdots, x_n \in E_1$. We define a real sequence (b_i) by

$$b_1 = 1,\ b_2 = 2 \quad\text{and}\quad b_i = 2b_{i-1} + b_{i-2}\ (i \ge 3).$$

First, we will introduce two lemmas before we prove the generalized Hyers -Ulam-Rassias stability of EQ. (1.3).

LEMMA 2.1. *Under the above assumptions, the following inequality*

$$\|f(kx) - k^2 f(x) + (k^2 - 1)f(0)\| \le \sum_{i=1}^{k-1} b_i \varphi((k-i)x, x, 0, \cdots, 0) \tag{2.4}$$

holds for any $x \in E_1$ and all integers $k \ge 2$.

PROOF. Setting $x_1 = x_2 = x$ and $x_3 = x_4 = \cdots = x_n = 0$ in (2.3) gives

$$\|f(2x) - 4f(x) + 3f(0)\| \le \varphi(x, x, 0, \cdots, 0)$$

for all $x \in E_1$. Now we make the induction hypothesis that (2.4) is true for some $k \ge 2$ and all $x \in E_1$. Then we have

$$\begin{aligned}
&\|f((k+1)x) - (k+1)^2 f(x) + ((k+1)^2 - 1)f(0)\| \\
&\le \|f((k+1)x) + f((k-1)x) - 2f(kx) - 2f(x) + 2f(0)\| \\
&\quad + \|2f(kx) - 2k^2 f(x) + 2(k^2 - 1)f(0)\| \\
&\quad + \|(k-1)^2 f(x) - f((k-1)x) - ((k-1)^2 - 1)f(0)\| \\
&\le \varphi(kx, x, 0, \cdots, 0) + \sum_{i=1}^{k-1} 2b_i \varphi((k-i)x, x, 0, \cdots, 0) \\
&\quad + \sum_{i=1}^{k-2} b_i \varphi((k-i-1)x, x, 0, \cdots, 0) \\
&\le \varphi(kx, x, 0, \cdots, 0) + 2b_1 \varphi((k-1)x, x, 0, \cdots, 0) \\
&\quad + \sum_{i=3}^{k} 2b_{i-1} \varphi((k-i+1)x, x, 0, \cdots, 0) \\
&\quad + \sum_{i=3}^{k} b_{i-2} \varphi((k-i+1)x, x, 0, \cdots, 0) \\
&\le \sum_{i=1}^{k} b_i \varphi((k-i+1)x, x, 0, \cdots, 0),
\end{aligned}$$

which completes the induction proof.

LEMMA 2.2. *Under the given assumptions, the following inequality*

$$\begin{aligned}&\left\|f(k^m x)-k^{2m}f(x)+(k^{2m}-1)f(0)\right\|\\&\le k^{2(m-1)}\sum_{i=1}^{k-1}\sum_{j=0}^{m-1}b_i k^{-2j}\varphi((k-i)k^j x,k^j x,0,\cdots,0)\end{aligned}$$

is true for any $x\in E_1$ and all $m,k\in\mathbb{N}$.

PROOF. By induction on m, we will prove the assertion. In view of Lemma 2.1, the assertion is true for $m=1$. Now we assume that the assertion is true for some m. By using Lemma 2.1 again, we obtain

$$\begin{aligned}&\left\|f(k^{m+1}x)-k^{2(m+1)}f(x)+(k^{2(m+1)}-1)f(0)\right\|\\&\le\left\|f(k^{m+1}x)-k^2f(k^m x)+(k^2-1)f(0)\right\|\\&\qquad+k^2\left\|f(k^m x)-k^{2m}f(x)+(k^{2m}-1)f(0)\right\|\\&\le\sum_{i=1}^{k-1}b_i\varphi((k-i)k^m x,k^m x,0,\cdots,0)\\&\qquad+k^{2m}\sum_{i=1}^{k-1}\sum_{j=0}^{m-1}b_i k^{-2j}\varphi((k-i)k^j x,k^j x,0,\cdots,0)\\&=k^{2m}\sum_{i=1}^{k-1}\sum_{j=0}^{m}b_i k^{-2j}\varphi((k-i)k^j x,k^j x,0,\cdots,0)\end{aligned}$$

