A NULL FOCAL THEOREM ON LORENTZ MANIFOLDS

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ABSTRACT. Let \( P \) be a spacelike \((n - 2)\)-dimensional submanifold of an \( n \)-dimensional Lorentz manifold \( M \) and let \( \sigma \) be a \( P \)-normal null geodesic with \( \text{Ric}(\sigma', \sigma') \geq m \), for any given nonpositive constant \( m \). We establish a sufficient condition such that there is a focal point of \( P \) along \( \sigma \).

1. Introduction

Let \( M^n \) be a Lorentz manifold of dimension \( n \) and let \( P \) be a spacelike \((n - 2)\)-dimensional submanifold of \( M \) with mean curvature vector field \( H \) of \( P \). Let \( \sigma \) be a null geodesic normal to \( P \) at \( p = \sigma(0) \) such that

\[
\begin{align*}
(1) & \quad h = \langle \sigma'(0), H_p \rangle > 0; \\
(2) & \quad \text{Ric}(\sigma', \sigma') \geq 0.
\end{align*}
\]

Then it is known [1, 6] that there is a focal point \( \sigma(r) \) of \( P \) along \( \sigma \) with \( 0 < r \leq 1/h \), provided \( \sigma \) is defined on this interval.

In the present paper, using the method in Y. Itokawa [4] we prove

THEOREM. Let \( m \) be any given nonpositive constant number. Let \( P \) be a spacelike \((n - 2)\)-dimensional submanifold in a Lorentz manifold \( M^n \) of dimension \( n \), with mean curvature vector field \( H \) of \( P \). Let \( \sigma \) be a null geodesic normal to \( P \) at \( p = \sigma(0) \) such that

\[
\begin{align*}
(1) & \quad h = \langle \sigma'(0), H_p \rangle > \sqrt{-\frac{m}{n - 2}}; \\
(2) & \quad \text{Ric}(\sigma', \sigma') \geq m.
\end{align*}
\]

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Then there is a focal point $\sigma(r)$ of $P$ along $\sigma$ with

$$\begin{cases} 
0 < r \leq \frac{\log \left[ \frac{(h + k)/(h - k)}{2k} \right]}{2k} & \text{if } m < 0, \\
0 < r \leq \frac{1}{h} & \text{if } m = 0
\end{cases},$$

where $k = \sqrt{-m/(n-2)}$, provided $\sigma$ is defined on this interval.

Note that in this theorem, if $m = 0$ then we have the same result as the above statement.

2. Preliminary results

Throughout this paper, $M$ will be as described in the first paragraph. Let $\nabla$ and $\langle \,,\rangle$ be the connection and metric tensor respectively of $M$, and let $R$ be the curvature tensor with respect to the connection $\nabla$ on $M$. As usual, the mean curvature vector field $H$ of a spacelike $(n-2)$-dimensional submanifold $P$ in $M^n$ is defined by

$$H_p = \frac{1}{n-2} \sum_{i=1}^{n-2} (\nabla e_i e_i) \perp$$

at $p \in P$, where $e_1, \ldots, e_{n-2}$ is any frame on $P$ at $p$.

The Ricci curvature tensor of $M$ is defined relative to a frame field by

$$\text{Ric}(X, Y) = \sum_i \varepsilon_i \langle R(X, E_i)E_i, Y \rangle,$$

where $\varepsilon_i = \langle E_i, E_i \rangle$.

It is well known [1, 6] that for a null vector $0 \neq v \in T_p M$,

$$\text{Ric}(v, v) = \sum_{i=1}^{n-2} \langle R(v, e_i)e_i, v \rangle,$$

where $e_1, \ldots, e_{n-2}$ are orthonormal spacelike vectors.

For a curve segment $\sigma : [0, b] \to M$ in a Lorentz manifold the integral

$$E(\sigma) = \frac{1}{2} \int_0^b \langle \sigma', \sigma'' \rangle \, ds$$
is called energy of $\sigma$. For a piecewise smooth variation $x$ of $\sigma$ let $E_x(t)$ be the value of $E$ on the longitudinal curve $s \mapsto x(s, t)$, so

$$E_x(t) = \frac{1}{2} \int_0^b \langle x_s, x_s \rangle \, ds.$$ 

By contrast with $L_x$, the function $E_x$ is always smooth without restriction on $x$. Thus $E$ can be used to study null geodesics.

