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# A NULL FOCAL THEOREM ON LORENTZ MANIFOLDS

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ABSTRACT. Let $P$ be a spacelike $(n-2)$-dimensional submanifold of an $n$-dimensional Lorentz manifold $M$ and let $\sigma$ be a $P$-normal null geodesic with $Ric(\sigma', \sigma') \geq m$, for any given nonpositive constant $m$. We establish a sufficient condition such that there is a focal point of $P$ along $\sigma$.

## 1. Introduction

Let $M^n$ be a Lorentz manifold of dimension $n$ and let $P$ be a spacelike $(n-2)$-dimensional submanifold of $M$ with mean curvature vector field $H$ of $P$. Let $\sigma$ be a null geodesic normal to $P$ at $p = \sigma(0)$ such that

$$(1) \quad h = \langle \sigma'(0), H_p \rangle > 0;$$
$$(2) \quad \mathrm{Ric}(\sigma', \sigma') \geq 0.$$

Then it is known [1, 6] that there is a focal point $\sigma(r)$ of $P$ along $\sigma$ with $0 < r \leq 1/h$, provided $\sigma$ is defined on this interval.

In the present paper, using the method in Y. Itokawa [4] we prove

THEOREM. *Let $m$ be any given nonpositive constant number. Let $P$ be a spacelike $(n-2)$-dimensional submanifold in a Lorentz manifold $M^n$ of dimension $n$, with mean curvature vector field $H$ of $P$. Let $\sigma$ be a null geodesic normal to $P$ at $p = \sigma(0)$ such that*

$$(1) \quad h = \langle \sigma'(0), H_p \rangle > \sqrt{\frac{-m}{n-2}};$$
$$(2) \quad \mathrm{Ric}(\sigma', \sigma') \geq m.$$

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*Then there is a focal point $\sigma(r)$ of $P$ along $\sigma$ with*

$$\left\{\begin{array}{ll} 0 < r \leq \dfrac{\log\left[(h+k)/(h-k)\right]}{2k} & \text{if} \quad m < 0, \\ 0 < r \leq \dfrac{1}{h} \quad \text{if} \quad m = 0 & \end{array}\right\},$$

*where $k = \sqrt{-m/(n-2)}$, provided $\sigma$ is defined on this interval.*

Note that in this theorem, if $m = 0$ then we have the same result as the above statement.

## 2. Preliminary results

Throughout this paper, $M$ will be as described in the first paragraph. Let $\overline{\nabla}$ and $\langle\, ,\, \rangle$ be the connection and metric tensor respectively of $M$, and let $R$ be the curvature tensor with respect to the connection $\overline{\nabla}$ on $M$. As usual, the *mean curvature vector field* $H$ of a spacelike $(n-2)$-dimensional submanifold $P$ in $M^n$ is defined by

$$H_p = \frac{1}{n-2} \sum_{i=1}^{n-2} (\overline{\nabla}_{e_i} e_i)^{\perp} \tag{2.1}$$

at $p \in P$, where $e_1, \ldots, e_{n-2}$ is any frame on $P$ at $p$.

The *Ricci curvature tensor* of $M$ is defined relative to a frame field by

$$\mathrm{Ric}(X,\, Y) = \sum_i \varepsilon_i \langle R(X,\, E_i)E_i,\, Y\rangle,$$

where $\varepsilon_i = \langle E_i,\, E_i\rangle$.

It is well known [1, 6] that for a null vector $0 \neq v \in T_pM$,

$$\mathrm{Ric}(v,\, v) = \sum_{i=1}^{n-2} \langle R(v,\, e_i)e_i,\, v\rangle, \tag{2.2}$$

where $e_1, \ldots, e_{n-2}$ are orthonormal spacelike vectors.

For a curve segment $\sigma : [0,\, b] \to M$ in a Lorentz manifold the integral

$$E(\sigma) = \frac{1}{2}\int_0^b \langle \sigma',\, \sigma'\rangle\, ds$$

is called *energy* of $\sigma$. For a piecewise smooth variation $x$ of $\sigma$ let $E_x(t)$ be the value of $E$ on the longitudinal curve $s \to x(s, t)$, so

$$E_x(t) = \frac{1}{2}\int_0^b \langle x_s, x_s \rangle \, ds.$$

By contrast with $L_x$, the function $E_x$ is always smooth without restriction on $x$. Thus $E$ can be used to study null geodesics.

