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ON THE HYERS-ULAM-RASSIAS STABILITY OF A GENERALIZED QUADRATIC EQUATION

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Abstract. In this paper we prove the stability of the generalized quadratic equation $f(x+y)+g(x-y)-2f(x)-2g(y)=0$ in the spirit of Hyers, Ulam and Rassias.

1. Introduction

In 1940, S. M. Ulam [22] had raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers [5] proved that if $f : V \to X$ is a mapping satisfying

$$\|f(x+y)-f(x)-f(y)\| \leq \delta$$

for all $x, y \in V$, where V and X are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T : V \to X$ such that

$$\|f(x)-T(x)\| \leq \delta$$

for all $x \in V$.

Throughout the paper, let V and X be a normed space and a Banach space respectively. Z. Gadja [3] and Th. M. Rassias [15] gave a generalization of the Hyers' result in the following way:

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THEOREM 1.1. *Let $f : V \to X$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \neq 1$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$(for all $x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique linear mapping $T : V \to X$ such that

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p$$

for all $x \in V$(for all $x \in V \setminus \{0\}$ if $p < 0$).

However, it was showed that a similar result for the case $p = 1$ does not hold([3,19]). Recently, Găvruta [4] also obtained a further generalization of the Hyers-Rassias's theorem([6, 7, 8, 9, 11, 14, 16]).

Lee and Jun [12, 13] obtained the Hyers-Ulam-Rassias stability of the Pexider equation of $f(x+y) = g(x) + h(y)$ ([10]):

THEOREM 1.2. *Let $f, g, h : V \to X$ be mappings. Assume that there exist $\theta \geq 0$ and $p \in [0, \infty)\setminus\{1\}$ such that*

$$\|f(x+y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in V$. Then there exists a unique additive mapping $T : V \to X$ such that

$$\|T(x) - f(x) + f(0)\| \leq \frac{4\theta}{|2^p - 2|}\|x\|^p + M$$

$$\|T(x) - g(x) + g(0)\| \leq \frac{(4 + 2^p)\theta}{|2^p - 2|}\|x\|^p + M$$

$$\|T(x) - h(x) + h(0)\| \leq \frac{(4 + 2^p)\theta}{|2^p - 2|}\|x\|^p + M$$

where $M = \|f(0) - g(0) - h(0)\|$ (if $1 < p$ then $M = 0$).

In 1983, the stability theorem for the quadratic functional equation

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0 \tag{1.1}$$

was proved F. Skof [21] for the function $f : V \to X$. In 1984, P. W. Cholewa [1] extended V by an Abelian group G in the Skof's result. We define any solution of (1.1) to be a quadratic function.

In 1992, S. Czerwik [2] gave a generalization of the Skof-Cholewa's result in the following way:

THEOREM 1.3. *Let $p(\neq 2)$, $\theta > 0$ be real numbers. Suppose that the function $f : V \to X$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

Then there exists exactly one quadratic function $g : V \to X$ such that

$$\|f(x) - g(x)\| \leq c + k\theta\|x\|^p$$

for all x in V if $p \geq 0$ and for all $x \in V \setminus \{0\}$ if $p < 0$, where: when $p < 2$, $c = \frac{\|f(0)\|}{3}$, $k = \frac{2}{4-2^p}$ and g is given by $g(x) = \lim_{n\to\infty} 4^{-n} f(2^n x)$ for all x in V. When $p > 2$, $c = 0$, $k = \frac{2}{2^p-4}$ and $g(x) = \lim_{n\to\infty} 4^n f(2^{-n}x)$ for all x in V. Also, if the mapping $t \to f(tx)$ from R to X is continuous for each fixed x in V, then $g(tx) = t^2 g(x)$ for all t in R.

Since then, the stability problem of the quadratic equation have been extensively investigated by a number of mathematician([17, 18, 20]). In this paper, we prove the stability of the generalized quadratic equation :

$$f(x+y) + g(x-y) - 2f(x) - 2g(y) = 0.$$

2. Main result

THEOREM 2.1. *Let $p < 0$, $\theta > 0$ be real numbers. Suppose that the functions $f, g : V \to X$ satisfy*

$$\|f(x+y) + g(x-y) - 2f(x) - 2g(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.1}$$

for all $x, y \in V \setminus \{0\}$. Then there exists exactly one quadratic function $Q : V \to X$ such that

$$\begin{aligned} &\|f(x) + g(y) - Q(x) - Q(y)\| \\ &\leq \frac{4\theta}{4-2^p}(\|x+y\|^p + \|x-y\|^p + 2\|x\|^p + 2\|y\|^p) \end{aligned} \tag{2.2}$$

for all $x, y \in V \setminus \{0\}$ with $x+y, x-y \in V \setminus \{0\}$. The function is given by

$$\begin{aligned} Q(x) &= \lim_{n\to\infty} \frac{f(2^n x)}{4^n} = \lim_{n\to\infty} \frac{f(-2^n x)}{4^n} \\ &= \lim_{n\to\infty} \frac{g(2^n x)}{4^n} = \lim_{n\to\infty} \frac{g(-2^n x)}{4^n} \end{aligned} \tag{2.3}$$

for all $x \in V$.

