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ON THE BONNET'S THEOREM FOR COMPLEX FINSLER MANIFOLDS

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ABSTRACT. In this paper, we investigate the topology of complex Finsler manifolds. For a complex Finsler manifold (M, F), we introduce a certain condition on the Finsler metric F on M. This is a generalization of the Kähler condition for the Hermitian metric. Under this condition, we can produce a Kähler metric on M. This enables us to use the usual techniques in the Kähler and Riemannian geometry.

We show that if the holomorphic sectional curvature of M is $\geq c^2 > 0$ for some $c > 0$, then $diam(M) \leq \frac{\pi}{c}$ and hence M is compact. This is a generalization of the Bonnet's theorem in the Riemannian geometry.

1. Introduction

Let M be an n-dimensional complex manifold with a local coordinate system (z^i), $i = 1, 2, \cdots, n$, where $z^i = x^i + \sqrt{-1}y^i$ so that (x^i, y^i), $i = 1, 2, \cdots, n$, is a local coordinate system of the underlying real manifold. We also use $x^{n+i} = y^i$, $i = 1, 2, \cdots, n$, so that (x^α), $\alpha = 1, 2, \cdots, 2n$ is a real local coordinate system. We will use (z^i, ζ^i), $i = 1, 2, \cdots, n$ as a local coordinate system for the holomorphic tangent bundle $T^{1,0}M$. Let J be the complex structure tensor of M defined by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\partial}{\partial x^i} \quad \text{for } i = 1, \cdots, n.$$

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Given a curve $c : [0, l] \to M$, we interpret $\dot{c}(t)$ as either

$$\sum_{i=1}^{n} \frac{dc^i}{dt} \frac{\partial}{\partial z^i} \qquad \text{or} \qquad \sum_{\alpha=1}^{2n} \frac{dc^\alpha}{dt} \frac{\partial}{\partial x^\alpha}.$$

depending on whether we regard M as a complex manifold or as a real manifold. They are identified by the identification map $\phi : T_pM \to T_p^{1,0}M$ defined by $\phi(v) = \frac{1}{2}(v - \sqrt{-1}Jv)$.

Complex Finsler metric is a generalization of the Hermitian metric in that we only require the structure on each tangent space to be a Hermitian norm rather than to be a Hermitian inner product. Formally,

Definition 1.1. A complex Finsler metric F on M is a function $F : T^{1,0}M \to \mathbb{R}$ satisfying:

(1) F is smooth outside of the zero section of $T^{1,0}M$;
(2) $F(z, \zeta) \geq 0$ and $F(z, \zeta) = 0$ if and only if $\zeta = 0$;
(3) $F(z, \lambda\zeta) = |\lambda| F(z, \zeta)$ for all $\lambda \in \mathbb{C}$;
(4) F is strongly pseudo-convex, i.e., $\left[\frac{\partial^2 F^2}{\partial \zeta^i \partial \bar{\zeta}^j}\right]$ is positive definite.

This structure is enough to define a length of a curve and in turn a distance between two points. This enables us to have an estimate on the length of a minimizing geodesic joining two points.

To a complex Finsler metric F on M, we can associate a real function F^o on M satisfying real analogue of (1)–(3) above. Indeed, $F^o : TM \to \mathbb{R}$ defined by $F^o(v) = F(\phi(v))$ for any real tangent vector v is a real Finsler metric without the condition that F^o is strongly convex. I.e., the real Hessian of $(F^o)^2$ is not necessarily positive definite.

Throughout this paper, we will denote by G the function F^2. This G is $(1,1)$-homogeneous in ζ, i.e., for all $(z, \zeta) \in T^{1,0}M$ and $\lambda \in \mathbb{C}$, $G(z, \lambda\zeta) = \lambda\bar{\lambda}G(z, \zeta)$. And we will use subscripts to denote the differentiations with respect to ζ variables. For example, $G_{i\bar{j}} = \frac{\partial^2 G}{\partial \zeta^i \partial \bar{\zeta}^j}$.