Let $P$ be a semi-Riemannian submanifold of $M$ and $q \in M$. Let $\Omega(P, q)$ be the set of all piecewise smooth curves $\sigma : [0, b] \to M$ that run from $P$ to $q$, and let $V^\perp(\sigma, P)$ be the vector space of all piecewise smooth vector fields $V$ on $\sigma$ with $V(0) \in T_{\sigma(0)}P$ and $V \perp \sigma'$. Then $E$ becomes a real-valued function on $\Omega(P, q)$. It is easy to check that *the critical points* of $E$ are exactly the normal geodesics from $P$ to $q$. If $\sigma \in \Omega(P, q)$ is such a geodesic, then strictly analogous to the index form $I_\sigma$ for $L$ is the *Hessian* $H_\sigma$ for $E$. We define $H_\sigma$ by

$$(2.3)\qquad H_\sigma(X, Y) = \int_0^b \{\langle X', Y' \rangle - \langle R(X, \sigma')\sigma', Y \rangle\} ds - \beta_{\sigma'}(X(0), Y(0))$$

for all $X, Y \in V^\perp(\sigma, P)$, where $\beta_{\sigma'}$ is the *second fundamental form* of $P$ defined by $\beta_{\sigma'}(v, w) = \langle \overline{\nabla}_v(w), \sigma' \rangle$.

Let $\sigma$ be a geodesic normal to $P$ at $p = \sigma(0)$. A Jacobi field on $\sigma$ satisfying $Y(0) \in T_pP$ and

$$(2.4)\qquad \beta_{\sigma'}(Y(0), w) = -\langle Y'(0), w \rangle$$

for all $w \in T_p(P)$ will be called an *P-Jacobi field.*

We define a *focal point* $\sigma(r), r \neq 0$, of $P$ on a null geodesic $\sigma$ normal to $P$ if there is a nontrivial $P$-Jacobi field $Y$ on $\sigma$ with $Y(r) = 0$.

To prove our theorem, We use the following lemma which is well known [1, 6].

LEMMA 2.1. *Let $P$ be a spacelike submanifold of a Lorentz manifold $M$, and let $\sigma : [0, b] \to M$ be a null geodesic segment normal to $P$ at $p = \sigma(0)$ with no $P$-focal points. Let $X \in V^\perp(\sigma, P)$ and let $Y \in V^\perp(\sigma, P)$ be a $P$-Jacobi field on $\sigma$ with $Y(b) = X(b)$. Then*

$$(2.5)\qquad H_\sigma(Y, Y) \le H_\sigma(X, X).$$

Here, we will use some notations in Y. Itokawa [4]. Let $P$ be a spacelike $(n-2)$-dimensional submanifold of $M$. For each P-Jacobi field $Y$ on $\sigma : [0, \infty) \to M$ normal to $P$, we construct a *modified field* $\bar{Y}$ defined by

$$\bar{Y}(s) = \begin{cases} Y(s) & \text{if } Y \text{ does not vanish on } (0,\, s), \\ 0 & \text{if } Y(t) = 0 \text{ for some } t,\ 0 < t \leq s. \end{cases}$$

Let $Y_1, \ldots, Y_{n-2}$ be a linearly independent set of $P$-Jacobi fields on the null geodesic $\sigma$ perpendicular to $\sigma$ such that $Y_1(0), \ldots, Y_{n-2}(0)$ is a basis for $T_{\sigma(0)}P$. Here, we define a function $g(s)$ by

$$g(s) = \frac{G^{1/2}[\bar{Y}_i(s)]}{G^{1/2}[Y_i(0)]} \tag{2.6}$$

where $G[v_i]$ is the Gram determinant made from the vectors $v_1, v_2, \ldots, v_{n-2}$. Then the function $g(s)$ remains valid for all $s$ even beyond the focal points of $P$.

