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RICCI CURVATURE FOR CONJUGATE AND FOCAL POINTS ON GRW SPACE-TIMES

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ABSTRACT. The authors compute the Ricci curvature of the GRW space-time to obtain two conditions for the conjugate points which appear as the Timelike Convergence Condition(TCG) and the Jacobi inequality. Moreover, under such two conditions, we obtain a lower bound of the length of a unit timelike geodesic for focal points emanating from the immersed spacelike hypersurface, the graph over the fiber in the GRW space-time.

1. Introduction

In [15] Myers showed that if $\mathrm{Ric}(\gamma', \gamma') \geq (n-1)k > 0$ for any unit speed geodesic γ and if the length of γ exceeds $\frac{\pi}{\sqrt{k}}$, then the geodesic γ contains a pair of conjugate points. We may also find some information for the conjugate points and focal points related to the Myers theorem in [4, 5, 6, 7, 9, 10, 11, 12, 13, 15]. In [7], P. Ehrlich and S.-B Kim generalized the condition for the conjugate points of the limit supremum of the integration of the Ricci curvature given by C. Chicone and P. Ehrlich [6] to the limit supremum of the integration of the Ricci curvature plus the trace of square of shear tensor along complete Riemannian or geodesically complete timelike geodesics. On the other hand, to obtain focal points along timelike geodesics orthogonal to the spacelike submanifold of codimension arbitrary, S.-B. Kim and D.-S. Kim [12] generalized the Myers-Galloway theorem on Lorentzian manifolds by using the submanifold index form technique (cf. [8, 9]). Further, S.-B. Kim, D.-S. Kim

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and Y.-T Jung [13] extended the Myers-Galloway theorem given in [12] to find a K-focal point for the spacelike submanifold K of codimension arbitrary by adapting mixed initial conditions of the scalar Riccati equations or Jacobi differential equations derived from the K-Jacobi tensors and their matrix Riccati equations.

Through a series of papers, L. Alias, A. Romero and M. Sanchez [2, 3] computed the Timelike Convergence Condition(TCC) on the Generalized Robertson Walker(GRW) space-time $M = (a,b) \times_f F$, that is, $\mathrm{Ric}(v,v) \geq 0$ for all timelike vector $v \in TM$ and then obtained that TCC holds in M if and only if $f'' \leq 0$ and $\mathrm{Ric}^F \geq (n-1)(ff'' - f'^2)$ where Ric^F is the Ricci tensor of F.

In this paper, we compute the Ricci curvature of the GRW space-time $M = (a,b) \times_f F$ to obtain two conditions for the conjugate points which appear as the two differential inequalities of the warping function f in M given by $\mathrm{Ric}^F \geq (n-1)(ff''-f'^2)g_F$ and the Jacobi inequality $f''+kf \leq 0$ for some constant $k > 0$. Moreover, using such two conditions, we obtain a lower bound of the length of a unit timelike geodesic for focal points emanating from the immersed spacelike hypersurface $x(F)$, the graph over the fiber in M.

2. Preliminaries

Let (F, g_F) be an n-dimensional Riemannian manifold and let $I = (a,b) \subset \mathbf{R}$ be an open interval in $\mathbf{R}$ endowed with the metric $-dt^2$. Throughout this paper, we will denote by M the $n+1$-dimensional product manifold $I \times F$ endowed with the Lorentzian metric g given by

$$g = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g_F),$$

where $f > 0$ is a smooth function on I, and π_I and π_F denote the projections onto I and F, respectively. That is, (M,g) is a warped product with base $(I, -dt^2)$, fiber (F, g_F) and warping function f. We will refer M as a *Generalized Robertson-Walker* (GRW) space-time. We write Ric^F for the pullback by π_F of the Ricci curvature of F. Note that if $\dim M = 4$ and the sectional curvature of the fiber are constant, then (M,g) is a classical Robertson-Walker space-time.

