ON THE HYERS-ULAM-RASSIAS STABILITY
OF A GENERALIZED QUADRATIC EQUATION

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ABSTRACT. In this paper we prove the stability of the generalized quadratic equation $f(x + y) + g(x - y) - 2f(x) - 2g(y) = 0$ in the spirit of Hyers, Ulam and Rassias.

1. Introduction

In 1940, S.M. Ulam [22] had raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

In 1941, Hyers [5] proved that if $f : V \rightarrow X$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in V$, where $V$ and $X$ are Banach spaces and $\delta$ is a given positive number, then there exists a unique additive mapping $T : V \rightarrow X$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in V$.

Throughout the paper, let $V$ and $X$ be a normed space and a Banach space respectively. Z. Gadja [3] and Th. M. Rassias [15] gave a generalization of the Hyers' result in the following way:

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Theorem 1.1. Let \( f : V \to X \) be a mapping such that \( f(tx) \) is continuous in \( t \) for each fixed \( x \). Assume that there exist \( \theta \geq 0 \) and \( p \neq 1 \) such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in V \) (for all \( x, y \in V \setminus \{0\} \) if \( p < 0 \)). Then there exists a unique linear mapping \( T : V \to X \) such that
\[
\|T(x) - f(x)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p
\]
for all \( x \in V \) (for all \( x \in V \setminus \{0\} \) if \( p < 0 \)).

However, it was showed that a similar result for the case \( p = 1 \) does not hold([3,19]). Recently, Găvruţa [4] also obtained a further generalization of the Hyers-Rassias’s theorem([6, 7, 8, 9, 11, 14, 16]).

Lee and Jun [12, 13] obtained the Hyers-Ulam-Rassias stability of the Pexider equation of \( f(x + y) = g(x) + h(y) \) ([10]):

Theorem 1.2. Let \( f, g, h : V \to X \) be mappings. Assume that there exist \( \theta \geq 0 \) and \( p \in [0, \infty) \setminus \{1\} \) such that
\[
\|f(x + y) - g(x) - h(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]
for all \( x, y \in V \). Then there exists a unique additive mapping \( T : V \to X \) such that
\[
\|T(x) - f(x) + f(0)\| \leq \frac{4\theta}{|2^p - 2|}\|x\|^p + M
\]
\[
\|T(x) - g(x) + g(0)\| \leq \frac{(4 + 2^p)\theta}{|2^p - 2|}\|x\|^p + M
\]
\[
\|T(x) - h(x) + h(0)\| \leq \frac{(4 + 2^p)\theta}{|2^p - 2|}\|x\|^p + M
\]
where \( M = \|f(0) - g(0) - h(0)\| \) (if \( 1 < p \) then \( M = 0 \)).

In 1983, the stability theorem for the quadratic functional equation
\[
(1.1) \quad f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0
\]
was proved F. Skof [21] for the function \( f : V \to X \). In 1984, P. W. Cholewa [1] extended \( V \) by an Abelian group \( G \) in the Skof’s result. We define any solution of (1.1) to be a quadratic function.

In 1992, S. Czerwik [2] gave a generalization of the Skof-Cholewa’s result in the following way:
Theorem 1.3. Let \( p(\neq 2), \theta > 0 \) be real numbers. Suppose that the function \( f : V \to X \) satisfies
\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \leq \theta (\|x\|^p + \|y\|^p).
\]
Then there exists exactly one quadratic function \( g : V \to X \) such that
\[
\|f(x) - g(x)\| \leq c + k\theta \|x\|^p
\]
for all \( x \) in \( V \) if \( p \geq 0 \) and for all \( x \in V \setminus \{0\} \) if \( p < 0 \), where: when \( p < 2 \),
\[
c = \frac{\|f(0)\|}{3}, k = \frac{2}{4 - 2p} \quad \text{and} \quad g(x) = \lim_{n \to \infty} 4^{-n} f(2^n x) \quad \text{for all } x \in V.
\]
When \( p > 2 \), \( c = 0 \), \( k = \frac{2}{2p - 3} \) and \( g(x) = \lim_{n \to \infty} 4^n f(2^{-n} x) \) for all \( x \) in \( V \). Also, if the mapping \( t \to f(t x) \) from \( R \) to \( X \) is continuous for each fixed \( x \) in \( V \), then \( g(t x) = t^2 g(x) \) for all \( t \) in \( R \).