Given a complex Finsler metric F on M, we define a length $L_F(c)$ of a curve $c : [0, l] \to M$ in M by

$$L_F(c) = \int_0^l F(c(t), \dot{c}(t))\, dt, \text{ where } \dot{c}(t) = \sum_{i=1}^{n} \frac{dc^i}{dt} \frac{\partial}{\partial z^i}.$$

DEFINITION 1.2. A geodesic for a complex Finsler metric F is a curve which is a critical point of L_F. More precisely, a curve $c : [0, l] \to M$ is a geodesic if for every variation $c_s : [0, l] \to M$, $-\epsilon < s < \epsilon$, of c,

$$\left.\frac{d}{ds}\right|_{s=0} L_F(c_s) = 0.$$

In terms of a local coordinate system (z, ζ) of $T^{1,0}M$, the geodesic equations for a complex Finsler metric F are

$$\frac{d^2c^i}{dt^2} + \sum_{k,l=1}^{n}\sum_{h=1}^{n} G^{i\bar{h}} \frac{\partial G_{k\bar{h}}}{\partial z^l}\frac{dc^k}{dt}\frac{dc^l}{dt} = 0, \qquad i = 1, 2, \cdots, n. \tag{1.1}$$

By the general theory of ordinary differential equations, we have

PROPOSITION 1.1. *Let F be a complex Finsler metric. Given $z \in M$ and nonzero $\zeta \in T_z^{1,0}M$, there exists a geodesic $c : (-\epsilon, \epsilon) \to M$ satisfying $c(0) = z$ and $\dot{c}(0) = \zeta$ for some $\epsilon > 0$.*

We also assume that the real Finsler metric associated to a complex Finsler metric is strongly convex. This is a usual assumption in the problem of classifying complex Finsler manifolds with constant holomorphic sectional curvature. For this, see [1]. If F^o is strongly convex, then well-established techniques in real Finsler geometry are readily applicable. Specifically, we have a version of the theorem of Hopf and Rinow that guarantees the existence of the minimizing geodesics joining any two points.

THEOREM 1.2. *Let (M, F) be a complete complex Finsler manifold with strongly convex F^o. Then any two points can be joined by a minimizing geodesic.*

Proof. Note that $L_F(c) = L_{F^o}(c)$. And apply the theorem of Hopf and Rinow in the real Finsler manifolds. □

For complete treatments on the Finsler geometry, we refer the reader to [2], [8], [1]. And for the geometry of the spaces beyond the Riemannian manifolds, we refer to [6], [4], [9].

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2. Main theorem

Let M be a complex manifold. The action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $T^{1,0}M \setminus \{O\}$ by scalar multiplication defines the projective holomorphic tangent bundle $\mathbb{P}T^{1,0}M$ of M by $\mathbb{P}T^{1,0}M = (T^{1,0}M \setminus \{O\})/\mathbb{C}^*$. Let $\tilde{\pi} : p^*T^{1,0}M \to \mathbb{P}T^{1,0}M$ be the pull-back bundle of the holomorphic tangent bundle $\pi : T^{1,0}M \to M$ by the canonical projection $p : \mathbb{P}T^{1,0}M \to M$. Since $G_{i\bar{j}}(z,\zeta)$ is a function defined on $\mathbb{P}T^{1,0}M$ and $[G_{i\bar{j}}]$ is positive definite, $[G_{i\bar{j}}]$ defines a Hermitian inner product on each fiber of $\tilde{\pi} : p^*T^{1,0}M \to \mathbb{P}T^{1,0}M$.