Observe that $\partial_t \in TM$ is a unit timelike vector field globally defined on M determining a time-orientation on M. If $x : S \to M$ is an immersed spacelike hypersurface in M, we may give the unit timelike vector field $N \in TM$ normal to S in the same orientation of ∂_t. Now, we are interested in the special hypersurface in M as the graph over the fiber F. If $u : F \to I$ is a smooth function, then the metric tensor induced on F by the natural embedding $x : F \to M$, $x(p) = (u(p), p)$, is given by

$$-du^2 + f^2(u)g_F, \tag{2.1}$$

and x is spacelike if and only if u satisfies $g_F(\nabla u, \nabla u) < f^2(u)$ on F, where ∇u denotes the gradient of u with respect to the metric g_F. In this case, we obtain that the unit timelike vector field normal to the graph in the same time-orientation of ∂_t is

$$N = \frac{f(u)}{\sqrt{f^2(u) - g_F(\nabla u, \nabla u)}}\Big(\partial_t + \frac{1}{f^2(u)}\nabla u\Big) \tag{2.2}$$

and the mean curvature function H associated to N is given by

$$\begin{aligned} nH\big(f^2(u) - g(\nabla u, \nabla u)\big)^{3/2} = {}&\big(f^2(u) - g(\nabla u, \nabla u)\big)\Big(nf'(u) + \frac{1}{f(u)}\Delta u\Big) \\ &+ \frac{1}{f(u)}\nabla^2 u(\nabla u, \nabla u) - f'(u)g(\nabla u, \nabla u), \end{aligned}$$

where Δ and ∇^2 denote the Laplacian and the Hessian of the Riemannian metric g_F, respectively(cf. [1, 2, 3]).

We may recall some formulas for the Ricci curvature tensor Ric of the GRW space-times M in [4, 5, 16]. Applying a timelike tangent vector field v of the type $v = \alpha\partial_t + v_1$ with v_1 tangent to F to compute the Ricci curvature Ric, we have

$$\begin{aligned} \mathrm{Ric}(\partial_t, \partial_t) &= -n\frac{f''(t)}{f(t)}, \\ \mathrm{Ric}(\partial_t, v_1) &= 0, \\ \mathrm{Ric}(v_1, v_1) &= \mathrm{Ric}^F(v_1, v_1) \\ &\quad + g(v_1, v_1)\Big(\frac{f''(t)}{f(t)} + (n-1)\big(\frac{f'(t)}{f(t)}\big)^2\Big), \end{aligned} \tag{2.3}$$

where v_1 is tangent to the fibers. Calculating the Ricci curvature on a GRW space-time and using (2.3), we also obtain

$$\mathrm{Ric}(v, v) = \mathrm{Ric}^F(v_1, v_1) + \Big(\frac{f''}{f} + (n-1)\big(\frac{f'}{f}\big)^2\Big)g(v_1, v_1) - n\frac{f''}{f}\alpha^2. \tag{2.4}$$

Inspired by the Lorentzian version of Myers Theorem given by J. Beem and P. Ehrlich ([4, 5]) and S.-B. Kim and D.-S. Kim [12], we apply the TCC and the spacelike hypersurfaces as in [2, 3] for conjugate and focal points of the GRW space-times

3. The main Theorems

J. Beem and P. Ehrlich obtained a upper bound of the acceleration of a reformed warping function $\phi = \ln f$ for the positive Ricci curvature by using the timelike tangent vector field $v = \partial_t + v_1$ on the GRW space-time M with the hypothesis $\mathrm{Ric}^F(v_1, v_1) \geq \rho g_F(v_1, v_1)$ for a constant $\rho \in \mathbf{R}$ (cf. Proposition 2.59 in [4] or Proposition 3.72 in [5]). But, if v is unit for the timelike tangent vector field $v = \partial_t + v_1$, then $g_F(v_1, v_1) = 0$ for all $v_1 \in TF$, which is impossible since g_F is positive definite. However, we cultivate their ideas by adapting the unit timelike tangent vector field v of the type $v = \alpha\partial_t + v_1$ to obtain the conditions for the conjugate points of (M, g) by computing the Ricci curvature Ric of the GRW space-time M

THEOREM 3.1. *Let $M = (a, b) \times_f F$ be a GRW space-time with $\dim F = n \geq 2$. Suppose there exists a positive constant $\rho \in \mathbf{R}$ such that $\mathrm{Ric}^F \geq \rho g_F$ on TF and let the warping function of (M, g) satisfies $f''(t) + kf(t) \leq 0$ for some positive constant k, Then, if $\gamma : [0, L] \to M$ is any unit speed timelike geodesic of length $L(\gamma)$ with $L(\gamma) \geq \pi/\sqrt{k}$ for some constant $k > 0$, the geodesic γ has a pair of conjugate points.*

