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A NOTE ON APPROXIMATION OF SOLUTIONS OF A K-POSITIVE DEFINITE OPERATOR EQUATIONS

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ABSTRACT. In this note we construct a sequence of Picard iterates suitable for the approximation of solutions of K-positive definite operator equations in arbitrary real Banach spaces. Explicit error estimate is obtained and convergence is shown to be as fast as a geometric progression.

1. Introduction

Let E be a real Banach space, E^* the dual space of E and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined for each $x \in E$ by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle .,. \rangle$ is the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single-valued. In the sequel we shall denote single-valued duality mapping by j.

In [1] Chidume and Aneke extended the notion of *K-positive definite* (Kpd) operators of Martyniuk [5] and Petryshyn ([6], [7]) from Hilbert spaces to arbitrary real Banach spaces. They called a linear unbounded operator A defined on a dense domain $D(A)$ in E a Kpd operator if there exist a continuously $D(A)$-invertible closed linear operator K with $D(A) \subseteq D(K)$ and a constant $c > 0$ such that for $j(Kx) \in J(Kx)$,

$$\langle Ax, j(Kx) \rangle \geq c\|Kx\|^2, \quad \forall x \in D(A). \tag{1}$$

Without loss of generality, we may assume $c \in (0, 1)$.

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In [1] (see also [2]) Chidume and Aneke proved:

THEOREM CA. *Let E be a real separable Banach space with a strictly convex dual E^* and let A be a Kpd operator with $D(A) = D(K)$. Suppose*

$$\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle,\ \forall x, y \in D(A). \tag{2}$$

Then there exists a constant $\alpha > 0$ such that for all $x \in D(A)$

$$\|Ax\| \le \alpha\|Kx\|. \tag{3}$$

Furthermore, the operator A is closed, $R(A) = E$ and the equation $Ax = f$ has a unique solution for any given $f \in E$.

For the special case of Theorem CA in which $E = L_P$(or ℓ_p) spaces, $2 \le p < \infty$, Chidume and Aneke constructed an iteration process which converges strongly to the unique solution of the equation $Ax = f$, provided that A and K commute. Recently, Chidume and Osilike [2] extended the convergence theorem of Chidume and Aneke [1] from L_p(or ℓ_p) spaces, $2 \le p < \infty$ to the more general real separable q-uniformly smooth Banach spaces, $1 < q < \infty$. Moreover, the commutativity assumption on A and K imposed in [1] was dropped in [2]. More recently Chuanzhi [3] proved convergence theorems for the iterative approximation of the solution of the *Kpd* operator equation $Ax = f$ in much more general separable uniformly smooth Banach spaces.

It is our purpose in this note to prove that the Picard iterates of a suitably defined operator converges strongly to the solution of the *Kpd* operator equation $Ax = f$ in the much more general setting of Theorem CA where E is a separable Banach space with a strictly convex dual. Explicit error estimate is obtained and convergence is shown to be as fast as a geometric progression. Our convergence theorem is valid in arbitrary real Banach spaces provided inequalities (2) and (3) of Theorem CA are satisfied and the equation $Ax = f$ has a solution.

2. Main results

Since K is continuously $D(A)$ invertible, there exists a constant $\beta > 0$ such that

$$\|Kx\| \ge \beta\|x\|, \quad \forall x \in D(K) = D(A). \tag{4}$$

In the sequel $c \in (0, 1)$, α and β are the constants appearing in inequalities (1), (3), and (4), respectively. We now prove the following:

THEOREM. *Let E be a real separable Banach space with a strictly convex dual and let $A : D(A) \subseteq E \to E$ be a Kpd operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky)\rangle = \langle Kx, j(Ay)\rangle$ for all $x, y \in D(A)$. Choose any $\epsilon \in \left(0, \frac{c^2}{(1+\alpha(1-c)+\alpha^2)}\right]$ and define $T_\epsilon : D(A) \subseteq E \to E$ by*

$$T_\epsilon x = x + \epsilon K^{-1} f - \epsilon K^{-1} Ax.$$

Then the Picard iteration method generated from an arbitrary $x_0 \in D(A)$ by

$$x_{n+1} = T_\epsilon x_n = T_\epsilon^n x_0$$

converges strongly to the solution of the equation $Ax = f$. Moreover, if x^* denotes the solution of the equation $Ax = f$, then

$$\|x_{n+1} - x^*\| \leq [1 - c\epsilon(1-c)]^n \beta^{-1} \|Kx_0 - Kx^*\|.$$

Proof. The existence of a unique solution to the equation $Ax = f$ follows from Theorem CA. Let x^* denote the solution. From (1), we obtain

$$\langle Ax - cKx, j(Kx)\rangle \geq 0$$

and it follows from Lemma 1.1 of Kato [4] that

$$\|Kx\| \leq \|Kx + \lambda(Ax - cKx)\|, \tag{5}$$

for all $x \in D(A)$ and for all $\lambda > 0$. Since

$$x_{n+1} = T_\epsilon x_n = x_n + \epsilon K^{-1} f - \epsilon K^{-1} Ax_n,$$

then

$$Kx_{n+1} = Kx_n + \epsilon f - \epsilon Ax_n = Kx_n + \epsilon Ax^* - \epsilon Ax_n \text{ (since } Ax^* = f).$$