Since then, the stability problem of the quadratic equation have been extensively investigated by a number of mathematician([17, 18, 20]). In this paper, we prove the stability of the generalized quadratic equation:

\[
f(x + y) + g(x - y) - 2f(x) - 2g(y) = 0.
\]

2. Main result

Theorem 2.1. Let \( p < 0, \theta > 0 \) be real numbers. Suppose that the functions \( f, g : V \to X \) satisfy
\[
(2.1) \quad \|f(x + y) + g(x - y) - 2f(x) - 2g(y)\| \leq \theta (\|x\|^p + \|y\|^p)
\]
for all \( x, y \in V \setminus \{0\} \). Then there exists exactly one quadratic function \( Q : V \to X \) such that
\[
\|f(x) + g(y) - Q(x) - Q(y)\|
\]
\[
\leq \frac{4\theta}{4 - 2p} (\|x + y\|^p + \|x - y\|^p + 2\|x\|^p + 2\|y\|^p)
\]
for all \( x, y \in V \setminus \{0\} \) with \( x + y, x - y \in V \setminus \{0\} \). The function is given by
\[
Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} = \lim_{n \to \infty} \frac{f(-2^n x)}{4^n}
\]
\[
= \lim_{n \to \infty} \frac{g(2^n x)}{4^n} = \lim_{n \to \infty} \frac{g(-2^n x)}{4^n}
\]
for all \( x \in V \).
Proof. If $f_1 = \frac{1}{\theta} f$ and $g_1 = \frac{1}{\theta} g$, then $f_1$ and $g_1$ satisfy

$$\|f_1(x + y) + g_1(x - y) - 2f_1(x) - 2g_1(y)\| \leq \|x\|^p + \|y\|^p \quad \text{for all } x, y \in V \setminus \{0\}$$

from (2.1). Hence we may assume that $\theta = 1$ without the loss of generality. From (2.1), we easily obtain

\begin{align*}
(2.4) \quad & \|f(2x) + g(0) - 2f(x) - 2g(x)\| \leq 2\|x\|^p, \\
(2.5) \quad & \|f(0) + g(2x) - 2f(x) - 2g(-x)\| \leq 2\|x\|^p, \\
(2.6) \quad & \|f(0) + g(-2x) - 2f(-x) - 2g(x)\| \leq 2\|x\|^p, \quad \text{and} \\
(2.7) \quad & \|f(-2x) + g(0) - 2f(-x) - 2g(-x)\| \leq 2\|x\|^p
\end{align*}

for all $x \in V \setminus \{0\}$. Let $U(x) = f(x) + f(-x) + g(x) + g(-x)$. From (2.4), (2.5), (2.6), and (2.7), we get

$$\|\frac{U(2x)}{4} - U(x)\| \leq 2\|x\|^p + \frac{1}{2}\|f(0) + g(0)\|$$

for all $x \in V \setminus \{0\}$.