Proof. From the Lorentzian version of Myers Theorem, we need a Ricci curvature condition that $\mathrm{Ric}(v, v) \geq nk > 0$ for any unit timelike vector v of the type $v = \alpha\partial_t + v_1 \in TM$. Since $g(\partial_t, \partial_t) = -1$, we have $\beta = g(v_1, v_1) = \alpha^2 - 1 \geq 0$. Also, since $\mathrm{Ric}(v, v) = \alpha^2 \mathrm{Ric}(\partial_t, \partial_t) + \mathrm{Ric}(v_1, v_1)$, putting $\phi = \ln f$, we have from (2.4) that

$$\begin{aligned}\mathrm{Ric}(v, v) =& \mathrm{Ric}^F(v_1, v_1) \\ & + (\beta - n\alpha^2)\phi''(t) + n(1 - \alpha^2)\phi'(t)^2.\end{aligned} \tag{3.1}$$

Since $g_F(v_1, v_1) = \beta e^{-2\phi}$, if $\mathrm{Ric}^F(v_1, v_1) \geq \rho g_F(v_1, v_1)$ for all $v_1 \in TF$, the term of (3.1) is following

$$\mathrm{Ric}(v, v) \geq \rho\beta e^{-2\phi} + ((1 - n)\beta - n)\phi''(t) - n\phi'(t)^2. \tag{3.2}$$

Thus, $\mathrm{Ric}(v, v) \geq nk > 0$ provided $\phi'' < G(\beta)$ for all $\beta \in [0, \infty)$, where

$$G(\beta) = \frac{\rho\beta e^{-2\phi} - n\phi'(t)^2 - nk}{(n - 1)\beta + n}.$$

Since $\rho > 0$, we have

$$G'(\beta) = \frac{n\rho e^{-2\phi} + n(n-1)\big((\phi'(t))^2 + k\big)}{((n-1)\beta + n)^2} > 0.$$

Hence, we can check that $G(\beta)$ is increasing and $G'(\beta)$ does not change sign in $[0,\infty)$. Thus, $G(\beta)$ has its minimum on $[0,\infty)$ for $\beta = 0$. Thus, we have

$$\phi(t)'' \leq -\phi'(t)^2 - k \text{ for all } t \in (a,b). \tag{3.3}$$

Since $\phi = \ln f$, we finally obtain the Jacobi inequality $f''(t) + kf(t) \leq 0$ for all $t \in (a,b)$. □

Now, using the warping function f, we may generalize the condition of the positive constant $\rho \in \mathbf{R}$ such that $\mathrm{Ric}^F \geq \rho g_F$ on TF to find the conditions for conjugate points of the GRW space-time M in Theorem 3.1 as follows.

THEOREM 3.2. *Let $M = (a,b) \times_f F$ be a GRW space-time and with $\dim F = n \geq 2$. Suppose (M,g) satisfies $\mathrm{Ric}^F(v_1,v_1) \geq (n-1)(f''f - f'^2)g_F(v_1,v_1)$ on TF and $f'' + kf \leq 0$ for some constant $k > 0$. Then, if $\gamma : [0,L] \to M$ is any unit speed timelike geodesic of length $L(\gamma)$ with $L(\gamma) \geq \pi/\sqrt{k}$, the geodesic γ has a pair of conjugate points.*

Proof. From the first part of the proof in Theorem 3.1, we obtain from (2.4) that

$$\mathrm{Ric}(v,v) = \mathrm{Ric}^F(v_1,v_1) + (\alpha^2 - 1)\Big(\frac{f''}{f} + (n-1)(\frac{f'}{f})^2\Big) - n\frac{f''}{f}\alpha^2. \tag{3.4}$$

Thus, we have

$$\mathrm{Ric} = \mathrm{Ric}^F(v_1,v_1) - (n-1)\Big(\frac{f''}{f} - (\frac{f'}{f})^2\Big)(\alpha^2 - 1) - n\Big(\frac{f''}{f}\Big).$$

Since $g_F(v_1,v_1) = \frac{1}{f^2}g(v_1,v_1) = \frac{1}{f^2}(\alpha^2 - 1) \geq 0$, we have the conditions for the conjugate points $\mathrm{Ric}^F(v_1,v_1) \geq (n-1)(f''f - f'^2)g_F(v_1,v_1)$ and $f'' + kf \leq 0$ to get the Ricci condition $\mathrm{Ric}(v,v) \geq nk > 0$. □

If F is a complete Riemannian manifold, the GRW space-time M is globally hyperbolic([4, 5]). From the Lorentzian version of Bonnet-Myers Theorem, we have the timelike diameter of (M,g) as follows.