REMARK. If $g(s) = 0$ for some $s > 0$, then there is a focal point $\sigma(r)$ of $P$ along $\sigma$ with $0 < r \leq s$.

## 3. Warped products

In this section, we construct model spaces called *warped products* [1, 6] which will be used for comparison purposes in the next section. Let $B$ be a domain in $(\mathbb{R}^2_1,\, du^2 - dv^2)$ containing a null geodesic segment $\gamma(s) = (s,\, s) : [0,\, b) \to B$, where $0 < b \leq \infty$. Let $f > 0$ be a continuous function on $B$ such that $f \circ \gamma : [0,\, b) \to \mathbb{R}$ is the solution to the differential equation

$$(f \circ \gamma)''(s) - k^2\, (f \circ \gamma)(s) = 0$$

with initial values

$$f \circ \gamma(0) = a_0 \quad \text{and} \quad (f \circ \gamma)'(0) = a_1,$$

where $k$ is a nonnegative constant number, $a_0 > 0$ and $a_1 < 0$. We know that the solution to the above differential equation is

$$f \circ \gamma(s) = \begin{cases} a_0 + a_1\, s & \text{if } k = 0, \\ a_0/(2k)\left[(k + a_1/a_0)e^{sk} + (k - a_1/a_0)e^{-sk}\right] & \text{if } k \neq 0. \end{cases} \tag{3.1}$$

Hence, for example we can take $B$ and $f$ satisfying the above conditions as follows:

$$\left\{\begin{array}{l} B=\{(u,\,v)|\,v\geq 0,\quad u+v<-\dfrac{2a_0}{a_1}\}, \\ f(u,\,v)=a_0+\dfrac{a_1(u+v)}{2} \end{array}\right\} \quad \text{if} \quad k=0$$

and

$$\left\{\begin{array}{l} B=\{(u,\,v)|\,v\geq 0,\quad u+v<\dfrac{\log\left[(a_1/a_0-k)/(a_1/a_0+k)\right]}{k}\}, \\ f(u,\,v)=\dfrac{a_0}{2k}\left[(k+\dfrac{a_1}{a_0})e^{(u+v)k/2}+(k-\dfrac{a_1}{a_0})e^{-(u+v)k/2}\right] \end{array}\right\}$$

if $0<k<-\frac{a_1}{a_0}$.

For a domain $B$ and a function $f$ as described above, we take a space which is diffeomorphic to

$$S^{n-2}_{\;f}\times B$$

and give it the metric

$$\langle\,,\,\rangle_{(x,\,y)}=f^2 g^{n-2}_{can}\otimes\ (du^2-dv^2)$$

at each $x\in S^{n-2}$ and $y\in B$. We denote the resulting Lorentz manifold with boundary by $\mathbb{M}$. Let us denote by $S^*_s$, the *parallel sphere* $S^{n-2}\times\gamma(s)$ and by $A^*(s)$, its area. Let $p^*\in\mathbb{M}$, $p^*=(x,\,\gamma(s))$ where $x\in S^{n-2}$ and $\gamma(s)\in B$. Then, a curve $\sigma^*:s\to(x,\,\gamma(s))\in\mathbb{M}$ is a null geodesic normal to $S^*_s$ at $p^*$ and we have

LEMMA 3.1. *Let $H_{p^*}$ be the mean curvature vector field of $S^*_s$ at $p^*$, and let $v$, $w$ be tangent vectors to $S^*_s$. Then we have*

(1) (3.2)
$$\mathrm{Ric}(\sigma^{*\prime},\,\sigma^{*\prime})\equiv-(n-2)k^2.$$

(2) *Each $S^*_s$ is totally umbilic and has parallel second fundamental form*

$$\beta_{\sigma^{*\prime}}(v,\,w)\equiv-\frac{(f\circ\gamma)'(s)}{f\circ\gamma(s)}\langle v,\,w\rangle. \tag{3.3}$$

*Hence*

$$\langle\sigma^{*\prime},\,H_{p^*}\rangle\equiv-\frac{(f\circ\gamma)'(s)}{(f\circ\gamma)(s)}. \tag{3.4}$$

