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CONFORMAL DEFORMATION ON A SEMI-RIEMANNIAN MANIFOLD (I)

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ABSTRACT. In this paper, we considered the uniqueness of positive time-solution to equation $\Box_g u(t,x) - c_n u(t,x) + c_n u(t,x)^{\frac{n+3}{n-1}} = 0$, where $c_n = \frac{n-1}{4n}$ and $\Box_g$ is the d'Alembertian for a Lorentzian warped manifold $M = [a,\infty) \times_f N$.

1. Introduction

In a recent study ([5, 6]), M. C. Leung has studied the problem of scalar curvature functions on Riemannian warped product manifolds and obtained partial results about the existence and nonexistence of Riemannian warped metric with some prescribed scalar curvature function. He has studied the uniqueness of positive solution to equation

$$\Delta_{g_0} u(x) + d_n u(x) = d_n u(x)^{\frac{n+2}{n-2}}, \tag{1.1}$$

where Δ_{g_0} is the Laplacian operator for an n-dimensional Riemannian manifold (N, g_0) and $d_n = \frac{n-2}{4(n-1)}$. Equation (1.1) is derived from the conformal deformation of Riemannian metric (cf. [1, 10, 14, 15]).

Similarly, let (N, g_0) be a compact Riemannian n-dimensional manifold with constant scalar curvature. We consider the $(n+1)$-dimensional Lorentzian warped manifold $M = [a,\infty) \times_f N$ with the metric $g = -dt^2 + f(t)^2 g_0$, where f is a positive function on $[a,\infty)$. Let $u(t,x)$ be a positive smooth function on M and let g have a constant scalar curvature equal to $+1$. If the conformal metric $g_c = u(t,x)^{\frac{4}{n-1}} g$ also has constant scalar curvature equal to $+1$, then $u(t,x)$ satisfies equation

$$\Box_g u(t,x) - c_n u(t,x) + c_n u(t,x)^{\frac{n+3}{n-1}} = 0, \tag{1.2}$$

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where $c_n = \frac{n-1}{4n}$ and $\Box_g$ is the d'Alembertian for a Lorentzian warped manifold $M = [a, \infty) \times_f N$.

In this paper, we study the uniqueness of positive solution to equation (1.2). In [5, 6], the author considered the scalar curvature of some Riemannian warped product and its conformal deformation of warped product metric. And also in [3], authors considered the existence of a nonconstant warping function on a Lorentzian warped product manifold such that the resulting warped product metric produces the constant scalar curvature when the fiber manifold has the constant scalar curvature. Indeed, in [3], authors proved that when the fiber manifold has the constant scalar curvature, then there is no obstruction of the existence of Lorentzian warped metric with constant scalar equal to +1. So we may assume that the Lorentzian warped product metric g has the constant scalar equal to +1.

2. Main results

In this section, we let (N, g_0) be a compact Riemannian n-dimensional manifold with $n \geq 3$ and without boundary. Then Theorem 5.4 in [2] implies the following proposition.

PROPOSITION 1. *Let $M = [a, \infty) \times_f N$ have a Lorentzian warped product metric $g = -dt^2 + f(t)^2 g_0$. Then the d'Alembertian $\Box_g$ is given by*

$$\Box_g = -\frac{\partial^2}{\partial t^2} - \frac{nf'(t)}{f(t)}\frac{\partial}{\partial t} + \frac{1}{f(t)^2}\Delta_x,$$

where Δ_x is the Laplacian on fiber manifold N.

By Proposition 1, equation (1.2) is changed into the following equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + \frac{nf'(t)}{f(t)}\frac{\partial u(t,x)}{\partial t} - \frac{1}{f(t)^2}\Delta_x u(t,x) + c_n u(t,x) - c_n u(t,x)^{\frac{n+3}{n-1}} = 0. \tag{2.1}$$

A positive solution to equation (1.2) *or* (2.1) *is called* nonspacelike(timelike or null) complete if the conformal metric $g_c = u^{\frac{4}{n-1}} g$ is a nonspacelike(timelike or null) complete Lorentzian metric on M. In this section we discuss whether the nonspacelike complete positive solution of equation (2.1) is unique. By Lemma 2 and Theorem 5 in [8] and Theorem 4.1 in [2], we have the following proposition.