Let $P$ be a semi-Riemannian submanifold of $M$ and $q \in M$. Let $\Omega(P, q)$ be the set of all piecewise smooth curves $\sigma : [0, b] \to M$ that run from $P$ to $q$, and let $V^\perp(\sigma, P)$ be the vector space of all piecewise smooth vector fields $V$ on $\sigma$ with $V(0) \in T_{\sigma(0)} P$ and $V \perp \sigma'$. Then $E$ becomes a real-valued function on $\Omega(P, q)$. It is easy to check that the critical points of $E$ are exactly the normal geodesics from $P$ to $q$. If $\sigma \in \Omega(P, q)$ is such a geodesic, then strictly analogous to the index form $I_\sigma$ for $L$ is the Hessian $H_\sigma$ for $E$. We define $H_\sigma$ by

$$H_\sigma(X, Y) = \int_0^b \left\{ \langle X', Y' \rangle - \langle R(X, \sigma') \sigma', Y \rangle \right\} \, ds$$

$$- \beta_{\sigma'}(X(0), Y(0))$$

for all $X, Y \in V^\perp(\sigma, P)$, where $\beta_{\sigma'}$ is the second fundamental form of $P$ defined by $\beta_{\sigma'}(v, w) = \langle \nabla_v(w), \sigma' \rangle$.

Let $\sigma$ be a geodesic normal to $P$ at $p = \sigma(0)$. A Jacobi field on $\sigma$ satisfying $Y(0) \in T_p P$ and

$$\beta_{\sigma'}(Y(0), w) = -\langle Y'(0), w \rangle$$

for all $w \in T_p(P)$ will be called an $P$-Jacobi field.

We define a focal point $\sigma(r), r \neq 0$, of $P$ on a null geodesic $\sigma$ normal to $P$ if there is a nontrivial $P$-Jacobi field $Y$ on $\sigma$ with $Y(r) = 0$.

To prove our theorem, We use the following lemma which is well known [1, 6].

**Lemma 2.1.** Let $P$ be a spacelike submanifold of a Lorentz manifold $M$, and let $\sigma : [0, b] \to M$ be a null geodesic segment normal to $P$ at $p = \sigma(0)$ with no $P$-focal points. Let $X \in V^\perp(\sigma, P)$ and let $Y \in V^\perp(\sigma, P)$ be a $P$-Jacobi field on $\sigma$ with $Y(b) = X(b)$. Then

$$H_\sigma(Y, Y) \leq H_\sigma(X, X).$$
Here, we will use some notations in Y. Itokawa [4]. Let $P$ be a space-like $(n-2)$-dimensional submanifold of $M$. For each $P$-Jacobi field $Y$ on $\sigma : [0, \infty) \rightarrow M$ normal to $P$, we construct a modified field $\bar{Y}$ defined by

$$\bar{Y}(s) = \begin{cases} Y(s) & \text{if } Y \text{ does not vanish on } (0, s), \\ 0 & \text{if } Y(t) = 0 \text{ for some } t, \; 0 < t \leq s. \end{cases}$$

Let $Y_1, \ldots, Y_{n-2}$ be a linearly independent set of $P$-Jacobi fields on the null geodesic $\sigma$ perpendicular to $\sigma$ such that $Y_1(0), \ldots, Y_{n-2}(0)$ is a basis for $T_{\sigma(0)} P$. Here, we define a function $g(s)$ by

$$(2.6) \quad g(s) = \frac{G^{1/2}[\bar{Y}_1(s)]}{G^{1/2}[\bar{Y}_1(0)]},$$

where $G[v_i]$ is the Gram determinant made from the vectors $v_1, v_2, \ldots, v_{n-2}$. Then the function $g(s)$ remains valid for all $s$ even beyond the focal points of $P$.

**Remark.** If $g(s) = 0$ for some $s > 0$, then there is a focal point $\sigma(r)$ of $P$ along $\sigma$ with $0 < r \leq s$.