Proof. If $f_1 = \frac{1}{\theta} f$ and $g_1 = \frac{1}{\theta} g$, then f_1 and g_1 satisfy

$$\begin{aligned} &\|f_1(x+y) + g_1(x-y) - 2f_1(x) - 2g_1(y)\| \\ &\leq \|x\|^p + \|y\|^p \qquad \text{for all} \quad x, y \in V \setminus \{0\} \end{aligned}$$

from (2.1). Hence we may assume that $\theta = 1$ without the loss of generality. From (2.1), we easily obtain

$$\|f(2x) + g(0) - 2f(x) - 2g(x)\| \leq 2\|x\|^p, \tag{2.4}$$

$$\|f(0) + g(2x) - 2f(x) - 2g(-x)\| \leq 2\|x\|^p, \tag{2.5}$$

$$\|f(0) + g(-2x) - 2f(-x) - 2g(x)\| \leq 2\|x\|^p, \quad \text{and} \tag{2.6}$$

$$\|f(-2x) + g(0) - 2f(-x) - 2g(-x)\| \leq 2\|x\|^p \tag{2.7}$$

for all $x \in V \setminus \{0\}$. Let $U(x) = f(x) + f(-x) + g(x) + g(-x)$. From (2.4), (2.5), (2.6), and (2.7), we get

$$\|\frac{U(2x)}{4} - U(x)\| \leq 2\|x\|^p + \frac{1}{2}\|f(0) + g(0)\|$$

for all $x \in V \setminus \{0\}$.

Applying an induction argument to n, we have

$$\|\frac{U(2^{n+1}x)}{4^{n+1}} - \frac{U(2^n x)}{4^n}\| \leq \frac{2 \cdot 2^{np}}{4^n}\|x\|^p + \frac{1}{2 \cdot 4^n}\|f(0) + g(0)\|$$

for all $x \in V \setminus \{0\}$. Hence

$$\begin{aligned} \|\frac{U(2^n x)}{4^n} - \frac{U(2^m x)}{4^m}\| &\leq \sum_{i=n}^{m-1} \|\frac{U(2^{i+1}x)}{4^{i+1}} - \frac{U(2^i x)}{4^i}\| \\ &\leq \sum_{i=n}^{m-1} (\frac{2 \cdot 2^{ip}}{4^i}\|x\|^p + \frac{1}{2 \cdot 4^i}\|f(0) + g(0)\|) \\ &\leq \frac{8 \cdot 2^{np}}{4^n(4 - 2^p)}\|x\|^p + \frac{2}{3 \cdot 4^n}\|f(0) + g(0)\| \end{aligned}$$

for all $m > n$ and $x \in V \setminus \{0\}$. This shows that $\{\frac{U(2^n x)}{4^n}\}$ is a Cauchy sequence. Since X is a Banach space, the sequence $\{\frac{U(2^n x)}{4^n}\}$ converges. Define $Q : V \to X$ by

$$4Q(x) = \lim_{n \to \infty} \frac{f(2^n x) + f(-2^n x) + g(2^n x) + g(-2^n x)}{4^n} \tag{2.8}$$

for all $x \in V$. From (2.4), (2.5), (2.6), and (2.7), we have the inequality

$$\begin{aligned} &\frac{1}{4^n}\|f(2^n x) + f(-2^n x) - g(2^n x) - g(-2^n x)\| \\ &\le \frac{2 \cdot 2^{(n-1)p}}{4^{n-1}}\|x\|^p + \frac{2\|g(0) - f(0)\|}{4^n} \end{aligned}$$

for all $x \in V \setminus \{0\}$. Hence

$$\lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x) - g(2^n x) - g(-2^n x)}{4^n} = 0. \tag{2.9}$$

