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ERROR ESTIMATION FOR NONLINEAR ELLIPTIC PROBLEMS USING THE h-p-MIXED FINITE ELEMENT METHOD IN 3 DIMENSIONAL SPACE

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ABSTRACT. The approximation properties for L^2-projection, Raviart-Thomas projection, and inverse inequality have been derived in 3 dimensional space. h-p-mixed finite element methods for strongly nonlinear second order elliptic problems are proposed and analyzed in $3D$. Solvability and convergence of the linearized problem have been shown through duality argument and fixed point argument. The analysis is carried out in detail using Raviart-Thomas-Nedelec spaces as an example.

1. Introduction

In this paper, we concern with a nonlinear boundary value problem to solve using mixed finite element method which computes both the scalar (pressure) and vector(flux) simultaneously with comparable accuracy. The h-version of finite element method for solving PDE, which reduces the meshsize h with the fixed degree of the approximating polynomials on each element fixed, has been used for a long time as a standard method. The p-version, which varies the degree of the polynomial defined on each element with a fixed mesh, has been studied and analyzed by Babuška, Szabo and Katz [5]. The h-p-version, the combination of the h- and p-versions, was first addressed by Babuška and Dorr [2]. Babuškaand Guo [3] also exhibited that the convergence rate of finite element solution to true solution for the h-p-version in $2D$ is much faster than that of h or p version only. So far, most papers deal with linear

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problems to analyze using h-p-version on finite element methods. Recently, only a few papers are presented about nonlinear elliptic problems: the h-version, the p-version and the h-p-version used in [21], [17], and [15, 16], respectively. We use the h-p-version of the mixed finite element method, to apply to our problem in 3-dimensional space.

Let $\Omega \subset\subset \mathbb{R}^3$ be an open domain with boundary $\partial\Omega$ throughout this paper. We want to analyze the following nonlinear Dirichlet problem:

$$\sum_{i=1}^{3} \frac{\partial a_i}{\partial x_i}(x,u,\nabla u) + a_0(x,u,\nabla u) = 0, \quad x \in \Omega, \tag{1.1}$$
$$u(x) = -g(x), \quad x \in \partial\Omega,$$

where the coefficients a_i, $0 \le i \le 3$, are twice continuously differentiable with bounded derivatives through the second order on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^3$. We assume that the quasi linear operator associated with ([1.1]) is elliptic with respect to the third variable ∇u.

The minimal surface problem is an example of the above form when $a(x,u,\nabla u) = -\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}$ and $a_0 = 0$. We also can apply above problem to find the equilibrium position of an elastic body when it is subjected to applied forces by taking $a(x,u,\nabla u) = T(x,u,\nabla u) = T(x,F)$ as the elastic nonlinear constitutive equation defining the first Piola-Kirchhoff stress tensor T as a known function of the deformation gradient $F = Id + \nabla u$.

We divide this paper into three sections. This is the first section which is the introduction to this paper. In this section, we showed divergence type second order elliptic problem we shall consider. In section 2, we present approximation properties devoted to analyze the applicability of the h-p-version to our elliptic problem in $3D$. L^2-projection property is used for estimation of scalar part and Raviart-Thomas-projection property is for vector part. Projection properties tell us the error bound of a projected value on the finite element space to original analytic function at a rate of degree p and mesh-size h. We also derive inverse inequalities between L^p norm and L^q norm with $2 \le p < q$ in finite element space. We use orthogonal Legendre polynomials as a basis of the approximating polynomial space we use to derive the above projection properties. In section 3, we first linearlize our problem using the second order Taylor expansion around finite element solution. Then, we show uniqueness and existence of finite element solution of the linearized solution. To show the solvability of our mixed finite element method, we employ Brouwer's fixed point theorem. At the end of this section, we asymptotically estimates error of approximating finite element solution to true solution with L^2-norm. We have a little bit heavy regularity constraint. A numerical solution exists if the true solution belongs to $H^{7/2+\varepsilon}(\Omega)$ when the function given as a boundary condition belongs to $H^{3+\varepsilon}(\partial\Omega)$ where $0 < \varepsilon \ll 1$. We can reduce regularity constraint by giving some restrictions between mesh size h and polynomial degree p. But, here, we will not consider that case.

Let $W^{k,p}(\Omega) = \{f \in L^p(\Omega) | D^\alpha f \in L^p(\Omega) \text{ if } |\alpha| \leq k\}$ be the Sobolev spaces equiped with the norm

$$\|f\|_{k,p,\Omega} = \Big(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

and the seminorm

$$|f|_{k,p,\Omega} = \Big(\sum_{|\alpha|= k} \|D^\alpha f\|^p_{L^p(\Omega)} \Big)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and k be nonnegative integer.

2. Approximation properties in 3-D

In this section, we state the three dimensional nonlinear boundary value problem. We shall study and present some approximation results which are essential to analyze the applicability of the h-p–version to this problem. We also derive error estimates for the approximate solution obtained with the mixed method.

Let $\mathcal{T}_N$ be a quasi-uniform family of meshes on Ω consisting of parallelepipeds E. h_E and ρ_E will denote the diameters of E and of the largest sphere that can be inscribed in E, respectively. Let $h_N = \max_{E\in\mathcal{T}_N}\{h_E\}$. We assume that there exist constants C_1 and C_2 independent of h_N such that for all $E \in \mathcal{T}_N$ and for all N,

$$\frac{h_N}{h_E} \leq C_1, \qquad \frac{h_E}{\rho_E} \leq C_2. \tag{2.1}$$

Moreover, we assume that each pair of E_1 and $E_2 \in \mathcal{T}_N$ has either an entire face or an edge in common or has empty intersection.

For $E \in \mathcal{T}_N$, let F_E be an affine invertible mapping such that $E = F_E(R)$, where $R = [-1,1]^3$ and

$$(x,y,z) = F_E((\xi,\zeta,\eta)) = B_E(\xi,\zeta,\eta)^T + b_E, \tag{2.2}$$

where B_E is a 3×3 matrix. With any scalar function $\tilde{v}$ defined on R (or ∂R) we associate the function v defined on E or ∂E given by

$$v = \tilde{v} \circ F_E^{-1} \qquad (\tilde{v} = v \circ F_E). \tag{2.3}$$

For vector-valued functions, the correspondence between $\tilde{\tau}$ defined on R and τ defined on E is given by

$$\tau = \frac{1}{J_E} B_E \tilde{\tau} \circ F_E^{-1} \qquad (\tilde{\tau} = J_E B_E^{-1} \tau \circ F_E), \tag{2.4}$$

where J_E is the absolute value of the determinant of B_E. Then it is easy to obtain the following results (see [10]). In this section we denote

$$W^p = Q^{p,p,p} \subset W \text{ where } W = L^2([-1,1]^3)$$

and

$$\mathcal{P}^p = \Big\{ f : f = \sum_{0 \le i+j+k \le p} c_{i,j,k} x^i y^j z^k \text{ where } c_{i,j,k} \in \mathbb{R} \Big\}.$$

LEMMA 2.1.

$$\begin{aligned} &(\text{div } \tilde{\tau}, \tilde{\phi})_R = (\text{div } \tau, \phi)_E \qquad \forall \tilde{\phi} \in L^2(R), \\ &\int_{\partial R} \tilde{\tau} \cdot \tilde{\nu} \tilde{\phi} d\tilde{S} = \int_{\partial E} \tau \cdot \nu \phi dS \qquad \forall \tilde{\phi} \in L^2(\partial R). \end{aligned} \tag{2.5}$$

We also have the following scaling results.

LEMMA 2.2. *For all integers $l \ge 0$,*

$$|\tilde{\tau}|_{l,p,R} \le C h_E^{l-1+n(1-1/p)} |\tau|_{l,p,E}, \tag{2.6}$$

$$|\tau|_{l,p,E} \le C h_E^{1-l+n(1/p-1)} |\tilde{\tau}|_{l,p,R}, \tag{2.7}$$

where the constant C depends on l but is independent of τ and h_E.