Let D be the Chern connection of the Hermitian vector bundle $\tilde{\pi} : p^*T^{1,0}M \to \mathbb{P}T^{1,0}M$. This is a unique connection of type $(1,0)$ which is compatible with the Hermitian structure. For this, see [3]. Following the idea of S. Kobayashi [5], we apply the techniques of the Hermitian geometry to the pull-back bundle $\tilde{\pi} : p^*T^{1,0}M \to \mathbb{P}T^{1,0}M$. Let $\{\omega_i{}^j\}$ and $\{\Omega_i{}^j\}$ be the connection forms and curvature forms of D with respect to a local frame $\{\frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^n}\}$. I.e.,

$$D\frac{\partial}{\partial z^i} = \sum_{j=1}^n \omega_i{}^j \frac{\partial}{\partial z^j} \qquad \text{and} \qquad D^2\frac{\partial}{\partial z^i} = \sum_{j=1}^n \Omega_i{}^j \frac{\partial}{\partial z^j}.$$

In terms of a local coordinate system (z^i, ζ^i) of $\mathbb{P}T^{1,0}M$,

$$\begin{aligned}
\omega_i{}^j &= G^{j\bar{h}}\frac{\partial G_{i\bar{h}}}{\partial z^k}dz^k + G^{j\bar{h}}\frac{\partial G_{i\bar{h}}}{\partial \zeta^k}d\zeta^k, \\
\Omega_i{}^j &= R_i{}^j{}_{k\bar{l}}dz^k \wedge d\bar{z}^l + P_i{}^j{}_{k\bar{l}}dz^k \wedge d\bar{\zeta}^l \\
&\quad + S_i{}^j{}_{k\bar{l}}d\zeta^k \wedge d\bar{z}^l + Q_i{}^j{}_{k\bar{l}}d\zeta^k \wedge d\bar{\zeta}^l,
\end{aligned}$$

where

$$\begin{aligned}
R_i{}^j{}_{k\bar{l}} &= -G^{j\bar{h}}\frac{\partial^2 G_{i\bar{h}}}{\partial z^k \partial \bar{z}^l} + G^{j\bar{h}}G^{a\bar{b}}\frac{\partial G_{a\bar{h}}}{\partial \bar{z}^l}\frac{\partial G_{i\bar{b}}}{\partial z^k}, \\
P_i{}^j{}_{k\bar{l}} &= -G^{j\bar{h}}\frac{\partial^2 G_{i\bar{h}}}{\partial z^k \partial \bar{\zeta}^l} + G^{j\bar{h}}G^{a\bar{b}}\frac{\partial G_{a\bar{h}}}{\partial \bar{\zeta}^l}\frac{\partial G_{i\bar{b}}}{\partial z^k}, \\
S_i{}^j{}_{k\bar{l}} &= -G^{j\bar{h}}\frac{\partial^2 G_{i\bar{h}}}{\partial \zeta^k \partial \bar{z}^l} + G^{j\bar{h}}G^{a\bar{b}}\frac{\partial G_{a\bar{h}}}{\partial \bar{z}^l}\frac{\partial G_{i\bar{b}}}{\partial \zeta^k}, \\
Q_i{}^j{}_{k\bar{l}} &= -G^{j\bar{h}}\frac{\partial^2 G_{i\bar{h}}}{\partial \zeta^k \partial \bar{\zeta}^l} + G^{j\bar{h}}G^{a\bar{b}}\frac{\partial G_{a\bar{h}}}{\partial \bar{\zeta}^l}\frac{\partial G_{i\bar{b}}}{\partial \zeta^k}.
\end{aligned}$$

Setting $R_{i\bar{j}k\bar{l}} = G_{h\bar{j}} R_{i\ k\bar{l}}^{\ h}$, etc., we obtain

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 G_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{a,b=1}^{n} G^{a\bar{b}} \frac{\partial G_{a\bar{j}}}{\partial \bar{z}^l} \frac{\partial G_{i\bar{b}}}{\partial z^k}, \text{ etc..}$$