as desired. □

In the next theorem, we shall prove the stability of the quadratic equation (1.3) for the case of $\psi_k(x_1,x_2,\cdots,x_n)<\infty$ for all $x_1,x_2,\cdots,x_n\in E_1$. We can quite similarly prove the theorem for the other case. Hence, we omit the proof of the theorem for $\tilde{\psi}_k(x_1,x_2,\cdots,x_n)<\infty$ for all $x_1,x_2,\cdots,x_n\in E_1$.

THEOREM 2.3. *Assume that a function $f:E_1\to E_2$ satisfies the inequality (2.3) for all $x_1,x_2,\cdots,x_n\in E_1$. If $\tilde{\psi}(x_1,x_2,\cdots,x_n)<\infty$ for all $x_1,x_2,\cdots,x_n\in E_1$, then we further assume that $f(0)=0$. There exists a unique quadratic function $Q:E_1\to E_2$ satisfying*

$$\|Q(x)-f(x)+f(0)\|\ \le\ \frac{1}{k^2}\Psi_k(x)\tag{2.5}$$

for all $x \in E_1$ *(when* $\psi_k(x_1, x_2, \cdots, x_n) < \infty$ *for all* $x_1, x_2, \cdots, x_n \in E_1$*), or*

$$\|Q(x) - f(x)\| \le \frac{1}{k^2}\tilde{\Psi}_k(x)$$

for all $x \in E_1$ *(when* $\tilde{\psi}_k(x_1, x_2, \cdots, x_n) < \infty$ *for all* $x_1, x_2, \cdots, x_n \in E_1$*), where*

$$\Psi_k(x) = \sum_{i=1}^{k-1} b_i \psi_k((k-i)x, x, 0, \cdots, 0)$$

and

$$\tilde{\Psi}_k(x) = \sum_{i=1}^{k-1} b_i \tilde{\psi}_k((k-i)x, x, 0, \cdots, 0).$$

If, moreover, f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for every fixed* $x \in E_1$*, then the* Q *satisfies*

$$Q(tx) = t^2 Q(x) \tag{2.6}$$

for all $x \in E_1$ *and* $t \in \mathbb{R}$*.*

PROOF. It follows from Lemma 2.2 that

$$\begin{aligned} &\left\| \frac{1}{k^{2m}} f(k^m x) - f(x) + \left(1 - \frac{1}{k^{2m}}\right) f(0) \right\| \\ &\le \frac{1}{k^2} \sum_{i=1}^{k-1} \sum_{j=0}^{m-1} b_i k^{-2j} \varphi((k-i)k^j x, k^j x, 0, \cdots, 0). \end{aligned} \tag{2.7}$$

First, we show that $\{k^{-2m} f(k^m x)\}$ is a Cauchy sequence: Let l, m be integers with $m > l > 0$. Then, by (2.7) and (2.1), we see

$$\begin{aligned} &\left\| \frac{1}{k^{2m}} f(k^m x) - \frac{1}{k^{2l}} f(k^l x) \right\| \\ &\le \frac{1}{k^{2l}} \left\| \frac{1}{k^{2(m-l)}} f(k^{m-l} k^l x) - f(k^l x) + \left(1 - \frac{1}{k^{2(m-l)}}\right) f(0) \right\| \\ &\quad + \left(\frac{1}{k^{2l}} - \frac{1}{k^{2m}}\right) \|f(0)\| \end{aligned}$$