LEMMA 2.3. *For any $\tilde{u} \in H^r(R)$, $r \ge 0$,*

$$\inf_{\tilde{w} \in \mathcal{P}^p(R)} \|\tilde{u} - \tilde{w}\|_{r,R} \le C h^{\mu - 3/2} \|u\|_{r,E}, \tag{2.8}$$

where $\mu = \min(p+1, r)$ and C depends on R, but is independent of p, h, and u.

Proof. For $r = 0$ the result follows from Lemma 4.2 in [4] by taking $\tilde{w} = 0$. Assume that r is integer. Then

$$\inf_{\tilde{w}\in\mathcal{P}^p(R)} \|\tilde{u} - \tilde{w}\|_{r,R} \le \inf_{\tilde{w}\in\mathcal{P}^p(R)} \{\|\tilde{u} - \tilde{w}\|_{\mu,R} + \sum_{i=\mu+1}^{r} |\tilde{u}|_{i,R} + \sum_{i=\mu+1}^{r} |\tilde{w}|_{i,R}\},$$

where $\sum_{i=\mu+1}^{r} |\tilde{w}|_{i,R} = 0$ if $r > p$.

Using Theorem 3.1.1 of [10],

$$\begin{aligned}\inf_{\tilde{w}\in\mathcal{P}^p(R)} \|\tilde{u} - \tilde{w}\|_{\mu,R} + \sum_{i=\mu+1}^{r} |\tilde{u}|_{i,R} &\le C\sum_{i=\mu}^{r} |\tilde{u}|_{i,R} \\ &\le C\sum_{i=\mu}^{r} h^{i-3/2}|u|_{i,E} \\ &\le Ch^{\mu-3/2}\|u\|_{r,E}.\end{aligned}$$

The result then follows from interpolation [24]. □

LEMMA 2.4. *Let $\tilde{\tau} \in (H^r(R))^3$ and $\tau \in (H^r(E))^3$, $r \ge 0$, be related by (2.4). Then,*

$$\inf_{\tilde{\mathbf{w}}\in(\mathcal{P}^p)^3} \|\tilde{\tau} - \tilde{\mathbf{w}}\|_{r,R} \le C(R) h_E^{\min(p+1,r)+1/2} \|\tau\|_{r,E}, \tag{2.9}$$

where C depends on r but is independent of h_E, p, and τ.

Proof. The above lemma is simply a vector form of (2.8). The proof follows in the same way, using the scaling relation (2.4) and Theorem 3.1.2 of [10]. □

2.1. Inverse inequalities

LEMMA 2.5. *The following inverse inequalities hold in three dimensions:*

$$\begin{aligned}&\|\chi\|_{0,q'} \le C(q, q', \Omega) p^{6/q-6/q'} \|\chi\|_{0,q}, \quad 1 \le q \le q' \le \infty, \\ &\quad \text{if} \quad \chi \in L^{q'}(\Omega) \bigcap W^p.\end{aligned} \tag{2.10}$$

2.2. The approximation properties of P^p

We shall use the L^2-projection onto W^p, $P^p : W \to W^p$, given by

$$(P^p w - w, \chi) = 0, \quad \chi \in W^p, \quad w \in W. \tag{2.11}$$

This has the following approximation property

$$\|P^p w - w\|_0 \le Q p^{-m} \|w\|_m. \tag{2.12}$$

Also, from Theorem 2.1 in [8] and using [6] (Theorem 6.2.4), we obtain

$$\|P^p w - w\|_{0,\infty} \le Q p^{-m+5/2} \|w\|_m. \tag{2.13}$$

It follows from (2.12) and (2.13), using interpolation theory, that

$$\begin{aligned} \|P^p w - w\|_{0,s} &\le Q p^{-m+5/2-5/s} \|w\|_m, \\ s \ge 2, &\quad 5/2 - 5/s \le m, \end{aligned} \tag{2.14}$$

where Q is independent of p.

2.3. R-T-N mixed finite elements in 3-D

We introduce the space $\tilde{Q}$ associated with the standard cube $R = [-1,1]^3$ in the (x,y,z)-plane. Given three integers $k, l, m \ge 0$, denote by $Q^{l,m,n}$ the space of all polynomials in three variables x, y, z of the form

$$f^{l,m,n}(x,y,z) = \sum_{i=0}^{l} \sum_{j=0}^{m} \sum_{k=0}^{n} c_{i,j,k} x^i y^j z^k, \quad c_{i,j,k} \in \mathbb{R}.$$

Now, we define the space $\tilde{Q}$ by

$$\tilde{Q} = \{\hat{q} = (q_1, q_2, q_3) : q_1 \in Q^{p+1,p,p}, \quad q_2 \in Q^{p,p+1,p}, \quad q_3 \in Q^{p,p,p+1}\}.$$

Note that, for $\hat{q} \in \tilde{Q}$, we have :

(i) div $\hat{q} \in Q^{p,p,p}$,

(ii) the restriction of $\hat{q} \cdot \nu$ to any face $\hat{S}$ of R is a polynomial of degree $\le k$ in two variables (see [22]).

LEMMA 2.6. *A function $\hat{q} \in \tilde{Q}$ is uniquely determined by* ([19], [20])*:*
(a) *the values of $\hat{q} \cdot \hat{\nu}$ at $(p+1)^2$ distinct points of each face $\hat{S}$ of R with no $p+2$ points aligned;*

or

(a′) *the moments $\int_{\hat{S}} (\hat{q} \cdot \nu_s) \hat{\rho} d\sigma$, $\quad \hat{\rho} \in Q^{p,p}$, for each face $\hat{S}$ of R; and*

(b) *the moments*

$$\int_R \hat{q}_1 x^i y^j z^k d\hat{x}, \quad 0 \le i \le p-1, \quad 0 \le j,k \le p,$$

$$\int_R \hat{q}_2 x^i y^j z^k d\hat{x}, \quad 0 \le j \le p-1, \quad 0 \le i,k \le p, \tag{2.15}$$

$$\int_R \hat{q}_3 x^i y^j z^k d\hat{x}, \quad 0 \le l \le p-1, \quad 0 \le i,j \le p.$$

2.4. The approximation properties of π^p

Recall that π^p is given locally (on every element E) by the following relations (see [22]):

$$\langle [\pi^p v - v] \cdot \vec{\nu_E}, \rho \rangle_{S_i} = 0, \quad \rho \in Q^{p,p}, \tag{2.16}$$

where $\langle \cdot, \cdot \rangle_E$, $1 \le i \le 6$, denotes the surface integral along each face S_i of the element E, and

$$(\pi^p v - v, \vec{\psi})_E = 0, \quad \vec{\psi} \in \bar{V}^p(E), \tag{2.17}$$

where

$$\bar{V}^p(E) = Q^{p-1,p,p} \times Q^{p,p-1,p} \times Q^{p,p,p-1}(E),$$

and $(\cdot,\cdot)_E$ denotes the standard $L^2(E)$-inner product. Now, let $\{L_i\}_{i\ge 0}$ denote the $L^2([-1,1])$-complete orthogonal Legendre polynomials. For any $v \in \mathbf{H}(div; R)$, let [18]

$$\begin{aligned} v(x,y,z) = \Big[&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{i,j,k} L_i(x) L_j(y) L_k(z), \\ &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} b_{i,j,k} L_i(x) L_j(y) L_k(z), \\ &\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} c_{i,j,k} L_i(x) L_j(y) L_k(z) \Big], \end{aligned} \tag{2.18}$$

and

$$\pi^p v(x,y,z) = \Big[\sum_{i=0}^{p+1} \sum_{j=0}^{p} \sum_{k=0}^{p} \tilde{a}_{i,j,k} L_i(x) L_j(y) L_k(z),$$

$$(2.19) \qquad \sum_{i=0}^{p} \sum_{j=0}^{p+1} \sum_{k=0}^{p} \tilde{b}_{i,j,k} L_i(x) L_j(y) L_k(z),$$

$$\sum_{i=0}^{p} \sum_{j=0}^{p} \sum_{k=0}^{p+1} \tilde{c}_{i,j,k} L_i(x) L_j(y) L_k(z)\Big].$$