Applying an induction argument to $n$, we have

$$\|\frac{U(2^{n+1}x)}{4^{n+1}} - \frac{U(2^nx)}{4^n}\| \leq \frac{2 \cdot 2^{np}}{4^n} \|x\|^p + \frac{1}{2 \cdot 4^n}\|f(0) + g(0)\|$$

for all $x \in V \setminus \{0\}$. Hence

\begin{align*}
\|\frac{U(2^nx)}{4^n} - \frac{U(2^mx)}{4^m}\| &\leq \sum_{i=n}^{m-1} \|\frac{U(2^{i+1}x)}{4^{i+1}} - \frac{U(2^ix)}{4^i}\| \\
&\leq \sum_{i=n}^{m-1} \left( \frac{2 \cdot 2^{ip}}{4^i} \|x\|^p + \frac{1}{2 \cdot 4^i}\|f(0) + g(0)\| \right) \\
&\leq \frac{8 \cdot 2^{np}}{4^n(4 - 2^p)} \|x\|^p + \frac{2}{3 \cdot 4^n}\|f(0) + g(0)\|
\end{align*}

for all $m > n$ and $x \in V \setminus \{0\}$. This shows that $\{\frac{U(2^nx)}{4^n}\}$ is a Cauchy sequence. Since $X$ is a Banach space, the sequence $\{\frac{U(2^nx)}{4^n}\}$ converges. Define $Q : V \to X$ by

\begin{equation}
(2.8) \quad 4Q(x) = \lim_{n \to \infty} \frac{f(2^nx) + f(-2^nx) + g(2^nx) + g(-2^nx)}{4^n}
\end{equation}
for all \( x \in V \). From (2.4), (2.5), (2.6), and (2.7), we have the inequality

\[
\frac{1}{4^n} \| f(2^n x) + f(-2^n x) - g(2^n x) - g(-2^n x) \| \\
\leq \frac{2 \cdot 2^{(n-1)p}}{4^{n-1}} \| x \|^p + \frac{2 \| g(0) - f(0) \|}{4^n}
\]

for all \( x \in V \setminus \{0\} \). Hence

\[
(2.9) \quad \lim_{n \to \infty} \frac{f(2^n x) + f(-2^n x) - g(2^n x) - g(-2^n x)}{4^n} = 0.
\]

From (2.8) and (2.9), we have

\[
2Q(x) = \lim_{n \to \infty} \frac{f(2^n x) + f(-2^n x)}{4^n} = \lim_{n \to \infty} \frac{g(2^n x) + g(-2^n x)}{4^n}
\]

for all \( x \in V \setminus \{0\} \). Replacing \( x \) by \( x + y \), \( y \) by \( x - y \) and dividing by 2 in (2.1), we have

\[
(2.10) \quad \| \frac{1}{2} f(2x) + \frac{1}{2} g(2y) - f(x + y) - g(x - y) \| \\
\leq \frac{1}{2} (\| x + y \|^p + \| x - y \|^p)
\]

where \( x + y, x - y \in V \setminus \{0\} \). From (2.1) and (2.11), we have

\[
\| f(x) + g(y) - \frac{1}{4} (f(2x) + g(2y)) \| \\
\leq \frac{1}{4} (\| x + y \|^p + \| x - y \|^p) + \frac{1}{2} (\| x \|^p + \| y \|^p)
\]

for all \( x, y \in V \setminus \{0\} \) with \( x + y, x - y \in V \setminus \{0\} \). Induction argument implies