Hence

$$Kx_n = Kx_{n+1} - \epsilon Ax^* + \epsilon Ax_n,$$

so that

$$
\begin{aligned}
&Kx_n - Kx^* \\
&= Kx_{n+1} - Kx^* - \epsilon Ax^* + \epsilon Ax_n \\
&= (1+\epsilon)(Kx_{n+1} - Kx^*) + \epsilon\Big[Ax_{n+1} - Ax^* - c(Kx_{n+1} - Kx^*)\Big] \\
&\quad -\epsilon(Kx_{n+1} - Kx^*) - \epsilon\Big[Ax_{n+1} - Ax^* - c(Kx_{n+1} - Kx^*)\Big] \\
&\quad -\epsilon Ax^* + \epsilon Ax_n \\
&= (1+\epsilon)\Big[K(x_{n+1} - x^*) + \frac{\epsilon}{1+\epsilon}[A(x_{n+1} - x^*) - cK(x_{n+1} - x^*)]\Big] \\
&\quad -\epsilon(1-c)K(x_{n+1} - x^*) - \epsilon(Ax_{n+1} - Ax_n) \\
&= (1+\epsilon)\Big[K(x_{n+1} - x^*) + \frac{\epsilon}{1+\epsilon}[A(x_{n+1} - x^*) - cK(x_{n+1} - x^*)]\Big] \\
&\quad -\epsilon(1-c)K(x_n - x^*) + \epsilon^2(1-c)(Ax_n - Ax^*) - \epsilon(Ax_{n+1} - Ax_n).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|K(x_n - x^*)\| \geq\ & \Big\|(1+\epsilon)\Big[K(x_{n+1} - x^*) \\
& + \frac{\epsilon}{1+\epsilon}[A(x_{n+1} - x^*) - cK(x_{n+1} - x^*)]\Big]\Big\| \\
& - \| - \epsilon(1-c)K(x_n - x^*) \\
& + \epsilon^2(1-c)(Ax_n - Ax^*) - \epsilon(Ax_{n+1} - Ax_n)\| \\
\geq\ & (1+\epsilon)\|K(x_{n+1} - x^*) + \frac{\epsilon}{1+\epsilon}[A(x_{n+1} - x^*) \\
& - cK(x_{n+1} - x^*)]\| - \epsilon(1-c)\|K(x_n - x^*)\| \\
& - \epsilon^2(1-c)\|Ax_n - Ax^*\| - \epsilon\|Ax_{n+1} - Ax_n\| \\
\geq\ & (1+\epsilon)\|K(x_{n+1} - x^*)\| - \epsilon(1-c)\|K(x_n - x^*)\| \\
& - \epsilon^2(1-c)\|Ax_n - Ax^*\| - \epsilon\|Ax_{n+1} - Ax_n\|, \\
& \text{(using (5))}
\end{aligned}
$$

so that

$$
\begin{aligned}
\|K(x_{n+1} - x^*)\| \leq\ & \frac{[1+\epsilon(1-c)]}{(1+\epsilon)}\|K(x_n - x^*)\| \\
& + \epsilon^2(1-c)\|Ax_n - Ax^*\| + \epsilon\|Ax_{n+1} - Ax_n\|.
\end{aligned} \tag{6}
$$

Since $\|Ax\| \leq \alpha\|Kx\|$, $\forall x \in D(A)$, we obtain

$$
\|Ax_n - Ax^*\| = \|A(x_n - x^*)\| \leq \alpha\|K(x_n - x^*)\| \tag{7}
$$

and

$$\begin{aligned} \|Ax_{n+1} - Ax_n\| &\leq \alpha \|K(x_{n+1} - x_n)\| \\ &= \alpha\epsilon \|A(x_n - x^*)\| \\ &\leq \alpha^2 \epsilon \|K(x_n - x^*)\|. \end{aligned} \tag{8}$$

Using (7) and (8) in (6), we obtain

$$\begin{aligned} \|K(x_{n+1} - x^*)\| &\leq \frac{[1 + \epsilon(1-c)]}{(1+\epsilon)} \|K(x_n - x^*)\| \\ &\quad + \alpha\epsilon^2(1-c)\|K(x_n - x^*)\| + \alpha^2\epsilon^2\|K(x_n - x^*)\| \\ &\leq [1 + \epsilon(1-c)][1 - \epsilon + \epsilon^2]\|K(x_n - x^*)\| \\ &\quad + \epsilon^2[\alpha(1-c) + \alpha^2]\|K(x_n - x^*)\| \\ &\leq [1 - \epsilon c + \epsilon^2]\|K(x_n - x^*)\| \\ &\quad + \epsilon^2[\alpha(1-c) + \alpha^2]\|K(x_n - x^*)\| \\ &= \Big[1 - \epsilon c + \epsilon^2[1 + \alpha(1-c) + \alpha^2]\Big]\|K(x_n - x^*)\| \\ &\leq [1 - \epsilon(1-c)c]\|K(x_n - x^*)\| \\ &\quad \left(\text{since } 0 < \epsilon \leq \frac{c^2}{[1 + \alpha(1-c) + \alpha^2]}\right) \\ &\leq \ldots \leq [1 - \epsilon(1-c)c]^n \|K(x_0 - x^*)\|. \end{aligned}$$

It now follows from inequality (4) that

$$\|x_{n+1} - x^*\| \leq [1 - \epsilon(1-c)c]^n \beta^{-1} \|K(x_0 - x^*)\| \to 0 \text{ as } n \to \infty,$$

completing the proof of the theorem. □

Remark. If E is an arbitrary real Banach space and $A : D(A) \subseteq E \to E$ is Kpd. It is clear from (1) that if $Ax = f$ has a solution, then the solution is unique. If inequalities (2) and (3) of Theorem CA are satisfied and $Ax = f$ has a solution, it is clear that our Picard iteration converges strongly to the solution with the explicit error estimate given in our Theorem above.

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