It follows from (2.17)-(2.19) that

$$(1) \qquad \begin{cases} a_{i,j,k} = \tilde{a}_{i,j,k}, & 0 \le i \le p-1, \quad 0 \le j \le p, \\ & \qquad\qquad\qquad\quad 0 \le k \le p, \\ b_{i,j,k} = \tilde{b}_{i,j,k}, & 0 \le i \le p, \qquad\quad 0 \le j \le p-1, \\ & \qquad\qquad\qquad\quad 0 \le k \le p, \\ c_{i,j,k} = \tilde{c}_{i,j,k}, & 0 \le i \le p, \qquad\quad 0 \le j \le p, \\ & \qquad\qquad\qquad\quad 0 \le k \le p-1. \end{cases}$$

Next, we see that (2.16), (2.18)-(2.20) imply that

$$(2.20) \qquad \begin{cases} \displaystyle\sum_{i=p}^{p+1} \tilde{a}_{i,j,k} L_i(\pm 1) = \sum_{i=p}^{\infty} a_{i,j,k} \quad L_i(\pm 1), \\ \qquad\qquad\qquad\qquad\qquad 0 \le j \le p,\ 0 \le k \le p, \\ \displaystyle\sum_{j=p}^{p+1} \tilde{b}_{i,j,k} L_j(\pm 1) = \sum_{j=p}^{\infty} b_{i,j,k} \quad L_j(\pm 1), \\ \qquad\qquad\qquad\qquad\qquad 0 \le i \le p,\ 0 \le k \le p, \\ \displaystyle\sum_{k=p}^{p+1} \tilde{c}_{i,j,k} L_k(\pm 1) = \sum_{k=p}^{\infty} c_{i,j,k} \quad L_k(\pm 1), \\ \qquad\qquad\qquad\qquad\qquad 0 \le i \le p,\ 0 \le j \le p. \end{cases}$$

Since $L_i(-1) = (-1)^i$ and $L_i(1) = 1$, (2.20) implies that

$$(2.21) \quad \begin{cases} \tilde{a}_{p,j,k} = \sum_{i=0}^{\infty} a_{2i+p,j,k}, \ \tilde{a}_{p+1,j,k} \\ \quad = \sum_{i=0}^{\infty} a_{2i+p+1,j,k}, \qquad 0 \le j \le p,\ 0 \le k \le p, \\ \tilde{b}_{i,p,k} = \sum_{j=0}^{\infty} b_{i,2j+p,k}, \ \tilde{b}_{i,p+1,k} \\ \quad = \sum_{j=0}^{\infty} b_{i,2j+p+1,k}, \qquad 0 \le i \le p,\ 0 \le k \le p, \\ \tilde{c}_{i,k,p} = \sum_{k=0}^{\infty} c_{i,j,2k+p}, \ \tilde{c}_{i,j,p+1} \\ \quad = \sum_{k=0}^{\infty} c_{i,j,2k+p+1}, \qquad 0 \le i \le p,\ 0 \le j \le p. \end{cases}$$

LEMMA 2.7. *Let $v \in V$ and $\pi^p v$ be its Raviart-Thomas projection in V^p given by* (2.16)-(2.17). *Then, if $v \in H^r(\Omega)^3$, we have*

$$(2.22) \qquad \|v - \pi^p v\|_0 \le Q p^{1/2-r} \|v\|_r, \ r > 1/2,$$

where $Q > 0$ is a constant independent of p and v but depending on r.

Proof. Assume that $\Omega = R$ and that the decomposition consists of just one element. Then, $v \in V$ and $\pi^p v \in V^p$ can be given, respectively, by (2.18) and (2.19). The following relation is a trivial consequence of well-known properties of the Legendre polynomials [18],

$$\begin{aligned} \|v - \pi^p v\|_0^2 = & \sum_{i=p}^{p+1} \sum_{j=0}^{p} \sum_{k=0}^{p} \frac{8(a_{i,j,k} - \tilde{a}_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)} \\ & + \sum_{i=0}^{p} \sum_{j=p}^{p+1} \sum_{k=0}^{p} \frac{8(b_{i,j,k} - \tilde{b}_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{p}\sum_{j=0}^{p}\sum_{k=p}^{p+1}\frac{8(c_{i,j,k}-\tilde{c}_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=0}^{p+1}\sum_{j=0}^{p}\sum_{k=p+1}^{\infty}\frac{8(a_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=p+1}^{\infty}\sum_{j=0}^{p+1}\sum_{k=0}^{p}\frac{8(b_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=0}^{p}\sum_{j=p+1}^{\infty}\sum_{k=0}^{p+1}\frac{8(c_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=0}^{p+1}\sum_{j=p+1}^{\infty}\sum_{k=0}^{\infty}\frac{8(a_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=0}^{\infty}\sum_{j=0}^{p+1}\sum_{k=p+1}^{\infty}\frac{8(b_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=p+1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{p+1}\frac{8(c_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=p+2}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{8(a_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=0}^{\infty}\sum_{j=p+2}^{\infty}\sum_{k=0}^{\infty}\frac{8(b_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
&+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=p+2}^{\infty}\frac{8(c_{i,j,k})^2}{(2i+1)(2j+1)(2k+1)}\\
=&\sum_{j=1}^{12} I_j.
\end{aligned}
\tag{2.25}
$$

Note that $I_4 - I_{12}$ can be bounded as follows:

$$
I_4 + I_7 + I_{10} \le 8p^{-2r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{a_{i,j,k}^2(1+i^2+j^2+k^2)^r}{(2i+1)(2j+1)(2k+1)}, \quad r \ge 0,
$$

while

$$I_5 + I_8 + I_{11} \le 8p^{-2r} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{b_{i,j,k}^2(1+i^2+j^2+k^2)^r}{(2i+1)(2j+1)(2k+1)}, \quad r \ge 0,$$

and

$$I_6 + I_9 + I_{12} \le 8p^{-2r} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{c_{i,j,k}^2(1+i^2+j^2+k^2)^r}{(2i+1)(2j+1)(2k+1)}, \quad r \ge 0.$$

Then, we shall see that

$$\begin{aligned} \|v\|_{r,R}^2 \ge & \sum_{0\le s+t+q\le r} \int\int\int_R (1-x^2)^s(1-y^2)^t \\ & \times (1-z^2)^q (\frac{\partial^{s+t+q} v}{\partial x^s \partial y^t \partial z^q})^2 \, dx\, dy\, dz \\ & \ge C(r) \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{a_{ijk}^2(1+i^2+j^2+k^2)^r}{(2i+1)(2j+1)(2k+1)}. \end{aligned} \tag{2.26}$$

