A NOTE ON APPROXIMATION OF SOLUTIONS OF A K-POSITIVE DEFINITE OPERATOR EQUATIONS

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ABSTRACT. In this note we construct a sequence of Picard iterates suitable for the approximation of solutions of $K$-positive definite operator equations in arbitrary real Banach spaces. Explicit error estimate is obtained and convergence is shown to be as fast as a geometric progression.

1. Introduction

Let $E$ be a real Banach space, $E^*$ the dual space of $E$ and let $J : E \to 2^{E^*}$ be the normalized duality mapping defined for each $x \in E$ by

$$J(x) = \{ f^* \in E^* : \langle x, f^* \rangle = \| x \|^2 = \| f^* \|^2 \},$$

where $\langle \cdot, \cdot \rangle$ is the generalized duality pairing. It is well known that if $E^*$ is strictly convex, then $J$ is single-valued. In the sequel we shall denote single-valued duality mapping by $j$.

In [1] Chidume and Aneke extended the notion of $K$-positive definite ($Kpd$) operators of Martynyuk [5] and Petryshyn ([6], [7]) from Hilbert spaces to arbitrary real Banach spaces. They called a linear unbounded operator $A$ defined on a dense domain $D(A)$ in $E$ a $Kpd$ operator if there exist a continuously $D(A)$-invertible closed linear operator $K$ with $D(A) \subseteq D(K)$ and a constant $c > 0$ such that for $j(Kx) \in J(Kx)$,

$$\langle Ax, j(Kx) \rangle \geq c \| Kx \|^2, \quad \forall x \in D(A).$$

(1)

Without loss of generality, we may assume $c \in (0, 1)$.

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In [1] (see also [2]) Chidume and Aneke proved:

**Theorem CA.** Let $E$ be a real separable Banach space with a strictly convex dual $E^*$ and let $A$ be a $Kpd$ operator with $D(A) = D(K)$. Suppose

$$\langle Ax, j(Ky) \rangle = \langle Kx, j(Ay) \rangle, \quad \forall x, y \in D(A).$$

Then there exists a constant $\alpha > 0$ such that for all $x \in D(A)$

$$\|Ax\| \leq \alpha \|Kx\|.$$ 

Furthermore, the operator $A$ is closed, $R(A) = E$ and the equation $Ax = f$ has a unique solution for any given $f \in E$.

For the special case of Theorem CA in which $E = L_p$ (or $\ell_p$) spaces, $2 \leq p < \infty$, Chidume and Aneke constructed an iteration process which converges strongly to the unique solution of the equation $Ax = f$, provided that $A$ and $K$ commute. Recently, Chidume and Osilike [2] extended the convergence theorem of Chidume and Aneke [1] from $L_p$ (or $\ell_p$) spaces, $2 \leq p < \infty$ to the more general real separable $q$-uniformly smooth Banach spaces, $1 < q < \infty$. Moreover, the commutativity assumption on $A$ and $K$ imposed in [1] was dropped in [2]. More recently, Chuanzhi [3] proved convergence theorems for the iterative approximation of the solution of the $Kpd$ operator equation $Ax = f$ in much more general separable uniformly smooth Banach spaces.

It is our purpose in this note to prove that the Picard iterates of a suitably defined operator converges strongly to the solution of the $Kpd$ operator equation $Ax = f$ in the much more general setting of Theorem CA where $E$ is a separable Banach space with a strictly convex dual. Explicit error estimate is obtained and convergence is shown to be as fast as a geometric progression. Our convergence theorem is valid in arbitrary real Banach spaces provided inequalities (2) and (3) of Theorem CA are satisfied and the equation $Ax = f$ has a solution.