From (2.8) and (2.9), we have

$$\begin{aligned} 2Q(x) &= \lim_{n\to\infty} \frac{f(2^n x) + f(-2^n x)}{4^n} \\ &= \lim_{n\to\infty} \frac{g(2^n x) + g(-2^n x)}{4^n} \end{aligned} \tag{2.10}$$

for all $x \in V \setminus \{0\}$. Replacing x by $x + y$, y by $x - y$ and dividing by 2 in (2.1), we have

$$\begin{aligned} &\|\frac{1}{2}f(2x) + \frac{1}{2}g(2y) - f(x+y) - g(x-y)\| \\ &\le \frac{1}{2}(\|x+y\|^p + \|x-y\|^p) \end{aligned} \tag{2.11}$$

where $x + y, x - y \in V \setminus \{0\}$. From (2.1) and (2.11), we have

$$\begin{aligned} &\|f(x) + g(y) - \frac{1}{4}(f(2x) + g(2y))\| \\ &\le \frac{1}{4}(\|x+y\|^p + \|x-y\|^p) + \frac{1}{2}(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $x, y \in V \setminus \{0\}$ with $x + y, x - y \in V \setminus \{0\}$. Induction argument implies

$$\begin{aligned} &\|f(x) + g(y) - \frac{1}{4^n}(f(2^n x) + g(2^n y))\| \\ &\le \frac{1}{4 - 2^p}(\|x+y\|^p + \|x-y\|^p + 2\|x\|^p + 2\|y\|^p). \end{aligned} \tag{2.12}$$

Hence

$$\begin{aligned}&\|\frac{1}{4^m}f(2^m x)+g(2^m y)-\frac{1}{4^{m+n}}(f(2^{m+n}x)+g(2^{m+n}y))\|\\ &\le \frac{2^{mp}}{4^m(4-2^p)}(\|x+y\|^p+\|x-y\|^p+2\|x\|^p+2\|y\|^p)\end{aligned}$$

for all $m,n\in N$ and $x,y\in V\setminus\{0\}$ with $x+y, x-y\in V\setminus\{0\}$. This shows that $\{\frac{1}{4^m}f(2^m x)+g(2^m y)\}$ is a Cauchy sequence and thus converges. Let

$$R(x,y)=\lim_{n\to\infty}\frac{f(2^n x)+g(2^n y)}{4^n}$$

for all $x,y\in V\setminus\{0\}$ with $x+y,x-y\in V\setminus\{0\}$. By the definition of $R(x,y)$ and (2.10), we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{f(2^n x)}{4^n}&=\frac{1}{2}(R(x,2x)-R(-x,2x)+\lim_{n\to\infty}\frac{f(2^n x)+f(-2^n x)}{4^n})\\ &=\frac{1}{2}(R(x,2x)-R(-x,2x)+2Q(x))\end{aligned}$$

for all $x\in V\setminus\{0\}$. Similarly we obtain

$$\lim_{n\to\infty}\frac{g(2^n x)}{4^n}=\frac{1}{2}(R(2x,x)-R(2x,-x)+2Q(x))$$

for all $x\in V\setminus\{0\}$. Replacing x by $2^n x$ and dividing by 4^n in (2.4), we have

$$\|\frac{f(2^{n+1}x)}{4^n}+\frac{g(0)}{4^n}-\frac{2f(2^n x)}{4^n}-\frac{2g(2^n x)}{4^n}\|\le\frac{2\cdot 2^{np}}{4^n}\|x\|^p \tag{2.13}$$

for all $n\in N$ and $x\in V\setminus\{0\}$. From (2.13), we have

$$\begin{aligned}&2\lim_{n\to\infty}\frac{f(2^n x)}{4^n}-2\lim_{n\to\infty}\frac{g(2^n x)}{4^n}\\ &=\lim_{n\to\infty}\frac{f(2^{n+1}x)}{4^n}-2\lim_{n\to\infty}\frac{f(2^n x)}{4^n}-2\lim_{n\to\infty}\frac{g(2^n x)}{4^n}\\ &=\ 0\end{aligned}$$

for all $x\in V\setminus\{0\}$. From this, we get

$$\lim_{n\to\infty}\frac{f(2^n x)}{4^n}=\lim_{n\to\infty}\frac{g(2^n x)}{4^n}\quad\text{for all}\quad x\in V. \tag{2.14}$$