Note that we can prove (2.26) as follows: Assume that s, q, r are nonnegative integers. Then (see [23]),

$$\begin{aligned} \|v\|_r^2 \ge & \sum_{0\le s+t+q\le r} \int\int\int_R (1-x^2)^s(1-y^2)^t \\ & \times (1-z^2)^q (\frac{\partial^{s+t+q} v}{\partial x^s \partial y^t \partial z^q})^2 \, dx\, dy\, dz \ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \\ & \sum_{0\le s+t+q\le r} \frac{8a_{i,j,k}^2}{(2i+1)(2j+1)(2k+1)} \frac{(i+s)!(j+t)!(k+q)!}{(i-s)!(j-t)!(k-q)!} \\ & \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \sum_{0\le s+t+q\le r} \frac{8a_{i,j,k}^2}{(2i+1)(2j+1)(2k+1)} A_i^s A_j^t A_k^q, \end{aligned}$$

where

$$A_m^n = \begin{cases} \frac{(m+n)!}{(m-n)!} & \text{if} \quad m \ge n \\ 0 & \text{if} \quad m < n \end{cases},$$

and using induction and interpolation we can prove the following:

$$\sum_{0\le s+t+q\le r} a_{i,j,k}^2 A_i^s A_j^t A_k^q \ge C(r) a_{i,j,k}^2 (1+i^2+j^2+k^2)^r.$$

Therefore, we have

$$I_4 + I_5 + \cdots + I_{12} \leq Qp^{-2r}\|v\|_r^2, \quad r \geq 0. \tag{2.27}$$

On the other hand, by following the line of proof in [18] analogously, we can show that

$$I_1 + I_2 + I_3 \leq Qp^{1-2s}\|v\|_s^2, \quad \text{for } s > 1. \tag{2.28}$$

Let now $v \in H^{1/2+\varepsilon}(\Omega)^3$. We also can show that

$$\|\pi^p v - v\|_0 \leq Qp^{-\varepsilon}\|v\|_{1/2+\varepsilon}, \tag{2.29}$$

$$I_1, I_2, I_3 \leq Qp^{-2\varepsilon}\|v\|_{1/2+\varepsilon}^2,$$

which yields (2.29). Using interpolation [25], it follows from (2.28) and (2.29) that, for s bounded away from $1/2$,

$$I_1 + I_2 + I_3 \leq Qp^{1-2s}\|v\|_s^2,$$

which, together with (2.23) and (2.27), concludes the proof for the case $\Omega = R$. For the case when Ω is a disjoint union of parallelograms, the result follows on each element by using affine mappings onto R. The proposition then follows by summing over all the elements. □

REMARK. This result differs from the one known for the h-version of the finite element method [17](1.5),

$$\|v - \pi^h v\|_0 \leq Qh^r\|v\|_r, \quad r > 1/2. \tag{2.30}$$

The constraint $r > 1/2$ (or $r \geq 1/2 + \varepsilon$) stems from the fact that, according to the trace theorem, this is the minimum requirement to ensure that v has a trace on the boundary which is an L^2-function (not just a distribution). In [17], the corresponding result required an additional half derivative on v ($r > 1$). In contrast, Proposition 2.1 assumes the minimum regularity. It is possible, however, that the bound still holds with the exponent of p replaced by $-r$, as suggested by (2.30).

COROLLARY 2.1. *For $s \geq 2$, $r > \max\{1/2, 5/2 - 5/s\}$,*

$$\|v - \pi^p v\|_{0,s} \leq Q(r,\Omega)p^{7/2-r-6/s}\|v\|_r, \tag{2.31}$$

where Q is dependent on r.

Proof. Let $\mathbf{P}^p$ be the L^2-projection $P^p \times P^p \times P^p : V \to V^p$. Then, the following analogue of (2.14) holds:

$$\|\mathbf{P}^p v - v\|_{0,s} \leq Qp^{-r+5/2-5/s}\|v\|_r, \quad s \geq 2, \quad 5/2 - 5/s \leq r. \tag{2.32}$$

Also,

$$\|\pi^p v - v\|_{0,s} \le \|\mathbf{P}^p v - \vec{v}\|_{0,s} + \|\pi^p v - \mathbf{P}^p v\|_{0,s}. \tag{2.33}$$

The second term in this expression may be bounded using the inverse inequality (2.10) as follows:

$$\begin{aligned} \|\pi^p v - \mathbf{P}^p v\|_{0,s} &\le Qp^{3-6/s}\|\pi^p v - \mathbf{P}^p \vec{v}\|_0 \\ &\le Qp^{3-6/s}(\|\mathbf{P}^p v - v\|_0 + \|\pi^p v - v\|_0). \end{aligned} \tag{2.34}$$

Combining (2.32)-(2.34) and using Lemma 2.7, we obtain the result. □

2.5. The h-p – version in 3-D

Let $V = \mathbf{H}(\text{div};\Omega) = \{v \in L^2(\Omega)^3 : \text{ div } v \in L^2(\Omega)\}$

be normed by

$$\|v\|_V = \|v\|_0 + \|\text{ div } v\|_0$$

and $W = L^2(\Omega)$.

In this section, we define, for each element $E \in \mathcal{T}_N$,

$$V^N(E) = \mathcal{Q}^{p_N+1,p_N,p_N}(E) \times \mathcal{Q}^{p_N,p_N+1,p_N}(E) \times \mathcal{Q}^{p_N,p_N,p_N+1}(E),$$

and let

$$W^N \times V^N \subset W \times V$$

be the Raviart-Thomas-Nedelec space of index $p_N \ge 0$, associated with this decomposition [12, 19, 20, 22] given by

$$V^N = \Big(\prod_{E\in\mathcal{T}_N} V^N(E)\Big)\bigcap\{f : \Omega \to \mathbb{R}^3 | f\cdot\nu_E = f\cdot\nu_{E'} \text{ on } E\bigcap E',\ \ E, E' \in \mathcal{T}_N\},$$

where ν_E denotes the outward unit normal vector along ∂E, $E \in \mathcal{T}_N$. We use the L^2-projection onto W^N, $P^N : L^2 \to W^N$ given by

$$(P^N w - w, \chi) = 0,\ \chi \in W^N, \quad w \in W. \tag{2.35}$$

We shall also use the R-T projection of V onto V^N, $\pi^N; V \to V^N$ [12]. (π^N is as defined in (2.16) and (2.17).)

Lemma 2.8. *Let $v \in H^r(\Omega)^3$ and $r > \{1/2, 5/2 - 5/s\}$, and let $\pi^N : V \to V^N$ be as defined above. Then,*

$$\begin{aligned} &\|\pi^N v - v\|_{0,s} \le Ch_N^{\min(p_N+1,r)-2+3/s} p_N^{7/2-r-6/s}\|v\|_r, \\ &v \in (H^r(\Omega))^3, \quad r > \max\{1/2, 5/2 - 5/s\}, \quad s \ge 2, \end{aligned} \tag{2.36}$$

where C is a constant independent of h_N, p_N, and s but depends on r.

Proof. Let $v \in (\mathbf{H}^r(\Omega))^3$, $r > \max\{1/2, 5/2 - 5/s\}$. Consider $\pi^N \omega = \omega$ for all $\omega \in \mathcal{P}^{p_N} \times \mathcal{P}^{p_N} \times \mathcal{P}^{p_N}$ as follows:

$$\begin{aligned} \|\pi^N v - v\|_{0,s,E} &\le Ch_N^{3/s-2}\|\pi^N(v-\omega) - (v-\omega)\|_{0,s,R} \\ &\le Ch_N^{3/s-2}p_N^{7/2-r-6/s}\|v-\omega\|_{r,R} \\ &\le Ch_N^{3/s-2+\mu}p_N^{7/2-r-6/s}\|\hat{v}\|_{r,E}, \end{aligned} \tag{2.37}$$

where $\mu = \min\{p_N + 1, r\} + 1/2$, and $E \in \mathcal{T}_N$. Taking summation over $E \in \mathcal{T}_N$ we see that

$$\|\pi^N \hat{v} - \hat{v}\|_{0,s;\Omega} \le Ch^{3/s-2+\mu}p_N^{7/2-r-6/s}\|v\|_r. \tag{2.38}$$

We have used Lemmas 2.4 and 2.1 in our proof. □

LEMMA 2.9. *The L^2-projection, P^N, defined above has the following property:*

$$\|w - P^N w\|_{0,s} \le Ch_N^{3/s-2+\mu}p_N^{-r+5/2-5/s}\|w\|_r, \tag{2.39}$$

where $\mu = \min(p_N + 1, r)$, $s \ge 2$, $5/2 - 5/s \le m$, $w \in H^r(\Omega)$. C is a constant independent of h_N, p_N, and w.

