Comm. Korean Math. Soc. **16** (2001), No. 1, pp. 113–118

ANOTHER METHOD OF CONSTRUCTION OF RIESZ BASES FOR MUTIRESOLUTION ANALYSES

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ABSTRACT. We discuss some conditions about the existence of the solution ϕ of the following integral equation

$$\phi(x) = \lambda \int h(2x-y)\phi(y)\,dy$$

and prove that the solution ϕ under certain conditions generates a multiresolution analysis.

1. Introduction

Multiresolution analysis starts the construction from an appropriate choice of the scaling function ϕ satisfying the following dilation equation

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x-k).$$

Then the closed subspace $V_j, j \in \mathbb{Z}$ spanned by the $\phi_{j,k}, k \in \mathbb{Z}$ with $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)$ satisfies the well-known properties [1, 3].

In oder to ensure the existence of the scaling function we must give some conditions on the filter function

$$H(\omega) = \frac{1}{2}\sum_{k\in\mathbb{Z}} h_k e^{-ik\omega}.$$

For example Daubeachies [2] studied polynomial filter and Zheng and Mingen [4] introduced some rational filters and constructed a large family of the wavelets.

Instead of the above dilation equation we starts from the following integral equation

$$\phi(x) = \lambda \int h(2x-y)\phi(y)\,dy. \tag{1}$$

Received July 20, 2000. Revised September 15, 2000.

2000 Mathematics Subject Classification: 42C30.

Key words and phrases: Riesz bases, multiresolution analysis.

In this paper we discuss some conditions about the existence of the solution ϕ of the above integral equation and prove that the solution ϕ under certain conditions generates a multiresolution analysis.

2. Main results

We denote the Fourier transform of ϕ by $\hat{\phi}$. By taking the Fourier transform on both sides of (2) we obtain

$$\hat{\phi}(\omega) = H_\lambda(\frac{\omega}{2})\hat{\phi}(\frac{\omega}{2}), \tag{2}$$

where $H_\lambda(\omega) = \frac{\lambda}{2}\hat{h}(\omega)$.

THEOREM 1. *If in (3) $h, \hat{h}'' \in L^2(\mathbb{R})$ and $\|\hat{h}\|_\infty \leq |\hat{h}(0)|(\neq 0)$, then $\prod_{k=1}^{\infty} H_\lambda(\frac{\omega}{2^k})$ converges to $\hat{\phi}$ uniformly on compact subsets of $\mathbb{R}$ and for $\lambda = \frac{2}{\hat{h}(0)}$, and its inverse Fourier transformation $\phi \in L^2(\mathbb{R})$ is a nontrivial solution of (1).*

PROOF. For any $a > 0$, we get the following Fourier series of $\hat{h}$ on $(-a, a)$

$$\hat{h}(\omega) = \sum_{k\in\mathbb{Z}} h_k e^{-i\omega\frac{k\pi}{a}}, \omega \in (-a, a).$$

Since

$$H_\lambda(0) = \frac{\lambda}{2}\sum_{k\in\mathbb{Z}} h_k, H_\lambda(0) = \frac{\lambda}{2}\hat{h}(0) = 1,$$

we have

$$H_\lambda(\omega) - 1 = \frac{\lambda}{2}\sum_{k\in\mathbb{Z}} h_k(e^{-i\omega\frac{k\pi}{a}} - 1).$$

Therefore

$$\begin{aligned}|H_\lambda(\omega) - 1| &\leq \frac{1}{\hat{h}(0)}\sum_{k\in\mathbb{Z}} |h_k| \quad |\sin(\tfrac{k\pi\omega}{2a})| \\ &\leq \frac{1}{\hat{h}(0)}\sum_{k\in\mathbb{Z}} |h_k| \quad |\tfrac{k\pi\omega}{2a}|^{\frac{1}{2}}.\end{aligned}$$

Since $h'' \in L^2(\mathbb{R})$, $|h_k| \leq O(\frac{1}{k^2})$ and

$$|H_\lambda(\omega) - 1| \leq C|\omega|^{\frac{1}{2}}.$$

Therefore

$$|H_\lambda(\frac{\omega}{2^k}) - 1| \leq C|\omega|^{\frac{1}{2}}\frac{1}{(\sqrt{2})^k}.$$

So $\prod_{k=1}^{\infty} H_\lambda(\frac{\omega}{2^k})$ converges to a continuous function $\hat{\phi}(\omega)$ uniformly on compact subsets of $\mathbb{R}$ and since $H_\lambda(0) = 1$, $\hat{\phi}(0) = 1$. On the other hand $\hat{h}(\omega) \leq |\hat{h}(0)|$ shows that

$$|H_\lambda(\omega)| = |\frac{\lambda}{2}||\hat{h}(\omega)| \leq 1.$$

Hence

$$|\hat{\phi}(\omega)| \leq |H_\lambda(\frac{\omega}{2})|$$

and $\hat{\phi} \in L^2(\mathbb{R})$ and its inverse Fourier transformation $\phi \in L^2(\mathbb{R})$ is a nontrivial solution of (1). □