DEFINITION 2.1. Let ζ be a nonzero holomorphic tangent vector at $z \in M$. The holomorphic sectional curvature H of ζ at $z \in M$ of a complex Finsler manifold (M, F) is

$$H(z,\zeta) = \frac{1}{F^4(z,\zeta)} \sum_{i,j,k,l=1}^{n} R_{i\bar{j}k\bar{l}} \zeta^i \bar{\zeta}^j \zeta^k \bar{\zeta}^l.$$

For a complex manifold M with a Kähler metric $g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz^i d\bar{z}^j$, its curvature tensor is

$$\mathcal{R}_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + g^{a\bar{b}} \frac{\partial g_{i\bar{b}}}{\partial z^k} \frac{\partial g_{a\bar{j}}}{\partial \bar{z}^l}$$

and the holomorphic sectional curvature $\mathcal{H}(z,\xi)$ of $\xi = (\xi^1, \cdots, \xi^n)$ is

$$\mathcal{H}(z,\xi) = \frac{1}{\|\xi\|^4} \mathcal{R}_{i\bar{j}k\bar{l}} \xi^i \bar{\xi}^j \xi^k \bar{\xi}^l,$$

where $\|\cdot\|$ is the norm associated to the Kähler metric g.

DEFINITION 2.2. A complex Finsler metric F on M is called pseudo-Kähler if $\frac{\partial G_{i\bar{j}}}{\partial z^k}(z,\zeta) = \frac{\partial G_{k\bar{j}}}{\partial z^i}(z,\zeta)$ for all $(z,\zeta) \in \mathbb{P}T^{1,0}M$.

If complex Finsler metric F is a priori a Hermitian metric, then the above condition is the usual Kähler condition. And for pseudo-Kähler Finsler metric F, by the Euler's identity, we have $\frac{\partial G_i}{\partial z^k} = \frac{\partial G_k}{\partial z^i}$.

The main goal of this paper is to prove the following

THEOREM 2.1. *Let (M, F) be a complete complex pseudo-Kähler Finsler manifold with strongly convex F^o. If the holomorphic sectional curvature of M is $\geq c^2 > 0$ for some $c > 0$, then $diam(M) \leq \frac{\pi}{c}$ and hence M is compact.*

3. Proof of the main theorem

We first establish the following propositions which are essential in proving Theorem 2.1.

PROPOSITION 3.1. *Given a geodesic $c_o : [0,l] \to M$ joining p and q in M, there is a C^1-variation $\alpha : [0,l] \times (-\epsilon,\epsilon) \to M$ of c_o, such that*

(1) *α is one-to-one,*
(2) *$\{\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\}$ are linearly independent for all $t \in [0,l]$, $s \in (-\epsilon,\epsilon)$,*
(3) *$\frac{\partial\alpha}{\partial s}(t,s)|_{s=0} = JT(t)$.*

Proof. By the theory of ordinary differential equations, for any $t \in [0,l]$, there exists $\epsilon(t) > 0$ such that a unique geodesic $\alpha_t(s)$ with the initial condition $\alpha_t(0) = c_0(t)$ and $\dot\alpha_t(0) = JT(t)$ is defined on $(-\epsilon(t),\epsilon(t))$. Since this $\epsilon(t)$ depends continuously on the initial datum and $[0,l]$ is compact, we can choose ϵ' such that $\alpha_t(s)$ is defined on $(-\epsilon',\epsilon')$ for all $t \in [0,l]$. Let $\alpha(t,s) = \alpha_t(s)$.

We know that $\{T, JT(t)\}$ is linearly independent over $\mathbb{R}$ along c_o, i.e., the sum of the squares of the determinants of all possible 2×2 minor matrices of the Jacobian matrix $J_{\mathbb{R}}(\alpha)$ of α is nonzero for $s=0$. So there exists $\epsilon > 0$ such that the rank of the Jacobian matrix $J_{\mathbb{R}}(\alpha)$ of α is 2 for every $s \in (-\epsilon,\epsilon)$. This implies that α is locally one-to-one on $[0,l]\times(-\epsilon,\epsilon)$. Note in particular that $\{\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial s}\}$ are linearly independent.