Proof. We can prove the lemma using the same methods as those used in Lemma 2.8 and, using (2.14), Lemma 2.3. □

We will also use the inverse-type inequalities

$$\begin{aligned} &\|\chi\|_{0,s} \le C(r,s,R)h_N^{3/s-3/r}p_N^{6/r-6/s}\|\chi\|_{0,r}, \\ &1 \le r \le s \le \infty, \quad \chi \in L^s(\Omega) \cap W^N (\text{ or } \ \chi \in L^2(\Omega)^3 \cap V^N). \end{aligned} \tag{2.40}$$

3. Solvability of linearized problem

In this section, we will show the solvability and convergence for the linearized version of (1.1) where Ω is a bounded convex domain in $\mathbb{R}^3$ with C^2-boundary $\partial\Omega$. The analysis we use here is basically similar to those employed by Milner [17] and Park [21]. The functions $a_i : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, $0 \le i \le 2$, are twice continuously differentiable with bounded derivatives through the second order. Also, assume that the quasi-linear operator associated with (1.1) is elliptic and assume to be symmetric. The mixed finite element method approximates at the same time the solution u of (1.1) and the flux

$$\sigma = -a(u, \nabla u) = -\big(a_1(u,\nabla u), a_2(u,\nabla u), a_3(u,\nabla u)\big) \tag{3.1}$$

which we assume to be in $C^{0,1}(\bar{\Omega})^2$ [7]. By the implicit function theorem, (3.1) can be inverted to obtain ∇u as a function of u and σ, say

$$\nabla u = -b(u, \sigma).$$

We now set

$$f(u,\sigma) = -a_0(u, \nabla u) = -a_0\big(u, -b(u,\sigma)\big)$$

in (1.1). Then, the mixed weak form of (1.1) we shall consider consists of finding $(u, \sigma) \in W \times V$ such that

$$\begin{aligned} (b(u,\sigma), v) - (\text{div } v, u) &= \langle g, v\cdot n\rangle \quad \forall v \in V, \\ (\text{div } \sigma, w) &= (f(u,\sigma), w) \quad \forall w \in W, \end{aligned} \tag{3.2}$$

where n denotes the unit outer normal vector to $\partial\Omega$. We now give a succinct description of the Raviart-Thomas-Nedelec mixed rectangular elements. Consider a quasi-uniform family $\{\mathcal{T}_N\}$ of decompositions of Ω by parallelepipeds E (with boundary elements allowed to have one curved face). The mixed finite element method is a discrete form of (3.2) and consists of finding $(u^N, \sigma^N) \in W^N \times V^N$ such that

$$\begin{aligned} (b(u^N,\sigma^N), v) - (\text{div } v, u^N) &= \langle g, v\cdot n\rangle \quad \forall v \in V^N, \\ (\text{div } \sigma^N, w) &= (f(u^N,\sigma^N), w) \quad \forall w \in W^N. \end{aligned} \tag{3.3}$$

In 1975, linearlization of a nonlinear problem using Newton method was proposed and analyzed by Douglas and Dupont [11] to get a numerical solution. Chen [9] applied and generalized that linearlization for mixed finite element method by nonlinear functional analysis. Park [21] employed and implemented the linearlization for the mixed method using second order Taylor approximation. Following [21], we use first and second order Taylor expansions of f and b with second order terms $\tilde{Q}_f$ and $\tilde{Q}_b$ respectively. Notice that $\tilde{Q}_f \in \mathbb{R}$ and $\tilde{Q}_b \in \mathbb{R}^3$. We obtain our first error equations by subtracting (3.3) from (3.2):

$$\begin{aligned} (b(u,\sigma) - b(u^N,\sigma^N), v) - (\text{div } v, u - u^N) &= 0 \quad \forall v \in V^N, \\ (\text{div } (\sigma - \sigma^N), w) &= (f(u,\sigma) - f(u^N,\sigma^N), w) \quad \forall w \in W^N. \end{aligned} \tag{3.4}$$

Recall that

$$\text{div} \cdot \pi^N = P^N \cdot \text{div } : \mathbf{H}^1(\Omega)^2 \to W^N.$$

Linearization of (3.4) at (u^N, σ^N), leads to the following form, which we will need for our fixed point argument:

$$\begin{aligned} &(B(u,\sigma)[\pi^N\sigma - \sigma^N], v) - (\text{div}v, P^N u - u^N) + (\mathbf{\Gamma}_1[P^N u - u^N], v) \\ =&(B(u,\sigma)[\pi^N\sigma - \sigma] + \mathbf{\Gamma}_1[P^N u - u] + \tilde{Q}_b(u - u^N, \sigma - \sigma^N), v) \ \forall v \in V^N, \end{aligned}$$

$$(\operatorname{div}(\pi^N\sigma - \sigma^N), w) - (\mathbf{\Gamma}_2[\pi^N\sigma - \sigma^N], w) - (\gamma[P^N u - u^N], w)$$
$$=(-\mathbf{\Gamma}_2[\pi^N\sigma - \sigma] - \gamma[P^N u - u] - \tilde{Q}_f(u - u^N, \sigma - \sigma^N), w) \quad \forall w \in W^N.$$

Here we have set [21]

$$B(u,\sigma) = b_\sigma(u,\sigma) = A^{-1}(u,\sigma), \quad \mathbf{\Gamma}_1 = b_u(u,\sigma), \quad \mathbf{\Gamma}_2 = f_\sigma(u,\sigma),$$

and $\gamma = f_u(u,\sigma)$.

Now define $M : H^2(\Omega) \to L^2(\Omega)$ by

$$\begin{aligned} Mw = & -\operatorname{div}(A(u,\sigma)\nabla w + A(u,\sigma)\mathbf{\Gamma}_1 w) + A(u,\sigma)\mathbf{\Gamma}_2 \cdot \nabla w \\ & - (\gamma - \mathbf{\Gamma}_2^T A(u,\sigma)\mathbf{\Gamma}_1)w \end{aligned} \tag{3.5}$$

and its formal adjoint M^* by

$$\begin{aligned} M^*\chi = & -\operatorname{div}(A(u,\sigma)\nabla\chi + A(u,\sigma)\mathbf{\Gamma}_2\chi) + A(u,\sigma)\mathbf{\Gamma}_1 \cdot \nabla\chi \\ & - (\gamma - \mathbf{\Gamma}_2^T A(u,\sigma)\mathbf{\Gamma}_1)\chi. \end{aligned}$$