**2. Main results**

Since $K$ is continuously $D(A)$ invertible, there exists a constant $\beta > 0$ such that

$$\|Kx\| \geq \beta \|x\|, \quad \forall x \in D(K) = D(A).$$

In the sequel $c \in (0, 1)$, $\alpha$ and $\beta$ are the constants appearing in inequalities (1), (3), and (4), respectively. We now prove the following:
Theorem. Let $E$ be a real separable Banach space with a strictly convex dual and let $A : D(A) \subseteq E \to E$ be a $Kpd$ operator with $D(A) = D(K)$. Suppose $\langle Ax, j(Ky) \rangle = \langle Kx, j(Ay) \rangle$ for all $x, y \in D(A)$. Choose any $\epsilon \in \left(0, \frac{c^2}{1+\alpha(1-c)+\alpha^2}\right)$ and define $T_\epsilon : D(A) \subseteq E \to E$ by

$$T_\epsilon x = x + \epsilon K^{-1}f - \epsilon K^{-1}Ax.$$  

Then the Picard iteration method generated from an arbitrary $x_0 \in D(A)$ by

$$x_{n+1} = T_\epsilon x_n = T_\epsilon^n x_0$$

converges strongly to the solution of the equation $Ax = f$. Moreover, if $x^*$ denotes the solution of the equation $Ax = f$, then

$$\|x_{n+1} - x^*\| \leq [1 - c\epsilon(1-c)]^n \beta^{-1}\|Kx_0 - Kx^*\|. $$

Proof. The existence of a unique solution to the equation $Ax = f$ follows from Theorem CA. Let $x^*$ denote the solution. From (1), we obtain

$$\langle Ax - cKx, j(Kx) \rangle \geq 0$$

and it follows from Lemma 1.1 of Kato [4] that

$$(5) \quad \|Kx\| \leq \|Kx + \lambda(Ax - cKx)\|,$$

for all $x \in D(A)$ and for all $\lambda > 0$. Since

$$x_{n+1} = T_\epsilon x_n = x_n + \epsilon K^{-1}f - \epsilon K^{-1}Ax_n,$$

then

$$Kx_{n+1} = Kx_n + \epsilon f - \epsilon Ax_n = Kx_n + \epsilon A x^* - \epsilon Ax_n \text{ (since } Ax^* = f).$$

Hence

$$Kx_n = Kx_{n+1} - \epsilon A x^* + \epsilon Ax_n,$$
so that

\[
Kx_n - Kx^* = Kx_{n+1} - Kx^* - \epsilon Ax^* + \epsilon Ax_n \\
= (1 + \epsilon)(Kx_{n+1} - Kx^*) + \epsilon \left[Ax_{n+1} - Ax^* - c(Kx_{n+1} - Kx^*) \right] \\
- \epsilon(Kx_{n+1} - Kx^*) - \epsilon \left[Ax_{n+1} - Ax^* - c(Kx_{n+1} - Kx^*) \right] \\
- \epsilon Ax^* + \epsilon Ax_n \\
= (1 + \epsilon)\left[K(x_{n+1} - x^*) + \frac{\epsilon}{1 + \epsilon} \left[A(x_{n+1} - x^*) - cK(x_{n+1} - x^*) \right] \right] \\
- \epsilon(1 - c)K(x_{n+1} - x^*) - \epsilon(Ax_{n+1} - Ax_n) \\
= (1 + \epsilon)\left[K(x_{n+1} - x^*) + \frac{\epsilon}{1 + \epsilon} \left[A(x_{n+1} - x^*) - cK(x_{n+1} - x^*) \right] \right] \\
- \epsilon(1 - c)K(x_n - x^*) + \epsilon^2(1 - c)(Ax_n - Ax^*) - \epsilon(Ax_{n+1} - Ax_n).
\]