3. Warped products

In this section, we construct model spaces called *warped products* [1, 6] which will be used for comparison purposes in the next section. Let $B$ be a domain in $(\mathbb{R}^2, du^2 - dv^2)$ containing a null geodesic segment $\gamma(s) = (s, s) : [0, b) \rightarrow B$, where $0 < b \leq \infty$. Let $f > 0$ be a continuous function on $B$ such that $f \circ \gamma : [0, b) \rightarrow \mathbb{R}$ is the solution to the differential equation

$$(f \circ \gamma)''(s) - k^2 (f \circ \gamma)(s) = 0$$

with initial values

$$f \circ \gamma(0) = a_0 \quad \text{and} \quad (f \circ \gamma)'(0) = a_1,$$

where $k$ is a nonnegative constant number, $a_0 > 0$ and $a_1 < 0$. We know that the solution to the above differential equation is

$$(3.1) \quad f \circ \gamma(s) = \begin{cases} a_0 + a_1 s & \text{if } k = 0, \\ a_0/(2k) \left[ (k + a_1/a_0)e^{sk} + (k - a_1/a_0)e^{-sk} \right] & \text{if } k \neq 0. \end{cases}$$
Hence, for example we can take $B$ and $f$ satisfying the above conditions as follows:

$$
\begin{align*}
B &= \{(u, v) | v \geq 0, \ u + v < -\frac{2a_0}{a_1}\}, \\
f(u, v) &= a_0 + \frac{a_1(u + v)}{2}
\end{align*}
$$

if $k = 0$

and

$$
\begin{align*}
B &= \{(u, v) | v \geq 0, \ u + v < \frac{\log \left(\frac{(a_1/a_0 - k)/(a_1/a_0 + k)}{k}\right)}{k}\}, \\
f(u, v) &= \frac{a_0}{2k} \left[ (k + \frac{a_1}{a_0})e^{(u+v)k/2} + (k - \frac{a_1}{a_0})e^{-(u+v)k/2} \right]
\end{align*}
$$

if $0 < k < -\frac{a_1}{a_0}$.

For a domain $B$ and a function $f$ as described above, we take a space which is diffeomorphic to $S^{n-2}_f \times B$ and give it the metric

$$
\langle \cdot, \cdot \rangle(x, y) = f^2 g^{n-2}_{\text{can}} \otimes (du^2 - dv^2)
$$

at each $x \in S^{n-2}$ and $y \in B$. We denote the resulting Lorentz manifold with boundary by $M$. Let us denote by $S^*_s$, the parallel sphere $S^{n-2} \times \gamma(s)$ and by $A^*(s)$, its area. Let $p^* \in M$, $p^* = (x, \gamma(s))$ where $x \in S^{n-2}$ and $\gamma(s) \in B$. Then, a curve $\sigma^* : s \to (x, \gamma(s)) \in M$ is a null geodesic normal to $S^*_s$ at $p^*$ and we have

**Lemma 3.1.** Let $H_{p^*}$ be the mean curvature vector field of $S^*_s$ at $p^*$, and let $v, w$ be tangent vectors to $S^*_s$. Then we have

1. (3.2) $\text{Ric}(\sigma'^*, \sigma''') \equiv -(n-2)k^2$.

2. Each $S^*_s$ is totally umbilic and has parallel second fundamental form

$$
(3.3) \quad \beta_{\sigma'^*}(v, w) \equiv -\frac{(f \circ \gamma)'(s)}{f \circ \gamma(s)} (v, w).
$$

Hence

$$
(3.4) \quad \langle \sigma'^*, H_{p^*} \rangle \equiv -\frac{(f \circ \gamma)'(s)}{(f \circ \gamma)(s)}.
$$
(3) For each \((x, \gamma(0)) \in S_0^*\), take \(Y_1^*, \ldots, Y_{n-2}^*\) to be linearly independent \(S_0^*\)-Jacobi fields on the null geodesic \(\sigma^* : s \to (x, \gamma(s))\) perpendicular to \(\sigma^*\) such that \(Y_1^*(0), \ldots, Y_{n-2}^*(0)\) is a basis for \(T_{\sigma^*(0)}S_0^*\). If we define a function \(g^*(s)\) by

\[
(3.5) \quad g^*(s) = \frac{G^{1/2}[Y_i^*(s)]}{G^{1/2}[Y_i^*(0)]},
\]

then

\[
(3.6) \quad A^*(s) = \int_{S_0^*} g^*(s) \text{dArea}_{S_0^*}(p^*) = (f \circ \gamma)^{n-2}(s) \theta(n - 2),
\]

where \(\theta(n - 2)\) is the volume of the euclidean sphere \((S^{n-2}, g^{(n-2)}_{\text{can}})\).