In the following we show that a nontrivial solution ϕ of the integral equation (1) with $\phi \in L^1(\mathbb{R}), \hat{\phi}(0) \neq 0$ will constitute a multiresolution analysis under the following conditions on $\hat{h}$,

$$\text{Supp } \hat{h} = [-\pi, \pi], \ \hat{h} \neq 0 \text{ on } (-\pi, \pi).$$

Let us denote $V_j = \overline{span}\{\phi(2^j - k) | k \in \mathbb{Z}\}, j \in \mathbb{Z}$.

THEOREM 2 (Monotonicity).

$$V_j \subset V_{j+1}.$$

PROOF. From the Fourier series of $\hat{h}(\omega/2)$ on $(-2\pi, 2\pi)$,

$$\hat{h}(\omega/2) = \sum_{k \in \mathbb{Z}} h_k e^{ki\frac{\omega}{2}}, \omega \in (-2\pi, 2\pi).$$

From (2) we have

$$\hat{\phi}(\omega) = \frac{\lambda}{2}\hat{h}(\omega/2)\hat{\phi}(\frac{\omega}{2}) = \frac{\lambda}{2}\sum_{k \in \mathbb{Z}} h_k e^{ki\frac{\omega}{2}}\hat{\phi}(\frac{\omega}{2}).$$

Taking the Fourier transform we obtain

$$\phi(x) = \lambda \sum_{k \in \mathbb{Z}} h_k \phi(2x - k). \tag{3}$$

Therefore

$$\phi(2^j x - l) = \lambda \sum_{k \in \mathbb{Z}} h_k \phi(2^{(j+1)}x - 2l - k).$$

So

$$V_j \subset V_{j+1}.$$

□

THEOREM 3 (Completeness). *$\{\phi(2^j x - k) | j, k \in \mathbb{Z}\}$ is a complete system.*

PROOF. From (4) we have

$$\phi(2^j x - \frac{l}{2}) = \lambda \sum_{k\in\mathbb{Z}} h_k \phi(2^{(j+1)}x - l - k).$$

Hence to prove that $\{\phi(2^j x - k) | j, k \in \mathbb{Z}\}$ is a complete system, it is sufficient to prove that $\{\phi(2^j x - \frac{l}{2}) | j, l \in \mathbb{Z}\}$ is a complete system. Suppose that $f \in L^2(\mathbb{R})$ satisfies

$$\langle \phi(2^j \cdot - \frac{k}{2}), f(\cdot) \rangle = 0, \quad \text{for all} \quad j, k \in \mathbb{Z}.$$

Then

$$\begin{aligned}
&\langle \phi(2^j \cdot - \frac{k}{2}), f(\cdot) \rangle \\
&= \lambda \int_{\mathbb{R}} < h(2^{j+1} \cdot - k - y), f(\cdot) > \phi(y)\, dy \\
&= \lambda \int_{\mathbb{R}} < 2^{-(j+1)} e^{-i\omega 2^{-(j+1)} \cdot (k+y)} \hat{h}(2^{-(j+1)}\omega), \hat{f}(\omega) >_\omega \phi(y)\, dy \\
&= \lambda 2^{-(j+1)} \int_{\mathbb{R}} \hat{h}(2^{-(j+1)}\omega) \overline{\hat{f}}(\omega) e^{-i\omega 2^{-(j+1)} k} \, [\int_{\mathbb{R}} \phi(y) e^{-i\omega 2^{-(j+1)} y}\, dy] \, d\omega \\
&= \sqrt{2\pi} 2^{-(j+1)} \lambda \int_{\mathbb{R}} \hat{h}(2^{-(j+1)}\omega) \overline{\hat{f}}(\omega) \hat{\phi}(2^{-(j+1)}\omega) e^{-i\omega 2^{-(j+1)} k}\, d\omega \\
&= \sqrt{2\pi} 2^{-(j+1)} \lambda \int_{\mathbb{R}} \hat{h}(2^{-(j+1)}\omega) \overline{\hat{f}}(\omega) \hat{\phi}(2^{-(j+1)}\omega) e^{-i\omega 2^{-(j+1)} k}\, d\omega \\
&= \sqrt{2\pi} \lambda \int_{\mathbb{R}} \hat{h}(\omega) \overline{\hat{f}}(2^{(j+1)}\omega) \hat{\phi}(\omega) e^{-i\omega k}\, d\omega \\
&= \sqrt{2\pi} \lambda \int_{-\pi}^{\pi} \hat{h}(\omega) \overline{\hat{f}}(2^{(j+1)}\omega) \hat{\phi}(\omega) e^{-i\omega k}\, d\omega \\
&= 0.
\end{aligned}$$