$$\leq \frac{1}{k^{2l}}\left[\frac{1}{k^2}\sum_{i=1}^{k-1}\sum_{j=0}^{m-l-1} b_i k^{-2j}\varphi((k-i)k^j k^l x, k^j k^l x, 0, \cdots, 0)\right] + \left(\frac{1}{k^{2l}} - \frac{1}{k^{2m}}\right)\|f(0)\|$$

$$\leq \frac{1}{k^2}\sum_{i=1}^{k-1}\sum_{j=l}^{m-1} k^{-2j} b_i(\varphi((k-i)k^j x, k^j x, 0, \cdots, 0) + \left(\frac{1}{k^{2l}} - \frac{1}{k^{2m}}\right)\|f(0)\| \to 0 \text{ as } l \to \infty.$$

Since E_2 is a Banach space, we may define a function $Q : E_1 \to E_2$ by

$$Q(x) = \lim_{m\to\infty} \frac{1}{k^{2m}} f(k^m x)$$

for any $x \in E_1$. By the definition of Q and (2.7) we can easily verify the validity of the inequality (2.5).

By replacing x_1, x_2, $\cdots$, and x_n in (2.3) by $k^m x_1$, $k^m x_2$ $\cdots$, and $k^m x_n$, respectively, and dividing the resulting inequality by k^{2m} and by using (2.1), we get

$$\left\|\frac{1}{k^{2m}} f\left(k^m \sum_{i=1}^{n} x_i\right) + \frac{1}{k^{2m}} \sum_{1\leq i<j\leq n} f(k^m(x_i - x_j)) - \frac{n}{k^{2m}}\sum_{i=1}^{n} f(k^m x_i)\right\| \leq \frac{1}{k^{2m}}\varphi(k^m x_1, k^m x_2, \cdots, k^m x_n) \to 0 \text{ as } m \to \infty,$$

which implies that Q is a quadratic function.

Now, let $Q' : E_1 \to E_2$ be another quadratic function which satisfies the inequality (2.5). Since Q and Q' are quadratic functions, we can easily show that

$$Q(k^m x) = k^{2m} Q(x) \text{ and } Q'(k^m x) = k^{2m} Q'(x) \tag{2.8}$$

for any $m \in \mathbb{N}$. Thus, it follows from (2.8), (2.5) and (2.1) that

$$\begin{aligned}
&\|Q(x) - Q'(x)\| \\
&= \frac{1}{k^{2m}}\|Q(k^m x) - Q'(k^m x)\| \\
&\leq \frac{1}{k^{2m}}\left(\|Q(k^m x) - f(k^m x) + f(0)\| + \left\|f(k^m x) - Q'(k^m x) - f(0)\right\|\right) \\
&\leq \frac{2}{k^{2m}}\frac{1}{k^2}\Psi_k(k^m x) \to 0 \text{ as } m \to \infty,
\end{aligned}$$

which implies that $Q(x) = Q^{'}(x)$ for all $x \in E_1$.

Finally, it can be proved that (2.6) is true if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$ (cf. [3]). □

In the following Corollary, the Hyers-Ulam-Rassias stability of EQ. (1.3) is proved.

COROLLARY 2.4. *If a function $f : E_1 \to E_2$ satisfies the functional inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\| \leq \varepsilon \tag{2.9}$$

for all $x_1, x_2, \cdots, x_n \in E_1$, then there exists exactly one quadratic function $Q : E_1 \to E_2$ such that

$$\|Q(x) - f(x)\| \leq \left(\frac{1}{3} + \frac{2}{(n+2)(n-1)}\right)\varepsilon \tag{2.10}$$

for all $x \in E_1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the Q satisfies (2.6) for all $x \in E_1$ and $t \in \mathbb{R}$.