Now we want to show that α is globally one-to-one for sufficiently small $\epsilon > 0$. Assume such ϵ does not exist. Then we have two sequences (t_n,s_n) and (t'_n,s'_n) for $n > N$ in $[0,l] \times [-\frac{1}{N},\frac{1}{N}]$ for some N such that $\alpha(t_n,s_n) = \alpha(t'_n,s'_n)$ and $(t_n,s_n) \neq (t'_n,s'_n)$ $-\frac{1}{n} < s_n, s'_n < \frac{1}{n}$. Since $[0,l] \times [-\frac{1}{N},\frac{1}{N}]$ is compact, they have converging subsequences with (t_o,s_o) and (t'_o,s'_o) as their limit points, respectively. Clearly, $s_o = s'_o = 0$. Next we will show that $t_o = t'_o$. By the continuity of α, we obtain that $\alpha(t_n,s_n)$ tends to $\alpha(t_o,0) = c_o(t_o)$ and that $\alpha(t'_n,s'_n)$ tends to $\alpha(t'_o,0) = c_o(t'_o)$. But by the assumption that $\alpha(t_n,s_n) = \alpha(t'_n,s'_n)$, we get $c_o(t_o) = c_o(t'_o)$ and hence $t_o = t'_o$. Thus the limit points are same.

On the other hand, α is locally one-to-one. So we can choose a neighborhood U of $(t_o,0) = (t'_o,0)$ where α is one-to-one. But there are two distinct points $(t_n,s_n) \neq (t'_n,s'_n)$ in U whose images under α are same. This is a contradiction. Therefore, we can choose $\epsilon > 0$ such that $\alpha : [0,l] \times (-\epsilon,\epsilon) \to M$ is one-to-one. □

Now modify the variation α. We define $c(t,s) = \alpha(t, s\sin\frac{\pi t}{l})$. Then we have

$$\begin{aligned}
&c(0,s) = \alpha(0,0) = c_o(0) = p \quad \text{and} \quad c(l,s) = \alpha(l,0) = c_o(l) = q,\\
&\frac{\partial c}{\partial t} = \frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial s}\frac{\pi s}{l}\cos\frac{\pi t}{l} \neq 0 \qquad \text{by (2) of Proposition 3.1,}\\
&\frac{\partial c}{\partial s} = \frac{\partial \alpha}{\partial s}\sin\frac{\pi t}{l},\\
&\left.\frac{\partial c}{\partial s}\right|_{s=0} = \frac{\partial \alpha}{\partial s}\sin\frac{\pi t}{l} = (\sin\frac{\pi t}{l})JT(t).
\end{aligned}$$

Next choose a continuous nowhere vanishing vector field X on some open set U containing $\{c(t,s) : t \in (0,l), s \in (-\epsilon,\epsilon)\}$ such that X agrees with $\frac{\partial c}{\partial t}(t,s)$, i.e.,

$$X(z) = \frac{\partial c}{\partial t}(t,s) \quad \text{if} \quad z = c(t,s).$$

For example, locally in the preferred real coordinate neighborhood, the image of $c(t,s)$ is considered as a real 2−dimensional slice L of $\mathbb{R}^{2n}$ and for z in the preferred real coordinate neighborhood, define $X(z)$ as $\frac{\partial c}{\partial t}(t,s)$ where $c(t,s)$ is the projection of z onto L and then use the partition of unity. Note that U can not contain p and q because of the continuity of X.