**Proof.** (1) The set of all lifts of all vector fields on \(S^{n-2}\) or \(B\) to \(S^{n-2}_f \times B\) is denoted as usual by \(\mathcal{L}(S^{n-2})\) and \(\mathcal{L}(B)\), respectively. If \(X, Y \in \mathcal{L}(S^{n-2})\) and \(U, V \in \mathcal{L}(B)\), then by using Koszul formula we see

\[
(3.7) \quad \begin{cases} 
\bar{\nabla}_U V \in \mathcal{L}(B) \text{ is the lift of } D_U V \text{ on } B. \\
\bar{\nabla}_U X = \bar{\nabla}_X U = \frac{U}{f} f X.
\end{cases}
\]

\[
o(\bar{\nabla}_X Y)^\perp = -\frac{\langle X, Y \rangle_f}{f} \text{grad } f.
\]

\[
\bar{\nabla}(\bar{\nabla}_X Y)^\top \in \mathcal{L}(S^{n-2}) \text{ is the lift of } \nabla_X Y \text{ on } S^{n-2}.
\]

By using (3.7), we have

\[
R(U, X)U = \bar{\nabla}_U \bar{\nabla}_X U - \bar{\nabla}_X \bar{\nabla}_U U - \bar{\nabla}_{[U, X]} U
\]

\[
= \bar{\nabla}_U \left\{ \frac{U}{f} f X \right\} - \frac{D_U f f}{f} X
\]

\[
= \frac{(UU - D_U U)f}{f} X.
\]

Let \(e_1, \ldots, e_{n-2}\) be orthonormal spacelike vectors tangent to \(S_0^*\) at \(p^*\).
Then since $\sigma^*$ is a null vector, (2.2) and (3.8) imply

$$\text{Ric}(\sigma^*, \sigma^*) = \sum_{i=1}^{n-2} \langle R(\sigma^*, e_i) e_i, \sigma^* \rangle$$

$$= - \sum_{i=1}^{n-2} \langle R(\sigma^*, e_i) \sigma^*, e_i \rangle$$

$$= -(n - 2) \gamma' \gamma' f$$

$$= -(n - 2) \frac{(f \circ \gamma)''(s)}{f \circ \gamma(s)}$$

$$= -(n - 2) k^2.$$

(2) \[ \beta_{\sigma^*^*}(v, w) = \langle (\nabla_v w), \sigma^*^* \rangle \]

$$= \langle - \frac{\langle v, w \rangle}{f} \text{grad} f, \gamma' \rangle$$

$$= - \frac{\langle v, w \rangle}{f} \gamma' f$$

$$= -(v, w) \frac{(f \circ \gamma)'(s)}{f \circ \gamma(s)}.$$

Hence, (2.1) implies

$$\langle \sigma^*, H_{\rho^*} \rangle = \langle \sigma^*, \frac{1}{n - 2} \sum_{i=1}^{n-2} (\nabla e_i e_i)^\perp \rangle$$

$$= \frac{1}{n - 2} \sum_{i=1}^{n-2} \beta_{\sigma^*^*}(e_i, e_i)$$

$$= -\frac{(f \circ \gamma)'(s)}{(f \circ \gamma)(s)}.$$  

(3) cf. [3, 5]. \hfill \square

4. Proof of Theorem

The purpose of this section is to prove our theorem as described in the first section. To prove our theorem, we need the next lemma which is a special case of the well-known Comparison Theorem of E. Heintze and H. Karcher [3].
**Lemma 4.1.** Let $P$ be a spacelike $(n-2)$-dimensional submanifold in Lorentz manifold $M^n$ with mean curvature vector field $H$ and let $\sigma$ be a null geodesic normal to $P$ at $p = \sigma(0)$. Let $\mathcal{M}$ be a warped product model as defined in the section 3, and let $g(s)$ and $g^*(s)$ be functions as (2.6) and (3.5) respectively. Suppose that

1. $\langle \sigma'(0), H_p \rangle \geq \langle \sigma^{*'}(0), H_{p^*} \rangle$,
2. $\text{Ric}(\sigma', \sigma') \geq \text{Ric}(\sigma^{*'}, \sigma^{*'})$.