PROPOSITION 2. *Let $M = [a, \infty) \times_f N$ have a Lorentzian warped product metric $g = -dt^2 + f(t)^2 g_0$. Then all future directed timelike(resp. null) geodesics are future complete if and only if for some $t_0 \in [a, \infty)$, $\int_{t_0}^{\infty} \frac{f(t)}{\sqrt{1+f(t)^2}}\, dt = \infty$ (resp. $\int_{t_0}^{\infty} f(t)\, dt = \infty$).*

If $u(t, x)$ is a positive function with only time-variable t, then equation (2.1) becomes

$$u''(t) + \frac{nf'(t)}{f(t)} u'(t) = c_n(u^{\frac{n+3}{n-1}}(t) - u(t)). \tag{2.2}$$

LEMMA 3. *Let $u(t)$ be a solution of equation (2.2) and $u(a) = 1$. We have two cases: that is, either*

i) *if there exists $t_1 > a$ such that $u(t_1) \geq 1$ and $u'(t_1) > 0$, then $u'(t) > 0$ for all $t > t_1$*
   or
ii) *if there is a point t_3 such that $u(t_3) < 1$ and $u'(t_3) < 0$, then $u'(t) < 0$ for all $t \geq t_3$.*

Proof. For case i), suppose not. Then there exists a point $t_2 > a$ such that $u(t_2) > 1$, $u'(t_2) = 0$ and $u''(t_2) \leq 0$, but equation (2.2) shows that this is not possible. For case ii), it is similar with the case i). □

The proof of following theorem is similar with that of Theorem 4.9 in [7].

THEOREM 4. *Let $u(t)$ be a positive solution of equation (2.2). Assume that there exist positive constants t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \leq C_0$ for all $t > t_0$. Then $u(t)$ is bounded from above.*

Proof. From equation (2.2) we have

$$\frac{(f^n u')'}{f^n} = c_n(u^{\frac{n+3}{n-1}} - u). \tag{2.3}$$

Let $\chi \in C_0^\infty([a,\infty))$ be a cut-off function. Multiplying both sides of equation (2.3) by $\chi^{n+1}u$ and then using integration by parts we obtain

$$(2.4)\quad c_n \int_a^\infty \chi^{n+1}u^2\,dt - \int_a^\infty (f^n u')(\frac{\chi^{n+1}u}{f^n})'\,dt = c_n \int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}\,dt.$$

We have

$$-(f^n u')(\frac{\chi^{n+1}u}{f^n})' = -(n+1)\chi^n u\chi' u' - \chi^{n+1}|u'|^2 + n\chi^{n+1}uu'\frac{f'}{f}.$$

Applying the Cauchy inequality we get

$$\begin{aligned} -(n+1)\chi^n u\chi' u' &= -2(\frac{n+1}{\sqrt{2}}\chi^{\frac{n+1}{2}-1}u\chi')(\frac{1}{\sqrt{2}}\chi^{\frac{n+1}{2}}u') \\ &\le \frac{(n+1)^2}{2}\chi^{n-1}u^2|\chi'|^2 + \frac{1}{2}\chi^{n+1}|u'|^2 \end{aligned}$$

and

$$\begin{aligned} n\chi^{n+1}uu'\frac{f'}{f} &= 2(\frac{n}{\sqrt{2}}\chi^{\frac{n+1}{2}}u\frac{f'}{f})(\frac{1}{\sqrt{2}}\chi^{\frac{n+1}{2}}u') \\ &\le \frac{n^2}{2}\chi^{n+1}(\frac{f'}{f})^2u^2 + \frac{1}{2}\chi^{n+1}|u'|^2. \end{aligned}$$

Together with (2.4) we obtain

$$\begin{aligned} &c_n \int_a^\infty \chi^{n+1}u^2dt + \frac{n^2}{2}\int_a^\infty (\frac{f'}{f})^2\chi^{n+1}u^2dt + \frac{(n+1)^2}{2}\int_a^\infty \chi^{n-1}u^2|\chi'|^2 \\ &\ge c_n \int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}\,dt. \end{aligned}$$

Applying Young's inequality and using the bound $|\frac{f'}{f}| \le C_0$, we have

$$(2.5)\qquad c_n \int_a^\infty \chi^{n+1}u^{\frac{2n+2}{n-1}}\,dt \le C' \int_a^\infty (|\chi'|^{n+1} + \chi^{n+1})\,dt,$$

where C' is a positive constant. Let $\chi \equiv 0$ on $[a,r] \cup [r+3,\infty)$ with $r > t_0$ and $\chi \equiv 1$ on $[r+1, r+2]$, $\chi \ge 0$ on $[a,\infty)$ and $|\chi'| \le \frac{1}{2}$. From equation (2.5) we have $\int_{r+1}^{r+2} u^{\frac{2n+2}{n-1}}dt \le C''$ for all $r > t_0$, where C'' is a constant independent on r. Therefore u is bounded from above. □