PROPOSITION 3.2. *On U, we have a Kähler metric defined by $g_{i\bar{j}}(z) = G_{i\bar{j}}(z, X(z))$ if (M,F) is a pseudo-Kähler Finsler manifold. We call this metric $g = g_{i\bar{j}}dz^i d\bar{z}^j$ the induced Kähler metric on U.*

Proof. By the definition of $g_{i\bar{j}}$, we have

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial G_{i\bar{j}}}{\partial z^k} + \sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \zeta^l}\frac{\partial X^l}{\partial z^k} + \sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \bar{\zeta}^l}\frac{\partial \overline{X}^l}{\partial z^k}.$$

Note that by the homogeneity of $G_{i\bar{j}}$,

$$\sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \zeta^l}(z, X(z))X^l = 0.$$

Hence we have

$$0 = \frac{\partial}{\partial z^k}\left(\sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \zeta^l}(z, X(z))X^l\right)$$
$$= \sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \zeta^l}\frac{\partial X^l}{\partial z^k} + \sum_{l=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^l \partial z^k}(z, X(z))X^l.$$

Note also that by the homogeneity of $\frac{\partial G_{i\bar{j}}}{\partial z^k}$,

$$\sum_{l=1}^{n}\frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^l \partial z^k}(z, X(z))X^l = 0.$$

Hence

$$\sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \zeta^l}\frac{\partial X^l}{\partial z^k} = 0.$$

And by the same token,

$$\sum_{l=1}^{n}\frac{\partial G_{i\bar{j}}}{\partial \bar{\zeta}^l}\frac{\partial \bar{X}^l}{\partial z^k} = 0.$$

Thus

$$\frac{\partial g_{i\bar{j}}}{\partial z^k} = \frac{\partial G_{i\bar{j}}}{\partial z^k} = \frac{\partial G_{k\bar{j}}}{\partial z^i} = \frac{\partial g_{k\bar{j}}}{\partial z^i}. \tag{3.1}$$ □

Now note that if c is a geodesic for a complex Finsler metric, then it is also a geodesic for the induced Kähler metric. Indeed, along a curve c, the induced Kähler metric g on U defined in Proposition 3.2 satisfies $g_{i\bar{j}}(c(t)) = G_{i\bar{j}}(c(t), \dot{c}(t))$ and (3.1). And so (1.1) reads

$$\frac{d^2c^i}{dt^2} + \sum_{k,l=1}^{n}\sum_{h=1}^{n} g^{i\bar{h}}\frac{\partial g_{k\bar{h}}}{\partial z^l}\frac{dc^k}{dt}\frac{dc^l}{dt} = 0\ , \qquad i = 1, 2, \cdots, n,$$

which is the geodesic equation for the Kähler metric g.

Next we will relate the notions of the holomorphic sectional curvatures of the complex Finsler metric and the induced Kähler metric.

PROPOSITION 3.3. *Along $c(t)$,*

$$H(c(t), \dot{c}(t)) = \frac{1}{F^4(c(t), \dot{c}(t))}R_{i\bar{j}k\bar{l}}\dot{c}^i\bar{\dot{c}}^j\dot{c}^k\bar{\dot{c}}^l.$$

And $H(c(t), \dot{c}(t)) = \mathcal{H}(c(t), \dot{c}(t))$ for $0 < t < l$.

Proof. Recall that $g_{i\bar{j}}(z) = G_{i\bar{j}}(z, X(z))$. As in the proof of Proposition 3.2,

$$\frac{\partial g_{a\bar{j}}}{\partial \bar{z}^l} = \frac{\partial G_{a\bar{j}}}{\partial \bar{z}^l} \qquad \text{and} \qquad \frac{\partial g_{i\bar{b}}}{\partial z^k} = \frac{\partial G_{i\bar{b}}}{\partial z^k}.$$

Next

$$\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} = \frac{\partial^2 G_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{h=1}^{n} \frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^h \partial \bar{z}^l} \frac{\partial X^h}{\partial z^k} + \sum_{h=1}^{n} \frac{\partial^2 G_{i\bar{j}}}{\partial \bar{\zeta}^h \partial \bar{z}^l} \frac{\partial \bar{X}^h}{\partial z^k}.$$

By the homogeneity of $\frac{\partial G_{i\bar{j}}}{\partial \bar{z}^l}$,

$$\frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^h \partial \bar{z}^l} X^h = 0.$$