COROLLARY 3.3. *Let $M = (a,b) \times_f F$ be a GRW space-time and let F be a complete Riemannian manifold with* $\dim F = n \geq 2$. *Then, if* $\mathrm{Ric}^F(v_1, v_1) \geq (n-1)(f''f - f'^2)g_F(v_1, v_1)$ *on TF and $f'' + kf \leq 0$ for some constant $k > 0$, we have* $\mathrm{diam}(M, g) \leq \pi/\sqrt{k}$.

From the Ricci curvature, it is easy to see that

$$\begin{aligned}
\mathrm{Ric}(N,N) &= \frac{f^2(u)}{f^2(u) - g_F(\nabla u, \nabla u)}\Big(\mathrm{Ric}(\partial_t, \partial_t) + \frac{\mathrm{Ric}(\nabla u, \nabla u)}{f^4(u)}\Big) \\
&= \frac{f^2(u)}{f^2(u) - g_F(\nabla u, \nabla u)}\Big[-n\frac{f''(u)}{f(u)} + \frac{1}{f^4(u)}\Big(\mathrm{Ric}^F(\nabla u, \nabla u) \\
&\quad + g(\nabla u, \nabla u)\big(\frac{f''}{f(u)} + (n-1)\frac{(f'(u))^2}{f(u)}\big)\Big)\Big]. \qquad (3.5)
\end{aligned}$$

Moreover, we may obtain some geometric inequalities for the focal points along the unit speed timelike geodesics orthogonal to the immersed spacelike hypersurfaces of the graphs over the fiber in a GRW space-time.

THEOREM 3.4. *Let (M,g) be a connected GRW space-time and suppose (F, g_F) is a Riemannian hypersurface of (M,g) with $\dim F = n \geq 2$ satisfying* $\mathrm{Ric}^F(v_1, v_1) \geq (n-1)(f''f - f'^2)g_F(v_1, v_1)$ *on TF and $f''(u) + kf(u) \leq 0$ for some constant $k > 0$. Then if $\gamma : [0, L] \to M$ is any unit speed timelike geodesic of length $L(\gamma)$ with $\gamma'(0) = N$ perpendicular at $\gamma(0)$ to the immersed spacelike hypersurfaces of M and if*

$$L(\gamma) \geq \frac{-4H + \sqrt{16H^2 + 4kn^2\pi^2}}{4nk}$$

for the mean curvature H associated to N, the geodesic γ has a focal point to $x(F)$.

Proof. To derive geometric consequences from this property, we set $E_{n+1}(t) = \gamma'(t)$ and $\{E_1(t), E_2(t), \cdots, E_n(t)\}$ be n spacelike parallel fields such that $\{E_1(0), \cdots, E_n(0)\}$ forms an orthonormal basis of $T_{\gamma(0)}x(F)$ and $\{E_1(t), E_2(t), \cdots, E_n(t)\}$ the orthonormal basis of $T_{\gamma(t)}x(F)$.

Set $W_i(t) = \cos(\frac{\pi t}{2L})E_i(t)$ for $i = 1, 2, \cdots, n$. Using the submanifold index form, we obtain

$$\sum_{i=1}^{n} I(W_i, W_i) = \sum_{i=1}^{n} g(S_{\gamma'(0)}W_i(0), W_i(0)) - \sum_{i=1}^{n} \int_0^L \Big[(\frac{\pi}{2L})^2 \sin^2(\frac{\pi t}{2L}) g(E_i, E_i) - \cos^2(\frac{\pi t}{2L}) g(R(E_i, \gamma')\gamma', E_i)\Big] dt$$