THEOREM 5. *Let (M,g) be a complete Lorentzian manifold with scalar curvature equal to 1. Let u be a positive smooth solution to equation (1.2) on $(M, g = -dt^2 + f^2(t)dx^2)$ such that the conformal metric $g_c = u^{\frac{4}{n-1}}g$ is a complete Lorentzian metric with scalar curvature equal to 1. Assume that $\lim_{t\to\infty} f(t) = \infty$ and there exist positive constants*

t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \leq C_0$ for all $t > t_0$. If $u(t,x)$ is a positive function with only time-variable t, then $u(t) \equiv 1$ on M.

Proof. If $u = u(t)$ is a solution of equation (2.2), then by Theorem 4 $u(t)$ is bounded from above. Suppose that there exists a point $t_1 \in [a, \infty)$ such that $u(t_1) > 1$. Then, by Omori-Yau maximum principle (c.f. [9]), there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} u(t_k) = \sup_{t\in[a,\infty)} u(t)$, $|u'(t_k)| \leq \frac{1}{k}$ and $u''(t_k) \leq \frac{1}{k}$. Since $\sup_{t\in[a,\infty)} > 1$, there exists a number $\epsilon > 0$ and K such that $c_n(u(t_k)^{\frac{n+3}{n-1}} - u(t_k)) > \epsilon$ for all $k > K$. This is a contradiction to equation (2.2), so $u(t) \leq 1$.

Suppose that there exists a point $t_2 \in [a, \infty)$ such that $u(t_2) < 1$. Assume that $u(t) \geq c$ for all $t > t_0$, where $c \in (0, 1)$ is a constant. Then we can find a sequence t_k' and a positive constant $\delta > 0$ such that $u(t_k') > 1 - \delta$ for all $k, \lim_{k\to\infty} u'(t_k) = 0$ and $u''(t_k) \geq 0$. But this contradicts equation (2.2). Therefore $\lim_{t\to\infty} u(t) = 0$ and we must have $u'(t_0) < 0$. The proof of Lemma 3 implies that we have $u'(t) < 0$ for all $t > t_0$.

There exist positive constants $t' > t_0$ and $C > 0$ such that for $t \geq t'$ we have

$$(f^n u')'(t) = c_n f^n(t)(u^{\frac{n+3}{n-1}}(t) - u(t)) \leq -Cf^n(t)u(t).$$

Integrating from t' to $t > t'$ we have

$$f^n(t)u'(t) \leq f^n(t')u'(t') - C\int_{t'}^t f^n(s)u(s)\,ds \leq -Cu(t)\int_{t'}^t f^n(s)\,ds,$$

as $u' \leq 0$. Therefore

$$\frac{u'(t)}{u(t)} \leq -C\frac{\int_{t'}^t f^n(s)ds}{f^n(t)}. \tag{2.6}$$

Using the bound $\frac{f'}{f} < C_0$ we have $(f^n)' \leq C_0 n f^n$. An integration gives

$$f^n(t) - f^n(t') \leq C_0 n \int_{t'}^t f^n(s)\,ds.$$

As $\lim_{t\to\infty} f(t) = \infty$, if t is large we have $\frac{1}{2}f^n(t) \leq C_0 \int_{t'}^t f^n(s)\,ds$, that is,

$$\frac{\int_{t'}^t f^n(s)ds}{f^n(t)} \geq c' \tag{2.7}$$

for all t large and for some positive constant c'. Equations (2.6) and (2.7) give $u(t) \leq \tilde{C}e^{-ct}$ for all t large enough, where $\tilde{C}$ is a positive constant. Thus the conformal metric $g_c = u^{\frac{4}{n-1}} g$ cannot be complete. It is a contradiction. Hence $u \equiv 1$ on $[a, \infty)$. □

We have the following result for non-time solutions.