Hence we have

$$\begin{aligned} 0 &= \frac{\partial}{\partial z^k} \left(\sum_{h=1}^{n} \frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^h \partial \bar{z}^l}(z, X(z)) X^h \right) \\ &= \sum_{h=1}^{n} \frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^h \partial \bar{z}^l} \frac{\partial X^h}{\partial z^k} + \sum_{h=1}^{n} \frac{\partial^3 G_{i\bar{j}}}{\partial \zeta^h \partial z^k \partial \bar{z}^l}(z, X(z)) X^h. \end{aligned}$$

And also by the homogeneity of $\frac{\partial^2 G_{i\bar{j}}}{\partial z^k \partial \bar{z}^l}$,

$$\sum_{h=1}^{n} \frac{\partial^3 G_{i\bar{j}}}{\partial \zeta^h \partial z^k \partial \bar{z}^l} X^h = 0.$$

Hence

$$\sum_{h=1}^{n} \frac{\partial^2 G_{i\bar{j}}}{\partial \zeta^h \partial \bar{z}^l} \frac{\partial X^h}{\partial z^k} = 0.$$

And by the same token,

$$\sum_{h=1}^{n} \frac{\partial^2 G_{i\bar{j}}}{\partial \bar{\zeta}^h \partial \bar{z}^l} \frac{\partial \bar{X}^h}{\partial z^k} = 0.$$

Thus

$$\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l}(z) = \frac{\partial^2 G_{i\bar{j}}}{\partial z^k \partial \bar{z}^l}(z, X(z)).$$

Therefore $\mathcal{R}_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}}$.

Since

$$F^2(c(t),\dot{c}(t)) = \sum_{i,j=1}^{n} G_{i\bar{j}}(c(t),\dot{c}(t))\dot{c}^i(t)\bar{\dot{c}}^j(t)$$
$$= \sum_{i,j=1}^{n} g_{i\bar{j}}(c(t))\dot{c}^i(t)\bar{\dot{c}}^j(t)\,,$$

we get $H(c(t),\dot{c}(t)) = \mathcal{H}(c(t),\dot{c}(t))$ for $0 < t < l$. □

Proof of Theorem 2.1. Let p and q be any two points in M. Then by the Theorem 1.2, there exists a minimizing geodesic $c_o : [0,l] \to M$ joining p and q. We will show that $L_F(c_o) \le \frac{\pi}{c}$. Then $d(p,q) \le \frac{\pi}{c}$ and $diam(M) \le \frac{\pi}{c}$, as asserted; in addition, because M is bounded and complete, it is compact.

Consider the variation $c(t,s)$ of c_o on page 309 such that

$$c(0,s) = c_o(0) = p \quad \text{and} \quad c(l,s) = c_o(l) = q,$$

and

$$T(t,s) = \frac{\partial c}{\partial t} = \frac{\pi s}{l}\cos\frac{\pi t}{l} \ne 0,$$
$$S(t,s) = \frac{\partial c}{\partial s} = (\sin\frac{\pi t}{l})\frac{\partial \alpha}{\partial s}\,,$$
$$S(t,0) = \left.\frac{\partial c}{\partial s}\right|_{s=0} = (\sin\frac{\pi t}{l})JT(t).$$

Since $L_F(c_o) \le L_F(c_s)$,

$$\left.\frac{d^2}{ds^2}\right|_{s=0} L_F(c_s) \ge 0.$$

Note that $L_F(c_s)$ is independent of the values of $F(c_s(t),\dot{c}_s(t))$ at $t = 0$ and l, i.e.,

$$L_F(c_s) = \int_0^l F(c_s(t),\dot{c}_s(t))\,dt$$
$$= \int_0^l F(c_s(t),\dot{c}_s(t))\chi_{(0,l)}\,dt.$$