\[
(2.12) \quad \| f(x) + g(y) - \frac{1}{4^n} (f(2^n x) + g(2^n y)) \| \\
\leq \frac{1}{4 - 2^p} (\| x + y \|^p + \| x - y \|^p + 2\| x \|^p + 2\| y \|^p).
\]
Hence
\[
\left\| \frac{1}{4^m} f(2^m x) + g(2^m y) - \frac{1}{4^{m+n}} (f(2^{m+n} x) + g(2^{m+n} y)) \right\|
\leq \frac{2^{mp}}{4^m (4 - 2^p)} \left( \|x + y\|^p + \|x - y\|^p + 2\|x\|^p + 2\|y\|^p \right)
\]
for all \( m, n \in \mathbb{N} \) and \( x, y \in V \setminus \{0\} \) with \( x + y, x - y \in V \setminus \{0\} \). This shows that \( \{ \frac{1}{4^m} f(2^m x) + g(2^m y) \} \) is a Cauchy sequence and thus converges. Let
\[
R(x, y) = \lim_{n \to \infty} \frac{f(2^n x) + g(2^n y)}{4^n}
\]
for all \( x, y \in V \setminus \{0\} \) with \( x + y, x - y \in V \setminus \{0\} \). By the definition of \( R(x, y) \) and (2.10), we obtain
\[
\lim_{n \to \infty} \frac{f(2^n x)}{4^n} = \frac{1}{2} \left( R(x, 2x) - R(-x, 2x) + \lim_{n \to \infty} \frac{f(2^n x) + f(-2^n x)}{4^n} \right)
\]
\[= \frac{1}{2} (R(x, 2x) - R(-x, 2x) + 2Q(x)) \]
for all \( x \in V \setminus \{0\} \). Similarly we obtain
\[
\lim_{n \to \infty} \frac{g(2^n x)}{4^n} = \frac{1}{2} (R(2x, x) - R(2x, -x) + 2Q(x))
\]
for all \( x \in V \setminus \{0\} \). Replacing \( x \) by \( 2^nx \) and dividing by \( 4^n \) in (2.4), we have
\[
(2.13) \quad \left\| \frac{f(2^{n+1} x)}{4^n} + \frac{g(0)}{4^n} - \frac{2f(2^n x)}{4^n} - \frac{2g(2^n x)}{4^n} \right\| \leq \frac{2 \cdot 2^{np}}{4^n} \|x\|^p
\]
for all \( n \in \mathbb{N} \) and \( x \in V \setminus \{0\} \). From (2.13), we have
\[
\lim_{n \to \infty} \frac{f(2^n x)}{4^n} = 2 \lim_{n \to \infty} \frac{f(2^n x)}{4^n} - 2 \lim_{n \to \infty} \frac{g(2^n x)}{4^n}
\]
\[= \lim_{n \to \infty} \frac{f(2^{n+1} x)}{4^n} - 2 \lim_{n \to \infty} \frac{f(2^n x)}{4^n} - 2 \lim_{n \to \infty} \frac{g(2^n x)}{4^n}
\]
\[= 0 \]
for all \( x \in V \setminus \{0\} \). From this, we get
\[
(2.14) \quad \lim_{n \to \infty} \frac{f(2^n x)}{4^n} = \lim_{n \to \infty} \frac{g(2^n x)}{4^n} \text{ for all } x \in V.
\]
Replacing $x$ by $2^n x$ and dividing by $4^n$ in (2.5), we have
\[ \| \frac{f(0)}{4^n} + \frac{g(2^{n+1} x)}{4^n} - \frac{2f(2^n x)}{4^n} - \frac{2g(-2^n x)}{4^n} \| \leq \frac{2}{4^n} \| x \|^p \]
for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. From this and (2.14), we obtain
\[ (2.15) \quad \lim_{n \to \infty} \frac{g(2^n x)}{4^n} = \lim_{n \to \infty} \frac{g(-2^n x)}{4^n} \]
for all $x \in V$. From (2.10), (2.14) and (2.15), we have
\[ (2.16) \quad \lim_{n \to \infty} \frac{f(2^n x)}{4^n} = \lim_{n \to \infty} \frac{f(-2^n x)}{4^n} = \lim_{n \to \infty} \frac{g(2^n x)}{4^n} = \lim_{n \to \infty} \frac{g(-2^n x)}{4^n} = Q(x) \]
for all $x \in V$. From the definition of $Q(x)$ and (2.16), we obtain
\[ (2.17) \quad Q(2x) = 4Q(x) \quad \text{and} \quad Q(x) = Q(-x) \]
for all $x \in V$. From (2.1), (2.16), and (2.17), we obtain similarly
\[ Q(x + y) + Q(x - y) - 2Q(x) - 2Q(y) = 0 \]
for all $x, y \in V$. From (2.12) and (2.16), we can easily obtain the inequality (2.2).