(3) *For each* $(x, \gamma(0)) \in S_0^*$, *take* $Y_1^*, \dots, Y_{(n-2)}^*$ *to be linearly independent* $S_0^*$*-Jacobi fields on the null geodesic* $\sigma^* : s \to (x, \gamma(s))$ *perpendicular to* $\sigma^*$ *such that* $Y_1^*(0), \dots, Y_{n-2}^*(0)$ *is a basis for* $T_{\sigma^*(0)} S_0^*$. *If we define a function* $g^*(s)$ *by*

$$g^*(s) = \frac{G^{1/2}[Y_i^*(s)]}{G^{1/2}[Y_i^*(0)]}, \tag{3.5}$$

*then*

$$A^*(s) = \int_{S_0^*} g^*(s)\, d\mathrm{Area}_{S_0^*}(p^*) = (f \circ \gamma)^{n-2}(s)\, \theta(n-2), \tag{3.6}$$

*where* $\theta(n-2)$ *is the volume of the euclidean sphere* $(S^{n-2}, g_{can}^{(n-2)})$.

*Proof.* (1)The set of all lifts of all vector fields on $S^{n-2}$ or $B$ to $S^{n-2}_f \times B$ is denoted as usual by $\mathcal{L}(S^{n-2})$ and $\mathcal{L}(B)$, respectively. If $X, Y \in \mathcal{L}(S^{n-2})$ and $U, V \in \mathcal{L}(B)$, then by using Koszul formula we see

$$\begin{cases} \circ\overline{\nabla}_U V \in \mathcal{L}(B) \text{ is the lift of } D_U V \text{ on } B. \\ \circ\overline{\nabla}_U X = \overline{\nabla}_X U = \frac{Uf}{f} X. \\ \circ(\overline{\nabla}_X Y)^{\perp} = -\frac{\langle X, Y\rangle}{f}\, grad\, f. \\ \circ(\overline{\nabla}_X Y)^{\top} \in \mathcal{L}(S^{n-2}) \text{ is the lift of } \nabla_X Y \text{ on } S^{n-2}. \end{cases} \tag{3.7}$$

By using (3.7), we have

$$\begin{aligned} R(U, X)U &= \overline{\nabla}_U \overline{\nabla}_X U - \overline{\nabla}_X \overline{\nabla}_U U - \overline{\nabla}_{[U, X]} U \\ &= \overline{\nabla}_U \{\frac{Uf}{f} X\} - \frac{D_U U f}{f} X \\ &= \frac{(UU - D_U U) f}{f} X. \end{aligned} \tag{3.8}$$

Let $e_1, \dots, e_{n-2}$ be orthonormal spacelike vectors tangent to $S_s^*$ at $p^*$.

Then since $\sigma^{*\prime}$ is a null vector, (2.2) and (3.8) imply

$$\begin{aligned}\mathrm{Ric}(\sigma^{*\prime}, \sigma^{*\prime}) &= \sum_{i=1}^{n-2}\langle R(\sigma^{*\prime}, e_i)e_i, \sigma^{*\prime}\rangle \\ &= -\sum_{i=1}^{n-2}\langle R(\sigma^{*\prime}, e_i)\sigma^{*\prime}, e_i\rangle \\ &= -(n-2)\frac{\gamma'\gamma' f}{f} \\ &= -(n-2)\frac{(f\circ\gamma)''(s)}{f\circ\gamma(s)} \\ &= -(n-2)k^2.\end{aligned}$$

(2)
$$\begin{aligned}\beta_{\sigma^{*\prime}}(v, w) &= \langle(\overline{\nabla}_v w), \sigma^{*\prime}\rangle \\ &= \langle -\frac{\langle v, w\rangle}{f}\, grad\, f, \gamma'\rangle \\ &= -\frac{\langle v, w\rangle}{f}\gamma' f \\ &= -\langle v, w\rangle\frac{(f\circ\gamma)'(s)}{f\circ\gamma(s)}.\end{aligned}$$