Then, for all such $s$,

$$g(s) \leq g^*(s).$$

**Proof.** Since the function $g^*(s)$ is strictly positive on the admissible interval, if $\tilde{Y}_i(s) = 0$ for any $i$, then the inequality follows trivially. If, on the other hand, $\tilde{Y}_i(s) \neq 0$ for all $i$, then $\tilde{Y}_i(s) = Y_i(s)$. In this case, let $k(s) = \log g^*(s) - \log g(s)$. Since $k(0) = 0$, it suffices to show that $k'(s) \geq 0$ for all $s$ such that $\sigma(s)$ comes before the first focal point $\sigma(r)$ of $P$ on $\sigma$. We fix $l$, $(0 \leq l < r)$. We may assume that $Y_1(l), \ldots, Y_{n-2}(l)$ are orthonormal and $Y_1^*(l), \ldots, Y_{n-2}^*(l)$ are orthonormal.

$$\{\log g(s)\}'(l) = \{\log G^{1/2} [Y_i(s)]\}'(l) = \sum_{i=1}^{n-2} (Y_i(l), Y_i'(l)) = \sum_{i=1}^{n-2} \left[ \langle Y_i(0), Y_i'(0) \rangle + \int_0^l \langle Y_i, Y_i' \rangle' \, ds \right] = \sum_{i=1}^{n-2} \left[ -\beta_{\sigma'}(Y_i(0), Y_i(0)) + \int_0^l \{\langle Y_i', Y_i' \rangle - \langle R(\sigma', \sigma') Y_i, Y_i \rangle \} \, ds \right] = \sum_{i=1}^{n-2} H_{\sigma}(Y_i, Y_i).$$

Similarly, we have

$$\{\log g^*(s)\}'(l) = \sum_{i=1}^{n-2} H_{\sigma^*}(Y_i^*, Y_i^*).$$

Let $E_1, \ldots, E_{n-2} \in V^\perp(\sigma, P)$ be spacelike parallel orthonormal vector fields on $\sigma$ such that $E_i(l) = Y_i(l)$ for all $i$, and let $E_1^*, \ldots, E_{n-2}^* \in W^\perp
$V^\perp(\sigma^*, S_0^*)$ be spacelike parallel orthonormal vector fields on $\sigma^*$ such that $E_i^*(l) = Y_i^*(l)$ for all $i$. If $Y_i^*(s) = \sum a_{ij}(s)E_j^*(s)$ on $\sigma^*$, then we define a vector field $X_i(s)$ on $\sigma$ by $X_i(s) = \sum a_{ij}(s)E_j(s)$. Then we have $|X_i(s)| = |Y_i^*(s)|$ and $|X_i(s)| = |Y_i(s)|$. By taking a suitable linear combination, we may assume $Y_i(l) = X_i(l)$. Since $H_\sigma(Y_i, Y_i) \leq H_\sigma(X_i, X_i)$ by Lemma 2.1, we have

$$\{\log g(s)\}'(l) = \sum_{i=1}^{n-2} H_\sigma(Y_i, Y_i) \leq \sum_{i=1}^{n-2} H_\sigma(X_i, X_i).$$

Hence, it remains to show that

$$\sum_{i=1}^{n-2} H_\sigma(X_i, X_i) \leq \sum_{i=1}^{n-2} H_{\sigma^*}(Y_i^*, Y_i^*).$$

From the definition (2.3) of Hessian form $H_\sigma$, we see

$$H_\sigma(X_i, X_i) = -\beta_{\sigma^*}(X_i(0), X_i(0))$$

$$+ \int_0^l \{\langle X_i', X_i' \rangle - \langle R(X_i, \sigma')\sigma', X_i \rangle \} ds,$$

$$H_{\sigma^*}(Y_i^*, Y_i^*) = -\beta_{\sigma^*}(Y_i^*(0), Y_i^*(0))$$

$$+ \int_0^l \{\langle Y_i'^*, Y_i'^* \rangle - \langle R(Y_i^*, \sigma'^*)\sigma'^*, Y_i^* \rangle \} ds.$$

Since $Y_i^*$ is a $S_0^*$-Jacobi field on $\sigma^*$ such that $Y_i^*(l) = E_i^*(l)$, we can know easily

$$Y_i^*(s) = (f \circ \gamma(s)/f \circ \gamma(l)) E_i^*(s),$$

and so

$$X_i(s) = (f \circ \gamma(s)/f \circ \gamma(l)) E_i(s)$$
on $\sigma$.