Now we have to prove the uniqueness. If $Q'$ is an another quadratic function satisfying (2.2), then
\[ \| 5Q(x) - 5Q'(x) \| \leq \frac{2}{4^n} \cdot \frac{2^{np}}{4^n} \cdot 4^n \| x \|^p + \| 3x \|^p + 2 \| x \|^p + 2 \| 2x \|^p \]
for all $n \in \mathbb{N}$ and $x \in V \setminus \{0\}$. Therefore
\[ Q(x) = Q'(x) \quad \text{for all} \quad x \in V. \]
THEOREM 2.2. Let \( p < 2, \theta > 0 \) be real numbers. Let \( \psi : V \to [0, \infty) \) be a mapping such that
\[
\psi(x) = \|x\|^p \quad \text{for } x \neq 0.
\]
Suppose that the functions \( f, g : V \to X \) satisfy
\[
\|f(x + y) + g(x - y) - 2f(x) - 2g(y)\| \leq \theta(\psi(x) + \psi(y))
\]
for all \( x, y \in V \). Then there exists exactly one quadratic function \( Q : V \to X \) such that
\[
\|f(x) + g(0) - Q(x)\| \leq \frac{4\theta}{4 - 2p} \psi(x) + \frac{2}{3} \theta \psi(0)
\]
for all \( x \in V \) and
\[
\|g(x) + f(0) - Q(x)\| \leq \frac{4\theta}{4 - 2p} \psi(x) + \frac{2}{3} \theta \psi(0)
\]
for all \( x \in V \). And then, the function \( Q \) is given by (2.3).

Proof. We may assume that \( \theta = 1 \) without the loss of generality. By the same method as in the proof of Theorem 2.1, we obtain the unique quadratic function \( Q : V \to X \) satisfying (2.2) and (2.3). From (2.18), we have
\[
\|f(x) + g(y) - \frac{1}{4}(f(2x) + g(2y))\|
\leq \frac{1}{4}(\psi(x + y) + \psi(x - y)) + \frac{1}{2}(\psi(x) + \psi(y))
\]
for all \( x, y \in V \). Replacing \( y = 0 \) on the both sides of (2.19), we obtain
\[
\|f(x) + g(0) - \frac{1}{4}(f(2x) + g(0))\| \leq \psi(x) + \frac{1}{2} \psi(0)
\]
for all \( x \in V \). Applying an induction argument to \( n \), we get
\[
\|f(x) + g(0) - \frac{1}{4^n}(f(2^n x) + g(0))\| \leq \frac{4}{4 - 2p} \psi(x) + \frac{2}{3} \psi(0)
\]
for all \( n \in N \) and \( x \in V \). From this, we get
\[
\|f(x) + g(0) - Q(x)\| \leq \frac{4}{4 - 2p} \psi(x) + \frac{2}{3} \psi(0)
\]
for all \( x \in V \). Similarly, we have
\[
\|g(x) + f(0) - Q(x)\| \leq \frac{4}{4 - 2p} \psi(x) + \frac{2}{3} \psi(0)
\]
for all \( x \in V \). \( \square \)
THEOREM 2.3. Let $p > 2, \theta > 0$ be real numbers. Suppose that the functions $f, g : V \to X$ satisfy

\begin{equation}
\|f(x + y) + g(x - y) - 2f(x) - 2g(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\end{equation}

for all $x, y \in V$. Then there exists exactly one quadratic function $Q : V \to X$ such that

\[ \|f(x) - f(0) - Q(x)\| \leq \frac{4\theta}{2^p - 4} \|x\|^p \]

for all $x \in V$ and

\[ \|g(x) - g(0) - Q(x)\| \leq \frac{4\theta}{2^p - 4} \|x\|^p \]

for all $x \in V$. The function is given by

\[ Q(x) = \lim_{n \to \infty} 4^n (f(2^{-n}x) - f(0)) = \lim_{n \to \infty} 4^n (g(2^{-n}x) - g(0)) \]

for all $x \in V$.