From [12], we know that the restrictions of the operator M and M^* to $H^2(\Omega)\bigcap H_0^1(\Omega)$ have bounded inverses, provided that $\partial\Omega$ is C^2. In our case it is only Lipschitz, but the result is still valid if we assume that (u,σ) can be extended to a pair $(\tilde{u},\tilde{\sigma})$ defined on a domain $\bar{\Omega}_0$ with a C^2-boundary, such that $\Omega \subset \Omega_0$ and $\text{meas}(\Omega_0 - \Omega)$ is arbitrary small [1, 14]. Then for any $\psi \in L^2(\Omega)$ there is a unique $\phi \in H^2(\Omega)\bigcap H_0^1(\Omega)$ such that $M\phi = \psi$ (and $M^*\phi = \psi$, respectively) and we have $\|\phi\|_2 \le C\|\psi\|_0$ if we assume that, for example, the zero order term of M^* is nonnegative, that is

$$\gamma \le \mathbf{\Gamma}_2^T A(u,\sigma)\mathbf{\Gamma}_1 - \operatorname{div}(A(u,\sigma)\mathbf{\Gamma}_2), \tag{3.6}$$

where $u \in C^{0,1}(\bar{\Omega})$ and $\sigma \in C^{0,1}(\bar{\Omega})^2$ [13], [21]. In this paper, we shall assume the structure condition (3.6) to employ our duality arguments. Let $\Phi : V^N \times W^N \to V^N \times W^N$ be given by $\Phi(\mu,\rho) = (\xi,\phi)$ where (ξ,ϕ) is the solution of the system,

$$\begin{aligned} & (B(u,\sigma)[\pi^N\sigma - \phi], v) - (\operatorname{div} v, P^N u - \xi) + (\mathbf{\Gamma}_1[P^N u - \xi], v) \\ = & (B(u,\sigma)[\pi^N\sigma - \sigma] + \mathbf{\Gamma}_1[P^N u - u] \\ & + \tilde{Q}_b(u - \mu, \sigma - \rho), v) \quad \forall v \in V^N, \\ & (\operatorname{div}(\pi^N\sigma - \phi), w) - (\mathbf{\Gamma}_2[\pi^N\sigma - \phi], w) - (\gamma[P^N u - \xi], w) \\ = & (-\mathbf{\Gamma}_2[\pi^N\sigma - \sigma] - \gamma[P^N u - u] \\ & - \tilde{Q}_f(u - \mu, \sigma - \rho), w) \quad \forall w \in W^N. \end{aligned} \tag{3.7}$$

Note that the left hand side of (3.7) corresponds to the mixed method for the operator M given by (3.5). In the next subsection, we will show that this system is uniquely solvable, so that the map Φ is well-defined.

The problem we want to study is to find $(y, q) \in W^N \times V^N$ such that

$$\begin{aligned} &(Bq, v) - (\mathrm{div} v, y) + (\mathbf{\Gamma}_1 y, v) = (l, v) \quad \forall v \in V^N, \\ &(\mathrm{div} q, w) - (\mathbf{\Gamma}_2 q, w) - (\gamma y, w) = (m, w) \quad \forall w \in W^N. \end{aligned} \tag{3.8}$$

Here we have assumed the structure condition (3.6). Note that this condition is reduced to $\gamma \leq 0$ if $\mathbf{\Gamma}_2 = 0$.

LEMMA 3.1. *Let $q \in V$, $l \in L^2(\Omega)^3$, and $m \in L^2(\Omega)$. If $y \in W^N$ satisfies the relation* (3.8), *then for sufficiently large p_N or small h_N,*

$$\|y\|_0 \leq Q(h_N^{1/2} p_N^{-1/2} \|q\|_0 + h_N^{3/2} p_N^{-2} \|\mathrm{div} q\|_0 + \|l\|_0 + \|m\|_0)$$

under the structure condition (3.6).

To prove the above Lemma, we use a duality argument.

LEMMA 3.2. *If $q \in V^N$ satisfies* (3.8), *then*

$$\|q\|_0 + \|\mathrm{div} q\|_0 \leq C(\|y\|_0 + \|l\|_0 + \|m\|_0).$$

Proof. To bound $\|q\|_0$, choose $v = q$, $w = y$ in (3.8) and add the resulting equations. The choice $w = \mathrm{div}\, q$ in (3.8)(b) gives the bound for $\|\mathrm{div}\, q\|_0$. □

LEMMA 3.3. *There exists one and only one solution of the system* (3.7) *under the structure condition* (3.6).

Proof. Existence follows from uniqueness since the system is linear. Assume $l = 0, m = 0$. Then Lemma 3.1 implies

$$\|y\|_0 \leq Q h_N^{1/2} p_N^{-1/2} \|q\|_V,$$

where $\|q\|_V = \|q\|_0 + \|\mathrm{div}\ q\|_0$. By Lemma 3.2, we have

$$\|q\|_V \leq c\|y\|_0 \Rightarrow \|y\|_0 \leq Q h_N^{1/2} p_N^{-1/2} \|q\|_V \leq Q h_N^{1/2} p_N^{-1/2} \|y\|_0,$$

which implies $y = 0$ for large p_N, or small h_N. Then, $q = 0$. □

Now it is clear from Theorem 3.3 that the map Φ defined via (3.7) is well-defined.

3.1. Existence and uniqueness

The solvability of (3.3) is now equivalent to showing that Φ has a fixed point.

THEOREM 3.1. *For p_N sufficiently large or h_N sufficiently small, Φ has a fixed point.*

To prove this, we shall need the following duality lemma.

LEMMA 3.4. *Let $\omega \in V$, $l \in L^2(\Omega)^3$, and $m \in L^2(\Omega)$. If $\tau \in W^N$ satisfies the relation*

$$(3.9)\qquad \begin{aligned} &(B(u,\sigma)\omega, v) - (\text{div}v, \tau) + (\mathbf{\Gamma}_1\tau, v) = (l, v) \quad \forall v \in V^N, \\ &(\text{div}\omega, w) - (\mathbf{\Gamma}_2\omega, w) - (\gamma\tau, w) = (m, w) \quad \forall w \in W^N, \end{aligned}$$

then there exists a constant $C > 0$, independent of p_N, h_N, such that

$$\|\tau\|_0 \leq C[h_N^{1/2} p_N^{-1/2}\|\omega\|_0 + h_N^{3/2} p_N^{-2}\|\text{div}\omega\|_0 + \|l\|_0 + \|m\|_0],$$

where $C = C(\theta, u, \sigma, \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \gamma, \Omega, \varepsilon)$.

Now let $\mathcal{V}^N = V^N$ with the stronger norm $\|v\|_{\mathcal{V}^N} = \|v\|_{0,4} + \|\text{div }v\|_0$, and let $\mathcal{W}^N = W^N$ with the stronger norm $\|w\|_{\mathcal{W}^N} = \|w\|_{0,4}$. We can now prove the existence of a solution of (3.3).

It follows from the Brouwer fixed point theorem that Theorem 3.1 is true, if we can show the following.

THEOREM 3.2. *For $\delta > 0$ sufficiently small (dependent of p_N and h_N), Φ maps the ball of radius δ of $\mathcal{W}^N \times \mathcal{V}^N$, centered at $(p^N u, \pi^N\sigma)$, into itself, provided the structure condition (3.6) holds.*

Proof. Let $\|\pi^N\sigma - \rho\|_{\mathcal{V}^N} \leq \delta$ and $\|P^N u - \mu\|_{\mathcal{W}^N} \leq \delta$. We apply Lemma 3.4 with $\tau = P^N u - y$, $\omega = \pi^N\sigma - q$,

$$l = B(u,\sigma)[\pi^N\sigma - \sigma] + \mathbf{\Gamma}_1[P^N u - u] + \tilde{Q}_b(u - \mu, \sigma - \rho)$$

and

$$m = -\mathbf{\Gamma}_2[\pi^N\sigma - \sigma] - \gamma[P^N u - u] - \tilde{Q}_f(u - \mu, \sigma - \rho).$$

Using Lemma 2.8, Lemma 2.9 and $2ab \leq a^2 + b^2$,

$$(3.10)\qquad \begin{aligned} \|P^N u - y\|_0 &\leq [h_N^{1/2} p_N^{-1/2}\|\pi^N\sigma - q\|_0 \\ &\quad + h_N^{3/2} p_N^{-2}\|\text{div }(\pi^N\sigma - q)\|_0 + \|l\|_0 + \|m\|_0] \\ &\leq C[\delta^2 + h_N^{r-1/2} p_N^{1/2-r}] \end{aligned}$$

for sufficiently large p_N with $r > 7/2$, where $C = C(\|\sigma\|_r, \|u\|_m, \mathbf{\Gamma}_1, \tilde{Q}_b, \tilde{Q}_f)$.