Hence, (2.1) implies

$$\begin{aligned}\langle\sigma^{*\prime}, H_{p^*}\rangle &= \langle\sigma^{*\prime}, \frac{1}{n-2}\sum_{i=1}^{n-2}(\overline{\nabla}_{e_i}e_i)^{\perp}\rangle \\ &= \frac{1}{n-2}\sum_{i=1}^{n-2}\beta_{\sigma^{*\prime}}(e_i, e_i) \\ &= -\frac{(f\circ\gamma)'(s)}{(f\circ\gamma)(s)}.\end{aligned}$$

(3) cf. [3, 5]. □

## 4. Proof of Theorem

The purpose of this section is to prove our theorem as described in the first section. To prove our theorem, we need the next lemma which is a special case of the well-known Comparison Theorem of E. Heintze and H. Karcher [3].

LEMMA 4.1. *Let $P$ be a spacelike $(n-2)$-dimensional submanifold in Lorentz manifold $M^n$ with mean curvature vector field $H$ and let $\sigma$ be a null geodesic normal to $P$ at $p = \sigma(0)$. Let $\mathbb{M}$ be a warped product model as defined in the section 3, and let $g(s)$ and $g^*(s)$ be functions as (2.6) and (3.5) respectively. Suppose that*

(1) $\langle \sigma'(0), H_p \rangle \geq \langle \sigma^{*\prime}(0), H_{p^*} \rangle$,

(2) $\mathrm{Ric}(\sigma', \sigma') \geq \mathrm{Ric}(\sigma^{*\prime}, \sigma^{*\prime})$.

*Then, for all such $s$,*

$$g(s) \leq g^*(s).$$

*Proof.* Since the function $g^*(s)$ is strictly positive on the admissible interval, if $\bar{Y}_i(s) = 0$ for any $i$, then the inequality follows trivially. If, on the other hand, $\bar{Y}_i(s) \neq 0$ for all $i$, then $\bar{Y}_i(s) = Y_i(s)$. In this case, let $k(s) = \log g^*(s) - \log g(s)$. Since $k(0) = 0$, it suffices to show that $k'(s) \geq 0$ for all $s$ such that $\sigma(s)$ comes before the first focal point $\sigma(r)$ of $P$ on $\sigma$. We fix $l, (0 \leq l < r)$. We may assume that $Y_1(l), \ldots, Y_{n-2}(l)$ are orthonormal and $Y_1^*(l), \ldots, Y_{n-2}^*(l)$ are orthonormal.

$$\begin{aligned}
&\{\log g(s)\}'(l) \\
=&\{\log G^{1/2}[Y_i(s)]\}'(l) \\
=&\sum_{i=1}^{n-2} \langle Y_i(l), Y_i'(l) \rangle \\
=&\sum_{i=1}^{n-2} \left[ \langle Y_i(0), Y_i'(0) \rangle + \int_0^l \langle Y_i, Y_i' \rangle' \, ds \right] \\
=&\sum_{i=1}^{n-2} \left[ -\beta_{\sigma'}(Y_i(0), Y_i(0)) + \int_0^l \{ \langle Y_i', Y_i' \rangle - \langle R(Y_i, \sigma')\sigma', Y_i \rangle \} ds \right] \\
=&\sum_{i=1}^{n-2} H_\sigma(Y_i, Y_i).
\end{aligned}$$

Similarly, we have

$$\{\log g^*(s)\}'(l) = \sum_{i=1}^{n-2} H_{\sigma^*}(Y_i^*, Y_i^*).$$

Let $E_1, \ldots, E_{n-2} \in V^\perp(\sigma, P)$ be spacelike parallel orthonormal vector fields on $\sigma$ such that $E_i(l) = Y_i(l)$ for all $i$, and let $E_1^*, \ldots, E_{n-2}^* \in$