Replacing x by $2^n x$ and dividing by 4^n in (2.5), we have

$$\|\frac{f(0)}{4^n} + \frac{g(2^{n+1}x)}{4^n} - \frac{2f(2^n x)}{4^n} - \frac{2g(-2^n x)}{4^n})\| \le \frac{2 \cdot 2^{np}}{4^n}\|x\|^p$$

for all $n \in N$ and $x \in V \setminus \{0\}$. From this and (2.14), we obtain

$$\lim_{n\to\infty} \frac{g(2^n x)}{4^n} = \lim_{n\to\infty} \frac{g(-2^n x)}{4^n} \tag{2.15}$$

for all $x \in V$. From (2.10), (2.14) and (2.15), we have

$$\begin{aligned} \lim_{n\to\infty} \frac{f(2^n x)}{4^n} &= \lim_{n\to\infty} \frac{f(-2^n x)}{4^n} \\ &= \lim_{n\to\infty} \frac{g(2^n x)}{4^n} \\ &= \lim_{n\to\infty} \frac{g(-2^n x)}{4^n} \\ &= Q(x) \end{aligned} \tag{2.16}$$

for all $x \in V$. From the definition of $Q(x)$ and (2.16), we obtain

$$Q(2x) = 4Q(x) \quad \text{and} \quad Q(x) = Q(-x) \tag{2.17}$$

for all $x \in V$. From (2.1), (2.16), and (2.17), we obtain similarly

$$Q(x+y) + Q(x-y) - 2Q(x) - 2Q(y) = 0$$

for all $x, y \in V$. From (2.12) and (2.16), we can easily obtain the inequality (2.2).

Now we have to prove the uniqueness. If Q' is an another quadratic function satisfying (2.2), then

$$\begin{aligned} &\|5Q(x) - 5Q'(x)\| \\ &\le \|\frac{Q(2^n x)}{4^n} + \frac{Q(2^n \cdot 2x)}{4^n} - \frac{f(2^n x)}{4^n} - \frac{g(2^n \cdot 2x)}{4^n}\| \\ &\quad + \|\frac{f(2^n x)}{4^n} + \frac{g(2^n \cdot 2x)}{4^n} - \frac{Q'(2^n x)}{4^n} - \frac{Q'(2^n \cdot 2x)}{4^n}\| \\ &\le 2 \cdot \frac{4}{4-2^p} \cdot \frac{2^{np}}{4^n}(\|x\|^p + \|3x\|^p + 2\|x\|^p + 2\|2x\|^p) \end{aligned}$$

for all $n \in N$ and $x \in V \setminus \{0\}$. Therefore

$$Q(x) = Q'(x) \quad \text{for all} \quad x \in V. \qquad \square$$

THEOREM 2.2. *Let $p < 2$, $\theta > 0$ be real numbers. Let $\psi : V \to [0, \infty)$ be a mapping such that*

$$\psi(x) = \|x\|^p \quad \text{for} \quad x \neq 0.$$

Suppose that the functions $f, g : V \to X$ satisfy

$$\|f(x+y) + g(x-y) - 2f(x) - 2g(y)\| \leq \theta(\psi(x) + \psi(y)) \tag{2.18}$$

for all $x, y \in V$. Then there exists exactly one quadratic function $Q : V \to X$ such that

$$\|f(x) + g(0) - Q(x)\| \leq \frac{4\theta}{4 - 2^p}\psi(x) + \frac{2}{3}\theta\psi(0)$$

for all $x \in V$ and

$$\|g(x) + f(0) - Q(x)\| \leq \frac{4\theta}{4 - 2^p}\psi(x) + \frac{2}{3}\theta\psi(0)$$

for all $x \in V$. And then, the function Q is given by (2.3).

Proof. We may assume that $\theta = 1$ without the loss of generality. By the same method as in the proof of Theorem 2.1, we obtain the unique quadratic function $Q : V \to X$ satisfying (2.2) and (2.3). From (2.18), we have

$$\begin{aligned} &\|f(x) + g(y) - \frac{1}{4}(f(2x) + g(2y))\| \\ &\leq \frac{1}{4}(\psi(x+y) + \psi(x-y)) + \frac{1}{2}(\psi(x) + \psi(y)) \end{aligned} \tag{2.19}$$

for all $x, y \in V$. Replacing $y = 0$ on the both sides of (2.19), we obtain

$$\|f(x) + g(0) - \frac{1}{4}(f(2x) + g(0))\| \leq \psi(x) + \frac{1}{2}\psi(0)$$

for all $x \in V$. Applying an induction argument to n, we get

$$\|f(x) + g(0) - \frac{1}{4^n}(f(2^n x) + g(0))\| \leq \frac{4}{4 - 2^p}\psi(x) + \frac{2}{3}\psi(0)$$

for all $n \in N$ and $x \in V$. From this, we get

$$\|f(x) + g(0) - Q(x)\| \leq \frac{4}{4 - 2^p}\psi(x) + \frac{2}{3}\psi(0)$$

for all $x \in V$. Similarly, we have

$$\|g(x) + f(0) - Q(x)\| \leq \frac{4}{4 - 2^p}\psi(x) + \frac{2}{3}\psi(0)$$

for all $x \in V$. □

THEOREM 2.3. *Let $p > 2$, $\theta > 0$ be real numbers. Suppose that the functions $f, g : V \to X$ satisfy*