THEOREM 6. *For an integer $n \geq 4$ let $g = -dt^2 + f^2(t)dx^2$ be a Lorentzian warped product metric on $M = [a, \infty) \times_f N$. Assume that $\lim_{t\to\infty} f(t) = \infty$ and there exist positive constants t_0 and C_0 such that $|\frac{f'(t)}{f(t)}| \leq C_0$ for all $t > t_0$. Let u be a positive smooth solution to equation (1.2) on (M, g). If there exist constants $\delta \in (0, 1)$ and $t_0 > 0$ such that $u(t, x) \leq 1 - \delta$ for $t \geq t_0$ and $x \in N$, then the conformal metric $g_c = u^{\frac{4}{n-1}} g$ is not complete.*

Proof. For $t \geq t_0$, we have $u(t, x) \leq 1 - \delta$ for some constant $\delta \in (0, 1)$. Therefore from equation (2.1) we get

$$c_n(u^{\frac{n+3}{n-1}} - u) \leq c_n[(1-\delta)^{\frac{4}{n-1}} - 1]u = -\epsilon u, \tag{2.8}$$

where $\epsilon = c_n[1 - (1-\delta)^{\frac{4}{n-1}}] > 0$.

For a fixed number $t \geq t_0$, we integrate equation (2.1) over N and use Green's formula and equation (2.8) to obtain

$$\frac{d^2}{dt^2}(\int_N u dx) + \frac{nf'(t)}{f(t)} \frac{d}{dt}(\int_N u dx) \leq -\epsilon \int_N u dx. \tag{2.9}$$

For $t \geq t_0$, let $U(t) = \int_N u(t, x) dx$. Then (2.9) can be written as $U''(t) + \frac{nf'(t)}{f(t)} U'(t) \leq -\epsilon U(t)$ for all $t \geq t_0$. Therefore $(f^n U')' \leq -\epsilon f^n U(t)$ for all $t \geq t_0$. Hence there exist positive constants C and c such that

$$U(t) = \int_N u dx \leq Ce^{-ct} \tag{2.10}$$

for all $t \geq t_0$. Assume that $n \geq 3$. We have

$$\int_N u^{\frac{2}{n-1}} dx \leq (\int_N u dx)^{\frac{2}{n-1}} \mathrm{vol}(N)^{\frac{n-3}{n-1}}, \tag{2.11}$$

where vol(N) is the volume of fiber manifold N in $M = [a, \infty) \times_f N$. Using (2.10) and (2.11) we have

$$\int_{t_0}^{\infty} \int_N u^{\frac{2}{n-1}} dx dt < \infty. \tag{2.12}$$

We claim that there exists $x_0 \in N$ such that $\int_{t_0}^{\infty} u^{\frac{2}{n-1}}(t, x_0)dt < \infty$. This means that the conformal metric $u^{\frac{4}{n-1}}(-dt^2 + f^2(t)dx^2)$ is not complete.

To prove the claim, by (2.12) and Fubini's theorem, there exists a positive integer C' such that for any positive integer k we have

$$\int_{t_0}^{k} \int_N u^{\frac{2}{n-1}}(t, x)\, dxdt = \int_N (\int_{t_0}^{k} u^{\frac{2}{n-1}}(t, x)dt)\, dx \leq C'.$$

For each integer $k > t_0$, there exists a point $x_k \in N$ such that

$$\int_{t_0}^{k} u^{\frac{2}{n-1}}(t, x_k)dt \leq \frac{C'}{\mathrm{vol}(N)} + 1. \tag{2.13}$$

A subsequence x_{k_i} converges to a point $x_0 \in N$. If $\int_{t_0}^{\infty} u^{\frac{2}{n-1}}(t, x_0)dt = \infty$, then there exists a positive integer k' such that $\int_{t_0}^{k'} u^{\frac{2}{n-1}}(t, x_0)dt > \frac{C'}{\mathrm{vol}(N)} + 2$.

As the function $\int_{t_0}^{k'} u^{\frac{2}{n-1}}(t, x)dt$ is continuous with respect to x, therefore in a neighborhood of x_0 we have

$$\int_{t_0}^{k'} u^{\frac{2}{n-1}}(t, x_0)dt > \frac{C'}{\mathrm{vol}(N)} + \frac{3}{2}. \tag{2.14}$$

As $\lim_{i\to\infty} x_{k_i} = x_0$ and for all i such that $k_i > k'$, x_{k_i} satisfies (2.14) and hence

$$\int_{t_0}^{k'} u^{\frac{2}{n-1}}(t, x_{k_i})dt > \frac{C'}{\mathrm{vol}(N)} + 1.$$

This contradicts with (2.13). The proof of the claim is completed. □

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