PROOF. If we put $\varphi(x_1, x_2, \cdots, x_n) = \varepsilon$ then φ satisfies the condition (2.1) for $k = 2$. Hence, it follows from Theorem 2.3 that there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\begin{aligned}
\|Q(x) - f(x) + f(0)\| &\leq \frac{1}{2^2}\psi(x, x, 0, \cdots, 0) \\
&\leq \frac{1}{2^2}\sum_{i=0}^{\infty} 2^{-2i}\varphi(2^i x, 2^i x, 0, \cdots, 0) \\
&= \frac{1}{2^2}\sum_{i=0}^{\infty}\frac{\varepsilon}{2^{2i}} \\
&= \frac{\frac{1}{4}\varepsilon}{1 - \frac{1}{4}} = \frac{1}{3}\varepsilon.
\end{aligned}$$

Since we get $\|f(0)\| \leq \frac{2\varepsilon}{n^2+n-2}$ by putting $x_1 = x_2 = \cdots = x_n = 0$ in (2.9), we see that the inequality (2.10) is true. The remaining part of the theorem can be easily proved by the same way as in the proof of Theorem 1 in the paper [2]. Hence, we here omit the proof. □

Similarly, as in the proof of Corollary 2.4, the Hyers-Ulam-Rassias stability for EQ. (1.3) is proved.

COROLLARY 2.5. *If a function $f : E_1 \to E_2$ satisfies the functional inequality*

$$\|Df(x_1, x_2, \cdots, x_n)\| \le \varepsilon \left(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p\right) \tag{2.11}$$

for some $0 < p < 2$ and for all $x_1, x_2, \cdots, x_n \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\|Q(x) - f(x)\| \le \frac{2\varepsilon}{4 - 2^p} \|x\|^p \tag{2.12}$$

for all $x \in E_1$. Moreover, if f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then the Q satisfies (2.6) for all $x \in E_1$ and $t \in \mathbb{R}$.

PROOF. If we put $\varphi(x_1, x_2, \cdots, x_n) = \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p)$ then φ satisfies the condition (2.1) for $k = 2$. Hence, it follows from Theorem 2.3 that there exists a unique quadratic function $Q : E_1 \to E_2$ such that

$$\begin{aligned} \|Q(x) - f(x) + f(0)\| &\le \frac{1}{2^2}\psi(x, x, 0, \cdots, 0) \\ &\le \frac{1}{2^2}\sum_{i=0}^{\infty} 2^{-2i}\varphi(2^i x, 2^i x, 0, \cdots, 0) \\ &= \frac{1}{2^2}\sum_{i=0}^{\infty} 2^{-2i}\varepsilon\left(\|2^i x\|^p + \|2^i x\|^p\right) \\ &= \frac{2}{2^2}\sum_{i=0}^{\infty} 2^{(p-2)i}\varepsilon\|x\|^p \\ &= \frac{\frac{1}{2}\varepsilon}{1 - 2^{p-2}}\|x\|^p = \frac{2\varepsilon}{4 - 2^p}\|x\|^p. \end{aligned}$$

Since we get $\|f(0)\| = 0$ by putting $x_1 = x_2 = \cdots = x_n = 0$ in (2.11), we see that the inequality (2.12) is true. The remaining part of the theorem can be easily proved by the same way as in the proof of Theorem 1 in the paper [3]. Hence, we here omit the proof. □

COROLLARY 2.6. *If a function $f : E_1 \to E_2$ satisfies the inequalities (2.11) for some $p > 2$ and for all $x_1, x_2, \cdots, x_n \in E_1$, then there exists a unique quadratic function $Q : E_1 \to E_2$ such that*

$$\|Q(x) - f(x)\| \le \frac{2\varepsilon}{2^p - 4}\|x\|^p$$

for all $x \in E_1$. *If, in addition,* f *is measurable or* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in E_1$, *then the quadratic function* Q *satisfies* (2.6) *for all* $x \in E_1$ *and* $t \in \mathbb{R}$.

PROOF. Since $\varphi(x_1, x_2, \cdots, x_n) = \varepsilon(\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p)$ satisfies the condition (2.2) with $k = 2$, it follows from Theorem 2.3 that the Corollary can be easily proved. □

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