$$\|f(x+y) + g(x-y) - 2f(x) - 2g(y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.20}$$

for all $x, y \in V$. Then there exists exactly one quadratic function $Q : V \to X$ such that

$$\|f(x) - f(0) - Q(x)\| \le \frac{4\theta}{2^p - 4}\|x\|^p$$

for all $x \in V$ and

$$\|g(x) - g(0) - Q(x)\| \le \frac{4\theta}{2^p - 4}\|x\|^p$$

for all $x \in V$. The function is given by

$$Q(x) = \lim_{n\to\infty} 4^n(f(2^{-n}x) - f(0)) = \lim_{n\to\infty} 4^n(g(2^{-n}x) - g(0))$$

for all $x \in V$.

Proof. We may assume that $\theta = 1$ without the loss of generality. From (2.20), we easily obtain

$$\|f(0) + g(0)\| = 0, \tag{2.21}$$

$$\|f(2x) + g(0) - 2f(x) - 2g(x)\| \le 2\|x\|^p, \tag{2.22}$$

$$\|f(0) + g(2x) - 2f(x) - 2g(-x)\| \le 2\|x\|^p, \tag{2.23}$$

$$\|f(x) + g(-x) - 2f(0) - 2g(x)\| \le \|x\|^p, \quad \text{and} \tag{2.24}$$

$$\|f(x) + g(x) - 2f(x) - 2g(0)\| \le \|x\|^p. \tag{2.25}$$

From (2.21), (2.22), and (2.25), we get

$$\begin{aligned}
&\|f(2x) - f(0) - 4(f(x) - f(0))\| \\
&\le \|f(2x) + g(0) - 4f(x) - 4g(0)\| + 3\|f(0) + g(0)\| \\
&\le \|f(2x) + g(0) - 2f(x) - 2g(x)\| \\
&\quad + 2\|f(x) + g(x) - 2f(x) - 2g(0)\| \\
&\le 4\|x\|^p.
\end{aligned} \tag{2.26}$$

Applying an induction argument to n, we have

$$\|(4^n(f(2^{-n}x) - f(0)) - 4^{n+1}(f(2^{-n-1}x) - f(0))\| \le \frac{4^{n+1}}{2^{(n+1)p}}\|x\|^p.$$

Hence

$$\|f(x) - f(0) - 4^n(f(2^{-n}x) - f(0))\| \le \frac{4}{2^p - 4}\|x\|^p. \tag{2.27}$$

Replacing x by $2^{-m}x$ and multiplying by 4^m in (2.27), we have

$$\begin{aligned}&\|4^m(f(2^{-m}x) - f(0)) - 4^{m+n}(f(2^{-m-n}x) - f(0))\|\\ &\le \frac{4^{m+1}}{2^{mp}(2^p - 4)}\|x\|^p\end{aligned}$$

for all $m, n \in N$. This shows that $\{4^n(f(2^{-n}x) - f(0))\}$ is a Cauchy sequence and thus converges from the completeness of X. Define $Q : V \to X$ by

$$Q(x) = \lim_{n\to\infty} 4^n(f(2^{-n}x) - f(0)).$$

From (2.27) and the definition of Q, we have the inequality

$$\|f(x) - f(0) - Q(x)\| \le \frac{4}{2^p - 4}\|x\|^p.$$

Replacing x by $2^{-n-1}x$ and multiplying by 4^n in (2.22), we have

$$\begin{aligned}&\|4^n(f(2^{-n}x) - f(0)) - 2\cdot 4^n(f(2^{-n-1}x) - f(0))\\ &\quad - 2\cdot 4^n(g(2^{-n-1}x) - g(0))\|\\ &= \|4^n(f(2^{-n}x) + g(0)) - 2f(2^{-n-1}x) - 2g(2^{-n-1}x))\|\\ &\quad + 4^n\|g(0) + f(0)\|\\ &\le \frac{2\cdot 4^n}{2^{(n+1)p}}\|x\|^p.\end{aligned} \tag{2.28}$$

Taking the limit in (2.28) as $n \to \infty$, we have

$$Q(x) = \lim_{n\to\infty} 4^n(g(2^{-n}x) - g(0)).$$

By the similar method as in (2.26) and (2.27), we obtain

$$\|g(x) - g(0) - Q(x)\| \le \frac{4}{2^p - 4}\|x\|^p$$

from (2.21), (2.23), and (2.24). The rest of proof is similar to the proof of Theorem 2.1. □

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