Proof. We may assume that $\theta = 1$ without the loss of generality. From (2.20), we easily obtain

(2.21) \[ \|f(0) + g(0)\| = 0, \]

(2.22) \[ \|f(2x) + g(0) - 2f(x) - 2g(x)\| \leq 2\|x\|^p, \]

(2.23) \[ \|f(0) + g(2x) - 2f(x) - 2g(-x)\| \leq 2\|x\|^p, \]

(2.24) \[ \|f(x) + g(-x) - 2f(0) - 2g(x)\| \leq \|x\|^p, \text{ and} \]

(2.25) \[ \|f(x) + g(x) - 2f(x) - 2g(0)\| \leq \|x\|^p. \]

From (2.21), (2.22), and (2.25), we get

\begin{align*}
\|f(2x) - f(0) - 4(f(x) - f(0))\| &
\leq \|f(2x) + g(0) - 4f(x) - 4g(0)\| + 3\|f(0) + g(0)\|
\leq \|f(2x) + g(0) - 2f(x) - 2g(x)\|
+ 2\|f(x) + g(x) - 2f(x) - 2g(0)\|
\leq 4\|x\|^p.
\end{align*}

(2.26)

Applying an induction argument to $n$, we have

\[ \|(4^n(f(2^{-n}x) - f(0)) - 4^{n+1}(f(2^{-n-1}x) - f(0)))\| \leq \frac{4^{n+1}}{2(n+1)p} \|x\|^p. \]
Hence
\[(2.27) \quad \|f(x) - f(0) - 4^n(f(2^{-n}x) - f(0))\| \leq \frac{4}{2^p - 4}\|x\|^p.\]

Replacing \(x\) by \(2^{-m}x\) and multiplying by \(4^m\) in (2.27), we have
\[
\|4^m(f(2^{-m}x) - f(0)) - 4^{m+n}(f(2^{-m-n}x) - f(0))\|
\leq \frac{4^{m+1}}{2^{mp}(2^p - 4)}\|x\|^p
\]
for all \(m, n \in \mathbb{N}\). This shows that \(\{4^n(f(2^{-n}x) - f(0))\}\) is a Cauchy sequence and thus converges from the completeness of \(X\). Define \(Q : V \rightarrow X\) by
\[Q(x) = \lim_{n \to \infty} 4^n(f(2^{-n}x) - f(0)).\]

From (2.27) and the definition of \(Q\), we have the inequality
\[
\|f(x) - f(0) - Q(x)\| \leq \frac{4}{2^p - 4}\|x\|^p.
\]

Replacing \(x\) by \(2^{-n-1}x\) and multiplying by \(4^n\) in (2.22), we have
\[
\|4^n(f(2^{-n}x) - f(0)) - 2 \cdot 4^n(f(2^{-n-1}x) - f(0))
- 2 \cdot 4^n(g(2^{-n-1}x) - g(0))\|
= \|4^n(f(2^{-n}x) + g(0)) - 2f(2^{-n-1}x) - 2g(2^{-n-1}x))\|
+ 4^n\|g(0) + f(0)\|
\leq \frac{2 \cdot 4^n}{2^{(n+1)p}}\|x\|^p.
\]

Taking the limit in (2.28) as \(n \to \infty\), we have
\[Q(x) = \lim_{n \to \infty} 4^n(g(2^{-n}x) - g(0)).\]

By the similar method as in (2.26) and (2.27), we obtain
\[
\|g(x) - g(0) - Q(x)\| \leq \frac{4}{2^p - 4}\|x\|^p
\]
from (2.21), (2.23), and (2.24). The rest of proof is similar to the proof of Theorem 2.1. \(\square\)
On the Hyers-Ulam-Rassias stability

References