Therefore

$$\sum_{i=1}^{n-2} \beta_{\sigma^*}(X_i(0), X_i(0)) = \left( \frac{f \circ \gamma(0)}{f \circ \gamma(l)} \right)^2 \sum_{i=1}^{n-2} \beta_{\sigma^*}(E_i(0), E_i(0))$$

$$= \left( \frac{f \circ \gamma(0)}{f \circ \gamma(l)} \right)^2 (n-2)(\sigma'(0), H_p).$$
Similarly,
\[
\sum_{i=1}^{n-2} \beta_{\sigma^{*'}}(Y_i^{*}(0), Y_i^{*}(0)) = \left( \frac{f \circ \gamma(0)}{f \circ \gamma(l)} \right)^2 \sum_{i=1}^{n-2} \beta_{\sigma^{*'}}(E_i^{*}(0), E_i^{*}(0)) = \left( \frac{f \circ \gamma(0)}{f \circ \gamma(l)} \right)^2 (n-2) \langle \sigma^{*'}(0), H_{p^*} \rangle.
\]

And since the \(X_i\)'s are spacelike, (2.2) implies
\[
\sum_{i=1}^{n-2} \langle R(X_i, \sigma') \sigma' , X_i \rangle = \sum_{i=1}^{n-2} \left( \frac{f \circ \gamma(s)}{f \circ \gamma(l)} \right)^2 \langle R(E_i, \sigma') \sigma' , E_i \rangle = \left( \frac{f \circ \gamma(s)}{f \circ \gamma(l)} \right)^2 \text{Ric}(\sigma', \sigma').
\]

Similarly, since the \(Y_i^{*}\)'s are also spacelike, (2.2) implies
\[
\sum_{i=1}^{n-2} \langle R(Y_i^{*}, \sigma^{*''}) \sigma^{*''} , Y_i^{*} \rangle = \sum_{i=1}^{n-2} \left( \frac{f \circ \gamma(s)}{f \circ \gamma(l)} \right)^2 \langle R(E_i^{*}, \sigma^{*''}) \sigma^{*''} , E_i^{*} \rangle = \left( \frac{f \circ \gamma(s)}{f \circ \gamma(l)} \right)^2 \text{Ric}(\sigma^{*''}, \sigma^{*''}).
\]

The above equalities and the hypotheses in the Theorem give
\[
\sum_{i=1}^{n-2} H_{\sigma}(X_i, X_i) \leq \sum_{i=1}^{n-2} H_{\sigma^{*'}}(Y_i^{*}, Y_i^{*}),
\]
which completes the proof of Lemma 4.1.

Let a warped product \(M\), a null geodesic \(\sigma^{*}\) and a function \(g^{*}(s)\) be as defined in the section 3 with the initial values \(a_0\) and \(a_1\) such that \(-a_1/a_0 = h\). Then (3.3) and (3.2) imply
\[
\langle \sigma^{*'}(0), H_{p^*} \rangle = -\frac{(f \circ \gamma)'(0)}{f \circ \gamma(0)} = -\frac{a_1}{a_0} = h,
\]
\[
\text{Ric}(\sigma^{*'}, \sigma^{*'}) = -(n-2)k^2 = -(n-2) \left( \sqrt{\frac{-m}{n-2}} \right)^2 = m.
\]
Hence, from the hypothesis in Theorem we obtain

\[(1) \quad \langle \sigma'(0), H_p \rangle = h = \langle \sigma^{*'}(0), H_{p^*} \rangle,\]
\[(2) \quad \text{Ric}(\sigma', \sigma') \geq m = \text{Ric}(\sigma^{*'}, \sigma^{*'}).\]

Therefore, Lemma 4.1 implies that for all such \(s\),

\[g(s) \leq g^*(s).\]

From (3.1), we know that if \(k = 0\), then \(f \circ \gamma(s) > 0\) only for

\[s < -\frac{a_0}{a_1} := \frac{1}{h}.\]

And if \(k \neq 0\), then \(h > k > 0\) from the hypothesis in Theorem. Hence, we also know from (3.1) that for \(b\) in the domain \([0, b)\) of \(f \circ \gamma\),

\[b \leq \frac{\log [(h + k)/(h - k)]}{2k}.
\]

As \(s\) approaches the value on the right side, \(f \circ \gamma(s)\) tends to 0 and so \(A^*(s)\) and \(g^*(s)\) also tend to 0 from (3.6). Since

\[g(s) \leq g^*(s),\]

\(g(s)\) also tends to 0 as \(s\) approaches the value on the right side. Remark in the section 2 implies that there is a focal point \(\sigma(r)\) of \(P\) along \(\sigma\) with

\[
\left\{
\begin{array}{ll}
0 < r \leq \frac{\log [(h + k)/(h - k)]}{2k} & \text{if} \quad m < 0, \\
0 < r \leq \frac{1}{h} & \text{if} \quad m = 0
\end{array}
\right\},
\]

provided \(\sigma\) is defined on this interval.

References


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