Therefore all the Fourier coefficients of $\hat{h}(\cdot)\overline{\hat{f}}(2^{(j+1)}\cdot)\hat{\phi}(\cdot)$ is 0, so

$$\hat{h}(\omega)\overline{\hat{f}}(2^{(j+1)}\omega)\hat{\phi}(\omega) = 0, \omega \in (-\pi, \pi) \; a.e. \tag{4}$$

On the other hand we have

$$\hat{\phi}(\omega) \neq 0, \text{ for all } \omega \in [-\pi, \pi]. \tag{5}$$

Indeed suppose that $\hat{\phi}(\omega') = 0$ for some $\omega' \in [-\pi, \pi]$. Then by (3) $\hat{\phi}(\frac{\omega'}{2}) = 0$. Inductively for any natural number, n $\hat{\phi}(\frac{\omega'}{2^n}) = 0$. Since $\phi \in L^1(\mathbb{R})$, $\hat{\phi}$ is continuous and $\hat{\phi}(0) = 0$ which is a contradiction to the assumption $\hat{\phi}(0) \neq 0$. From (4) for all $j \in \mathbb{Z}$,

$$\hat{f}(2^{(j+1)}\omega) = 0, \; \omega \in (-\pi, \pi) \; a.e.$$

Hence $\hat{f} = 0$ *a.e.* and $f = 0$ *a.e.* □

THEOREM 4 (Approximation).

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\} \text{ and } \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

PROOF. Since $\{\phi(2^j x - k)|j, k \in \mathbb{Z}\}$ is a complete system, $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$. If $f \in \bigcap_{j\in\mathbb{Z}} V_j$, then

$$f(x) = \sum_{k\in\mathbb{Z}} \alpha_{jk}\phi(2^j x - k), \text{ for all } j \in \mathbb{Z}. \tag{6}$$

On the other hand by taking the Fourier transform on (6)

$$\hat{f}(\omega) = H_j(\omega)\hat{\phi}(2^{-j}\omega), \tag{7}$$

where

$$H_j(\omega) = 2^{-j}\sum_{k\in\mathbb{Z}} \alpha_{jk}e^{-i\omega 2^{-j}k}.$$

Since *Supp* $\hat{\phi} \subset [-2\pi, 2\pi]$ by (2), for $\omega \neq 0$ there exists $j \in \mathbb{Z}$ such that $\hat{\phi}(2^{-j}\omega) = 0$. By (7) we $\hat{f}(\omega) = 0$, for $\omega \neq 0$. Hence $f = 0$. □

THEOREM 5 (Riesz basis). *$\{\phi(x - k)|k \in \mathbb{Z}\}$ constitutes a Riesz basis for V_0.*

PROOF. For any $f \in V_0$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} \alpha_k\phi(x - k),$$

we take the Fourier transform and obtain

$$\hat{f}(\omega) = H(\omega)\hat{\phi}(\omega),$$

where $H(\omega) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-i\omega k}$. Therefore

$$\begin{aligned}
\|f\|^2 &= \int_{\mathbb{R}} |H(\omega)|^2|\hat{\phi}(\omega)|^2\, d\omega \\
&= \sum_{l\in\mathbb{Z}} \int_{(2l-1)\pi}^{(2l+1)\pi} |H(\omega)|^2|\hat{\phi}(\omega)|^2\, d\omega \\
&= \sum_{l\in\mathbb{Z}} \int_{-\pi}^{\pi} |H(\omega)|^2|\hat{\phi}(\omega + 2l\pi)|^2\, d\omega \\
&= \int_{-\pi}^{\pi} |H(\omega)|^2 G(\omega)\, d\omega,
\end{aligned}$$

where $G(\omega) = \sum_{l\in\mathbb{Z}} |\hat{\phi}(\omega + 2l\pi)|^2$. Since $Supp\ \hat{\phi} \subset [-2\pi, 2\pi]$ by (2) and $\hat{\phi}$ is continuous, for any $\omega \in \mathbb{R}$ $G(\omega)$ is computed at most two sums and

$$G(\omega) \leq 2M,$$

where M is an bound of $\hat{\phi}$. On the other hand by (4) and the continuity of $\hat{\phi}$,

$$G(\omega) \geq |\hat{\phi}(\omega)|^2 \geq m^2 > 0, \omega \in [-\pi, \pi],$$

where m is the minimum of $\hat{\phi}$ on $[-\pi, \pi]$. Moreover since G is a 2π-periodic function,

$$G(\omega) \geq m^2, \text{ for all } \omega \in \mathbb{R}.$$

From the above equality we have constants $A, B > 0$ such that

$$A \int_{-\pi}^{\pi} |H(\omega)|^2\, d\omega \leq \|f\|^2 \leq B \int_{-\pi}^{\pi} |H(\omega)|^2\, d\omega.$$

So the proof is completed. □

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