By applying Lemma 3.2 to (3.9), we have

$$\begin{aligned} &\|\pi^N\sigma - q\|_0 + \|\text{div }(\pi^N\sigma - q)\| \\ (3.11) \qquad &\leq C[\|P^N u - y\|_0 + \|l\|_0 + \|m\|_0] \\ &\leq C[h_N^{r-1/2}p_N^{1/2-r} + \delta^2]. \end{aligned}$$

Using Lemma 2.10, we have,

$$\|\pi^N\sigma - q\|_{\mathcal{V}^N} \leq Ch_N^{-3/4}p_N^{3/2}[h_N^{r-1/2}p_N^{1/2-r} + \delta^2],$$

while (3.10) and (3.11) imply that

$$\|P^N u - y\|_{\mathcal{W}} \leq Ch_N^{-3/4}p_N^{3/2}[\delta^2 + h_N^{r-1/2}p_N^{1/2-r}].$$

Therefore,

$$\|P^N u - y\|_{\mathcal{W}^N} + \|\pi^N\sigma - q\|_{\mathcal{V}^N} \leq Ch_N^{-3/4}p_N^{3/2}[\delta^2 + h_N^{r-1/2}p_N^{1/2-r}],$$

where $C = C(\|\sigma\|_r, \|u\|_m, \mathbf{\Gamma}_1, \tilde{Q}_b, \tilde{Q}_f)$. We want to choose p_N, h_N, and δ so that $I = [2C_1h_N^{r-5/4}p_N^{2-r}, \frac{1}{2C_1}h_N^{3/4}p_N^{-3/2}]$ is not empty with $r = 7/2 + \varepsilon$. Then, for $\delta = 2C_1h_N^{r-5/4}p_N^{2-r}$, we have $\|P^N u - y\|_{\mathcal{W}^N} \leq \delta$ and $\|\pi^N\sigma - q\|_{\mathcal{V}^N} \leq \delta$. □

We shall prove a uniqueness result provided that the coefficients a_i, $i = 0, 1, 2$ of (1.1) are three times continuously differentiable.

THEOREM 3.3. *If p_N is sufficiently large or h_N sufficiently small, there is a unique solution of (3.3) near the solution $\{u, \sigma\}$ of (3.2) under the structure condition (3.6).*

Proof. Let $(u_i^N, \sigma_i^N) \in \mathcal{W}^N \times \mathcal{V}^N$, $i = 1, 2$ be the solution of (3.3) and let

$$U = u_1^N - u_2^N, \quad \mathbf{\Sigma} = \sigma_1^N - \sigma_2^N, \quad \zeta^i = \sigma - \sigma_i^N, \quad \xi^i = u - u_i^N; \quad i = 1, 2.$$

Rewrite (3.3) as

$$\begin{aligned} &(b(u,\sigma) - b(u_2^N, \sigma_2^N), v) - (\text{div } v, U) \\ = &\ (b(u,\sigma) - b(u_1^N, \sigma_1^N), v), \ \forall v \in \mathcal{V}^N, \\ &(\text{div}\mathbf{\Sigma}, w) = (f(u_1^N, \sigma_1^N) - f(u_2^N, \sigma_2^N), w), \ \forall w \in \mathcal{W}^N. \end{aligned}$$

By the linearization, we obtain

$$\begin{aligned}&(B(u,\sigma)\mathbf{\Sigma}, v) - (\text{div } v, U) + (\mathbf{\Gamma}_1 U, v)\\ &= (\tilde{Q}_b(\xi^2,\zeta^2) - \tilde{Q}_b(\xi^1,\zeta^1), v),\ \forall v \in \mathcal{V}^N,\end{aligned}$$

$$\begin{aligned}&(\text{div } \mathbf{\Sigma}, w) - (\mathbf{\Gamma}_2 \cdot \Sigma, w) - (\gamma U, w)\\ &= (\tilde{Q}_b(\xi^2,\zeta^2) - \tilde{Q}_b(\xi^1,\zeta^1), w),\ \forall w \in \mathcal{W}^N.\end{aligned}$$

It follows from Lemma 3.2 that

$$\begin{aligned}\|\mathbf{\Sigma}\|_0 + \|\text{div } \mathbf{\Sigma}\|_0 \ \le\ & C[\|U\|_0 + \|\tilde{Q}_b(\xi^1,\zeta^1) - \tilde{Q}_b(\xi^2,\zeta^2)\|_0\\ &+\|\tilde{Q}_f(\xi^1,\zeta^1) - \tilde{Q}_f(\xi^2,\zeta^2)\|_0].\end{aligned}$$

Also, by Lemma 3.1, we see that

$$\begin{aligned}\|U\|_0 \ \le\ & Q\{h_N^{1/2} p_N^{-1/2}\|\mathbf{\Sigma}\|_0 + h_N^{3/2} p_N^{-2}\|\text{div } \mathbf{\Sigma}\|_0 + \|\tilde{Q}_b(\xi^1,\zeta^1)\\ &-\tilde{Q}_b(\xi^2,\zeta^2)\|_0 + \|\tilde{Q}_f(\xi^1,\zeta^1) - \tilde{Q}_f(\xi^1,\zeta^1)\|_0\}.\end{aligned}$$

We now want to show that

$$\begin{aligned}&\|\tilde{Q}_b(\xi^1,\zeta^1) - \tilde{Q}_b(\xi^2,\zeta^2)\|_0 \le C h_N^{2r-4} p_N^{7-2r}(\|\mathbf{\Sigma}\|_0 + \|U\|_0),\\ &\|\tilde{Q}_b(\xi^1,\zeta^1) - \tilde{Q}_b(\xi^2,\zeta^2)\|_0 \le C h_N^{2r-4} p_N^{7-2r}(\|\mathbf{\Sigma}\|_0 + \|U\|_0).\end{aligned} \tag{3.12}$$

To show (3.12), use the definition of the quadratic form $\tilde{Q}_f$[21]. It follows that

$$\begin{aligned}&\|\tilde{Q}_f(\xi^2,\zeta^2) - \tilde{Q}_f(\xi^1,\zeta^1)\|_0\\ &\le C[\|\xi^1\|_{0,\infty} + \|\xi^2\|_{0,\infty} + \|\xi^1\|_{0,\infty}^2 + \|\zeta^1\|_{0,\infty}\\ &\quad + \|\xi^1\|_{0,\infty}\|\zeta^1\|_{0,\infty} + \|\zeta^2\|_{0,\infty} + \|\zeta^2\|_{0,\infty}^2][\|\mathbf{\Sigma}\|_0 + \|U\|_0].\end{aligned}$$

By the inverse estimates (2.40), and the δ chosen,

$$\begin{aligned}\|P^N u - u^N\|_{0,\infty} &\le h_N^{-3/4} p_N^{3/2}\|P^N u - u^N\|_{0,4}\\ &\le h_N^{-3/4} p_N^{3/2}\delta\\ &\le K h_N^{r-2} p_N^{7/2-r},\end{aligned}$$

$$\begin{aligned}\|\pi^N \sigma - \sigma^N\|_{0,\infty} &\le h_N^{-3/4} p_N^{3/2}\|\pi^N \sigma - \sigma^N\|_{0,4}\\ &\le h_N^{-3/4} p_N^{3/2}\delta\\ &\le K h_N^{r-2} p_N^{7/2-r}.\end{aligned}$$

So, we have

$$\begin{aligned}\|\xi^i\|_{0,\infty} = \|u_i - u_i^N\|_{0,\infty} &\le \|u_i - P^N u_i\|_{0,\infty} + \|P^N u_i - u_i^N\|_{0,\infty} \\ &\le K\left[h_N^{m-2} p_N^{-m+5/2}\|u\|_m + h_N^{r-2} p_N^{7/2-r}\right] \\ &\to 0\end{aligned}$$

and

$$\begin{aligned}\|\zeta^i\|_{0,\infty} = \|\sigma_i - \sigma_i^N\|_{0,\infty} &\le \|\sigma_i - \pi^N \sigma_i\|_{0,\infty} + \|\pi^N \sigma_i - \sigma_i^N\|_{0,\infty} \\ &\le K\left[h_N^{r-2} p_N^{7/2-r}\|\sigma\|_r + h_N^{r-2} p_N^{7/2-r}\right] \\ &\to 0\end{aligned}$$

as $p_N \to \infty$ and $h_N \to 0$. We have now shown (3.12). By Lemmas 3.1 and 3.2, this concludes the proof of the theorem. □

3.2. L^2-error estimates

In this subsection we will estimate the error of the solution of the h-p–version for the problem (1.1).