for $i = 1, 2, \cdots, n$. If $\mathrm{Ric}(\gamma', \gamma') \geq nk > 0$ for all $t \in [0, L]$ and $L \geq -4H + \sqrt{16H^2 + 4n^2\pi^2 k}/4nk$, we have $\sum_{i=1}^{n} I(W_i, W_i) \geq 0$. Hence, $I(W_i, W_i) \geq 0$ for some $i \in \{1, 2, \cdots, n\}$. On the other hand, if $\gamma|_{[0,L]}$ has no focal points, we should have that $I(W_i, W_i) < 0$ for all i from the Lorentzian index lemma(cf. Theorem 10.22 in [5], or Theorem 34 in [16]). Thus, γ has a focal point if

$$L \geq \frac{-4H + \sqrt{16H^2 + 4kn^2\pi^2}}{4nk}. \qquad \square$$

COROLLARY 3.5. *Let $M = (a, b) \times_f F$ be a connected GRW space-time and let F a complete Riemannian manifold. Let $x : F \to M$ be a compact immersed spacelike hypersurface with* $\dim F = n \geq 2$. *For any unit speed maximal timelike geodesic γ of length L with $\gamma'(0) = N$ perpendicular at $\gamma(0)$ to the spacelike hypersurfaces $x(F)$, suppose* $\mathrm{Ric}^F(v_1, v_1) \geq (n-1)(f''f - f'^2)g_F(v_1, v_1)$ *on TF and $f''(u) + kf(u) \leq 0$ for some constant $k > 0$. Then*

$$\mathrm{diam}_{x(F)}(M, g) \leq \frac{-4H + \sqrt{16H^2 + 4kn^2\pi^2}}{4nk}$$

for the mean curvature H associated to N

REMARK. If $x : F \to M$ is an immersed compact maximal spacelike hypersurface of M, then $\mathrm{diam}_{x(F)}(M, g) \leq \pi/2\sqrt{k}$, which is the just half of $\mathrm{diam}(M, g)$ in Corollary 3.3.

References

[1] Luis J. Alias, A. Romero, and M. Sanchez, *Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems*, Tohoku Math. J. **49** (1997), 337–345.

[2] ———, *Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes*, General Relativity and Gravitation **27** (1995), no. 1, 71–84.

[3] ———, *Spacelike hypersurfaces of constant mean curvature in spatially closed Lorentzian manifolds*, WOGDA'94, Proceeding of III Fall's Workshop: Differential Geometry and its Applications (1994), 197–207.

[4] J. Beem and P. Ehrlich, *Global Lorentzian Geometry*, Marcel Dekker Pure and Applied Mathematics **67** (1981).

[5] J. Beem, P. Ehrlich, and K. Easley, *Global Lorentzian Geometry, Second Revised edition*, Marcel Dekker Pure and Applied Mathematics (1996).

[6] C. Chicone and P. Ehrlich, *Line integration of Ricci curvature and conjugate points in Lorentzian and Riemannian manifolds*, Manuscripta Math. **31** (1981), 297–316.

[7] P. Ehrlich and S.-B. Kim, *From the Riccati inequality to the Raychaudhuri equation*, Contemporary Math. Series **170** (1994), 65–78.

[8] P. Ehrlich and M. Sanchez, *Some semi-Riemannian volume comparison theorems*, preprint.

[9] J.-H. Eschenburg and J. J. O'Sullivan, *Jacobi tensors and Ricci curvature*, Math. Ann. **252** (1980), 1–26.

[10] T. T. Frankel, *On the size of a rotating fiuid mass in general relativity*, Thesis, University of California, Sandiago (1979).

[11] G. J. Galloway, *A generalization of Myers theorem and an application to relativistic cosmology*, J. of Diff. Geom. **14** (1979), 105–116.

[12] S.-B. Kim and D.-S. Kim, *A focal Myers-Galloway theorem on space-time*, J. Korean Math. Soc. **31** (1994), no. 1, 97–110.

[13] S.-B. Kim, D.-S. Kim, and Y.-T. Jung, *Jacobi Tensors and Focal Points for Spacelike Submanifolds*, Kyungpook Math. J. **35** (1996), no. 3, 633–638.

[14] W. Kühnel, *Conformal transformations between Einstein spaces*, Conformal Geometry (edited by R. Kulkarni and U. Pinkal), Vieweg Publishing Company, Braunschweig (1988), 105–146.

[15] S. B. Myers, *Riemannian manifolds with positive mean curvature*, Duke Math. J. **8** (1941), 401–404.

[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press Pure and Applied Mathematics (1983).

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