$V^{\perp}(\sigma^*, S_0^*)$ be spacelike parallel orthonormal vector fields on $\sigma^*$ such that $E_i^*(l) = Y_i^*(l)$ for all $i$. If $Y_i^*(s) = \sum a_{ij}(s)E_j^*(s)$ on $\sigma^*$, then we define a vector field $X_i(s)$ on $\sigma$ by $X_i(s) = \sum a_{ij}(s)E_j(s)$. Then we have $|X_i(s)| = |Y_i^*(s)|$ and $|X_i'(s)| = |Y_i^{*\prime}(s)|$. By taking a suitable linear combination, we may assume $Y_i(l) = X_i(l)$. Since $H_\sigma(Y_i, Y_i) \leq H_\sigma(X_i, X_i)$ by Lemma 2.1, we have

$$\{\log g(s)\}'(l) = \sum_{i=1}^{n-2} H_\sigma(Y_i, Y_i) \leq \sum_{i=1}^{n-2} H_\sigma(X_i, X_i).$$

Hence, it remains to show that

$$\sum_{i=1}^{n-2} H_\sigma(X_i, X_i) \leq \sum_{i=1}^{n-2} H_{\sigma^*}(Y_i^*, Y_i^*).$$

From the definition (2.3) of Hessian form $H_\sigma$, we see

$$\begin{aligned} H_\sigma(X_i, X_i) = & -\beta_{\sigma'}(X_i(0), X_i(0)) \\ & + \int_0^l \{\langle X_i', X_i'\rangle - \langle R(X_i, \sigma')\sigma', X_i\rangle\} ds, \end{aligned}$$

$$\begin{aligned} H_{\sigma^*}(Y_i^*, Y_i^*) = & -\beta_{\sigma^{*\prime}}(Y_i^*(0), Y_i^*(0)) \\ & + \int_0^l \{\langle Y_i^{*\prime}, Y_i^{*\prime}\rangle - \langle R(Y_i^*, \sigma^{*\prime})\sigma^{*\prime}, Y_i^*\rangle\} ds. \end{aligned}$$

Since $Y_i^*$ is a $S_0^*$-Jacobi field on $\sigma^*$ such that $Y_i^*(l) = E_i^*(l)$, we can know easily

$$Y_i^*(s) = (f \circ \gamma(s)/f \circ \gamma(l))\, E_i^*(s),$$

and so

$$X_i(s) = (f \circ \gamma(s)/f \circ \gamma(l))\, E_i(s) \text{ on } \sigma.$$

Therefore

$$\begin{aligned} \sum_{i=1}^{n-2} \beta_{\sigma'}(X_i(0), X_i(0)) &= \left(\frac{f \circ \gamma(0)}{f \circ \gamma(l)}\right)^2 \sum_{i=1}^{n-2} \beta_{\sigma'}(E_i(0), E_i(0)) \\ &= \left(\frac{f \circ \gamma(0)}{f \circ \gamma(l)}\right)^2 (n-2)\langle \sigma'(0), H_p\rangle. \end{aligned}$$

Similarly,

$$\sum_{i=1}^{n-2}\beta_{\sigma^{*\prime}}(Y_i{}^*(0),\, Y_i{}^*(0)) = \left(\frac{f\circ\gamma(0)}{f\circ\gamma(l)}\right)^2 \sum_{i=1}^{n-2}\beta_{\sigma^{*\prime}}(E_i{}^*(0),\, E_i{}^*(0))$$
$$= \left(\frac{f\circ\gamma(0)}{f\circ\gamma(l)}\right)^2 (n-2)\langle\sigma^{*\prime}(0),\, H_{p^*}\rangle.$$

And since the $X_i$s are spacelike, (2.2) implies

$$\sum_{i=1}^{n-2}\langle R(X_i,\,\sigma')\sigma',\, X_i\rangle = \sum_{i=1}^{n-2}\left(\frac{f\circ\gamma(s)}{f\circ\gamma(l)}\right)^2 \langle R(E_i,\,\sigma')\sigma',\, E_i\rangle$$
$$= \left(\frac{f\circ\gamma(s)}{f\circ\gamma(l)}\right)^2 \mathrm{Ric}(\sigma',\,\sigma').$$