THEOREM 3.4. *If the solution of* (1.1) *is regular enough that* $u \in H^{r+1}(\Omega)$ *and* $\sigma \in H^r(\Omega)^3$, *where* $r \ge 7/2$, *then we have the following error estimates.*

$$\begin{aligned}\|u - u^N\|_W \le Q\{& h_N^{r-1/4} p_N^{1/4-r}\|\sigma\|_r(\|\sigma\|_r + 1) \\ &+ h_N^{m-5/4} p^{5/4-m}\|u\|_m(\|u\|_m + 1)\},\end{aligned}$$

$$\begin{aligned}\|\sigma - \sigma^N\|_V \le Q\{& h_N^{r-5/4} p_N^{2-r}\|\sigma\|_{r+1}(\|\sigma\|_r + 1) \\ &+ h_N^{m-5/4} p_N^{5/4-m}\|u\|_m(\|u\|_m + 1)\}.\end{aligned}$$

Proof. Let $\zeta = \sigma - \sigma^N$, $\xi = u - u^N$, $\theta = \pi^N\sigma - \sigma^N$, and $\tau = P^N u - u^N$. Rewrite (3.3) as

$$\begin{aligned}&(B(u,\sigma)\theta, v) - (\mathrm{div}\ v, \tau) + (\mathbf{\Gamma}_1\tau, v) \\ &= (B(u,\sigma)[\pi^N\sigma - \sigma] + \mathbf{\Gamma}_1[P^N u - u] + \tilde{Q}_b(\xi,\zeta), v) \quad \forall v \in V^N,\end{aligned}$$

$$\begin{aligned}&(\mathrm{div}\ \theta, w) - (\mathbf{\Gamma}_2\theta, w) - (\gamma\tau, w) \\ &= (-\mathbf{\Gamma}_2[\pi^N\sigma - \sigma] - \gamma[P^N u - u] - \tilde{Q}_f(\xi,\zeta), w) \quad \forall w \in W^N.\end{aligned}$$

Then, just as in Theorem 5.1 in [15]

$$\begin{aligned}
&\|\theta\|_0 + \|\text{div }\theta\|_0 \\
&\le C\{\|\tau\|_0 + \|\tau\|_{0,4}^2 + \|\theta\|_{0,4}^2 + \|\pi^N\sigma - \sigma\|_{0,4}^2 \\
&\quad + \|\pi^N\sigma - \sigma\|_0 + \|P^N u - u\|_{0,4}^2 + \|P^N u - u\|_0\} \\
&\le C\{\|\tau\|_0 + h_N^{-3/4} p_N^{3/2}\|\tau\|_{0,4}\|\tau\|_{0,2} + \|\theta\|_{0,4} h_N^{-3/4} p_N^{3/2}\|\theta\|_0 \\
&\quad + \|\pi^N\sigma - \sigma\|_{0,4}^2 + \|\pi^N\sigma - \sigma\|_0 + \|P^N u - u\|_{0,4}^2 + \|P^N u - u\|_0\} \\
&\le C\{\|\tau\|_0 + h_N^{r-2} p_N^{7/2-r}\|\tau\|_0 + h_N^{r-2} p_N^{7/2-r}\|\theta\|_0 \\
&\quad + \|\pi^N\sigma - \sigma\|_{0,4}^2 + \|\pi^N\sigma - \sigma\|_0 + \|P^N u - u\|_{0,4}^2 + \|P^N u - u\|_0\} \\
&\le C\{\|\tau\|_0 + h_N^{r-2} p_N^{7/2-r}\|\tau\|_0 + h_N^{r-2} p_N^{7/2-r}\|\theta\|_0 \\
&\quad + h_N^{r-5/4} p_N^{2-r}\|\sigma\|_r(\|\sigma\|_r + 1) + h_N^{m-5/4} p_N^{5/4-m}\|u\|_m(\|u\|_m + 1)\}.
\end{aligned}$$

By taking p_N large or h_N small,

$$\begin{aligned}
\|\theta\|_0 + \|\text{div }\theta\|_0 \le C\{\|\tau\|_0 + h_N^r p_N^{1/2-r}\|\sigma\|_r(\|\sigma\|_r + 1) \\
+ h_N^m p_N^{-m}\|u\|_m(\|u\|_m + 1)\}.
\end{aligned} \tag{3.13}$$

Likewise, by Lemma 3.1,

$$\begin{aligned}
\|\tau\|_0 \quad &\le \quad Q(h_N p_N^{-1/2}\|\theta\|_0 + h_N^2 p_N^{-2}\|\text{div }\theta\|_0 + \|l\|_0 + \|m\|_0) \\
&\le \quad Q\{h_N p_N^{-1/2}\|\theta\|_0 + h_N^2 p_N^{-2}\|\text{div }\theta\|_0 + h_N^{r-1} p_N^{5/2-r}\|\tau\|_0 \\
&\qquad + h_N^{r-1} p_N^{5/2-r}\|\theta\|_0 + h_N^r p_N^{1/2-r}\|\sigma\|_r(\|\sigma\|_r + 1) \\
&\qquad + h_N^m p_N^{-m}\|u\|(\|u\|_m + 1)\},
\end{aligned}$$

and, by taking p_N large or h_N small,

$$\begin{aligned}
\|\tau\|_0 \quad \le \quad & Q\{h_N p_N^{-1/2}\|\theta\|_0 + h_N^2 p_N^{-2}\|\text{div }\theta\|_0 + h_N^{r-1} p_N^{5/2-r}\|\tau\|_0 \\
& + h_N^{r-1} p_N^{5/2-r}\|\theta\|_0 + h_N^r p_N^{1/2-r}\|\sigma\|_r(\|\sigma\|_r + 1) \\
& + h_N^m p_N^{-m}\|u\|(\|u\|_m + 1)\}.
\end{aligned} \tag{26}$$

Substituting (3.14) into (3.13), we find

$$\|\theta\|_V \le Q\{h_N^r p_N^{1/2-r}\|\sigma\|_r(\|\sigma\|_r + 1) + h_N^m p_N^{-m}\|u\|_m(\|u\|_m + 1)\}, \tag{3.14}$$

and, substituting (3.14) into (3.14),

$$\|\tau\|_0 \le Q\{h_N^r p_N^{1/2-r}\|\sigma\|_r(\|\sigma\|_r + 1) + h_N^m p_N^{-m}\|u\|_m(\|u\|_m + 1)\}. \tag{3.15}$$

By combining (3.14), (3.15) with Lemmas 2.5 and 2.9, we have

$$\|u - u^N\|_W \leq Q\{h_N^{r-1/4} p_N^{1/4-r} \|\sigma\|_r(\|\sigma\|_r + 1) + h_N^{m-5/4} p^{5/4-m} \|u\|_m(\|u\|_m + 1)\},$$

$$\|\sigma - \sigma^N\|_V \leq Q\{h_N^{r-5/4} p_N^{2-r} \|\sigma\|_{r+1}(\|\sigma\|_r + 1) + h_N^{m-5/4} p_N^{5/4-m} \|u\|_m(\|u\|_m + 1)\}.$$

□

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