Hence

\[
\|K(x_n - x^*)\| \geq \left\| (1 + \epsilon)\left[K(x_{n+1} - x^*) \\
+ \frac{\epsilon}{1 + \epsilon} \left[A(x_{n+1} - x^*) - cK(x_{n+1} - x^*) \right] \right] \right\| \\
- \| \epsilon(1 - c)K(x_n - x^*) \\
+ \epsilon^2(1 - c)(Ax_n - Ax^*) - \epsilon(Ax_{n+1} - Ax_n)\| \\
\geq (1 + \epsilon)\|K(x_{n+1} - x^*) + \frac{\epsilon}{1 + \epsilon} \left[A(x_{n+1} - x^*) \\
- cK(x_{n+1} - x^*) \right] \| - \epsilon(1 - c)\|K(x_n - x^*)\| \\
- \epsilon^2(1 - c)\|Ax_n - Ax^*\| - \epsilon\|Ax_{n+1} - Ax_n\| \\
\geq (1 + \epsilon)\|K(x_{n+1} - x^*)\| - \epsilon(1 - c)\|K(x_n - x^*)\| \\
- \epsilon^2(1 - c)\|Ax_n - Ax^*\| - \epsilon\|Ax_{n+1} - Ax_n\|; \\
\text{(using (5))}
\]

so that

\[
\|K(x_{n+1} - x^*)\| \leq \frac{1 + \epsilon(1 - c)}{1 + \epsilon} \|K(x_n - x^*)\| \\
+ \epsilon^2(1 - c)\|Ax_n - Ax^*\| + \epsilon\|Ax_{n+1} - Ax_n\|. 
\]

Since \(\|Ax\| \leq \alpha\|Kx\|, \forall x \in D(A),\) we obtain

\[
\|Ax_n - Ax^*\| = \|A(x_n - x^*)\| \leq \alpha\|K(x_n - x^*)\|
\]

(7)
and

\[ ||Ax_{n+1} - Ax_n|| \leq \alpha ||K(x_{n+1} - x_n)|| \]
\[ = \alpha \epsilon ||A(x_n - x^*)|| \]
\[ \leq \alpha^2 \epsilon ||K(x_n - x^*)||. \]

Using (7) and (8) in (6), we obtain

\[ ||K(x_{n+1} - x^*)|| \leq \frac{[1 + \epsilon(1 - c)]}{(1 + \epsilon)} ||K(x_n - x^*)|| \]
\[ + \epsilon^2(1 - c)||K(x_n - x^*)|| + \alpha^2 \epsilon^2 ||K(x_n - x^*)|| \]
\[ \leq [1 + \epsilon(1 - c)][1 - \epsilon + \epsilon^2] ||K(x_n - x^*)|| \]
\[ + \epsilon^2[\alpha(1 - c) + \alpha^2] ||K(x_n - x^*)|| \]
\[ \leq [1 - \epsilon c + \epsilon^2][1 + \alpha(1 - c) + \alpha^2] ||K(x_n - x^*)|| \]
\[ + \epsilon^2[\alpha(1 - c) + \alpha^2] ||K(x_n - x^*)|| \]
\[ = \left[ 1 - \epsilon c + \epsilon^2[1 + \alpha(1 - c) + \alpha^2] \right] ||K(x_n - x^*)|| \]
\[ \leq [1 - \epsilon(1 - c)c] ||K(x_n - x^*)|| \]
\[ \left( \text{since } 0 < \epsilon \leq \frac{c^2}{[1 + \alpha(1 - c) + \alpha^2]} \right) \]
\[ \leq ... \leq [1 - \epsilon(1 - c)c]^n ||K(x_0 - x^*)||. \]

It now follows from inequality (4) that

\[ ||x_{n+1} - x^*|| \leq [1 - \epsilon(1 - c)c]^n \beta^{-1} ||K(x_0 - x^*)|| \rightarrow 0 \text{ as } n \rightarrow \infty, \]

completing the proof of the theorem. \( \Box \)

**Remark.** If \( E \) is an arbitrary real Banach space and \( A : D(A) \subseteq E \rightarrow E \) is \( Kpd \). It is clear from (1) that if \( Ax = f \) has a solution, then the solution is unique. If inequalities (2) and (3) of Theorem CA are satisfied and \( Ax = f \) has a solution, it is clear that our Picard iteration converges strongly to the solution with the explicit error estimate given in our Theorem above.

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