Similarly, since the $Y_i^*$s are also spacelike, (2.2) implies

$$\sum_{i=1}^{n-2}\langle R(Y_i^*,\,\sigma^{*\prime})\sigma^{*\prime},\, Y_i^*\rangle = \sum_{i=1}^{n-2}\left(\frac{f\circ\gamma(s)}{f\circ\gamma(l)}\right)^2 \langle R(E_i^*,\,\sigma^{*\prime})\sigma^{*\prime},\, E_i^*\rangle$$
$$= \left(\frac{f\circ\gamma(s)}{f\circ\gamma(l)}\right)^2 \mathrm{Ric}(\sigma^{*\prime},\,\sigma^{*\prime}).$$

The above equalities and the hypotheses in the Theorem give

$$\sum_{i=1}^{n-2} H_\sigma(X_i,\, X_i) \le \sum_{i=1}^{n-2} H_{\sigma^*}(Y_i^*,\, Y_i^*),$$

which completes the proof of Lemma 4.1. □

Let a warped product $\mathbb{M}$, a null geodesic $\sigma^*$ and a function $g^*(s)$ be as defined in the section 3 with the initial values $a_0$ and $a_1$ such that $-a_1/a_0 = h$. Then (3.3) and (3.2) imply

$$\langle\sigma^{*\prime}(0),\, H_{p^*}\rangle = -\frac{(f\circ\gamma)'(0)}{f\circ\gamma(0)} = -\frac{a_1}{a_0} = h,$$
$$\mathrm{Ric}(\sigma^{*\prime},\,\sigma^{*\prime}) = -(n-2)k^2 = -(n-2)\left(\sqrt{\frac{-m}{n-2}}\right)^2 = m.$$

Hence, from the hypothesis in Theorem we obtain

(1) $\langle \sigma'(0), H_p \rangle = h = \langle \sigma^{*\prime}(0), H_{p^*} \rangle$,

(2) $\mathrm{Ric}(\sigma', \sigma') \geq m = \mathrm{Ric}(\sigma^{*\prime}, \sigma^{*\prime})$.

Therefore, Lemma 4.1 implies that for all such $s$,

$$g(s) \leq g^*(s).$$

From (3.1), we know that if $k = 0$, then $f \circ \gamma(s) > 0$ only for

$$s < -\frac{a_0}{a_1} := \frac{1}{h}.$$

And if $k \neq 0$, then $h > k > 0$ from the hypothesis in Theorem. Hence, we also know from (3.1) that for $b$ in the domain $[0, b)$ of $f \circ \gamma$,

$$b \leq \frac{\log[(h+k)/(h-k)]}{2k}.$$

As $s$ approaches the value on the right side, $f \circ \gamma(s)$ tends to 0 and so $A^*(s)$ and $g^*(s)$ also tend to 0 from (3.6). Since

$$g(s) \leq g^*(s),$$

$g(s)$ also tends to 0 as $s$ approaches the value on the right side. Remark in the section 2 implies that there is a focal point $\sigma(r)$ of $P$ along $\sigma$ with

$$\left\{ \begin{array}{ll} 0 < r \leq \dfrac{\log[(h+k)/(h-k)]}{2k} & \text{if} \quad m < 0, \\ 0 < r \leq \dfrac{1}{h} \quad \text{if} \quad m = 0 & \end{array} \right\},$$

provided $\sigma$ is defined on this interval.

## References


[1] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry*, Dekker. New York, 1996.

[2] J. Cheeger and S. T. Yau, *A lower bound for the heat kernel*, Comm. Pure Appl. Math. **34** (1981), 465–480.

[3] E. Heintze and H. Karcher, *A general comparison theorem with application to volume estimates for submanifolds*, Ann. Sci. Ecole Norm. Sup. **11** (1978), 459–470.

[4] Y. Itokawa, *Distance Spheres and Myers-type Theorems for Manifolds with lower bounds on the Ricci curvature*, Illinois J. Math. **34** (1990), no. 4, 693–705.
[5] S. Markvorsen, *A Ricci curvature criterion for compactness of riemannian manifolds*, Arch. Math. **39** (1981), 85–91.
[6] B. O'Neill, *Semi-Riemannian Geometry with applications to relativity*, Academic Press. New York, 1983.

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