And hence we can use the induced Kähler metric on U to find the second variation formula of arc length. Let $\langle\ ,\ \rangle$ denote the real part of the induced Kähler metric $g_{i\bar{j}}(z) = G_{i\bar{j}}(z, X(z))$ and ∇ be its Levi-Civita connection. For simplicity, $\langle\ ,\ \rangle = 0$ at p and q. Thus

$$L_F(c_s) = \int_0^l \langle T, T\rangle^{1/2}\, dt$$

and the second variation formula of length in Riemannian metric $\langle\ ,\ \rangle$ of U is

$$\frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} L_F(c_s) = \lim_{\delta\to 0}\left\{ \langle T, \nabla_S S\rangle\Big|_{\delta}^{l-\delta} + \langle S, \nabla_T S\rangle\Big|_{\delta}^{l-\delta} \right\} - \int_0^l \langle S, \nabla_T\nabla_T S + \mathcal{R}(T,S)T\rangle + \big(T\langle T, S\rangle\big)^2\, dt.$$

We will show that the boundary terms are zero. Because c_o is a geodesic in U with respect to $\langle\ ,\ \rangle$ and g is a Kähler metric on U, $\nabla_T JT = J\nabla_T T = 0$. Hence

$$\begin{aligned}\nabla_T S &= \nabla_T(\sin\tfrac{\pi t}{l})JT(t)\\ &= \tfrac{\pi}{l}(\cos\tfrac{\pi t}{l})JT(t) + (\sin\tfrac{\pi t}{l})\nabla_T JT(t)\\ &= \tfrac{\pi}{l}(\cos\tfrac{\pi t}{l})JT(t).\end{aligned}$$

So we get

$$\begin{aligned}\langle S, \nabla_S T\rangle\Big|_{\delta}^{l-\delta} &= \langle (\sin\tfrac{\pi t}{l})JT(t), \tfrac{\pi}{l}(\cos\tfrac{\pi t}{l})JT(t)\rangle\Big|_{\delta}^{l-\delta}\\ &= \tfrac{\pi}{l}(\sin\tfrac{\pi t}{l})(\cos\tfrac{\pi t}{l})\Big|_{\delta}^{l-\delta},\end{aligned}$$

which tends to 0 as δ goes to 0.

Next since $S = (\sin\frac{\pi t}{l})\frac{\partial\alpha}{\partial s}$ and $\frac{\partial\alpha}{\partial s} = \sum_{i=1}^{2n}(\frac{d\alpha^i}{dt})\frac{\partial}{\partial x^i}$, we get

$$\begin{aligned}\nabla_S S &= (\sin^2\tfrac{\pi t}{l})\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial s}\\ &= (\sin^2\tfrac{\pi t}{l})\left(\frac{d\alpha^i}{dt}\left(\frac{\partial}{\partial x^i}\frac{d\alpha^j}{dt}\right)\frac{\partial}{\partial x^j} + \frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right).\end{aligned}$$

Note that for a Kähler metric, its canonical Hermitian connection and its Levi-Civita connection coincide under a suitable identification of tangent vectors, i.e., if $\tilde{\nabla}$ is the canonical Hermitian connection of U and $\phi : T_xM \to T_x^{1,0}M$ is the identification map on page 1,

$$\phi(\nabla_X Y) = \tilde{\nabla}_{\phi(X)}\phi(Y).$$

For this, see [7]. Thus

$$\begin{aligned}\phi(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}) &= \tilde{\nabla}_{\frac{\partial}{\partial z^i}}\frac{\partial}{\partial z^j}\\ &= \Gamma^k_{ij}\frac{\partial}{\partial z^k}\\ &= g^{k\bar{m}}\frac{\partial g_{j\bar{m}}}{\partial z^i}\frac{\partial}{\partial z^k}\\ &= G^{k\bar{m}}(z,X(z))\frac{\partial G_{j\bar{m}}}{\partial z^i}(z,X(z))\frac{\partial}{\partial z^k},\end{aligned}$$

where the last equation comes from (3.1).

Now

$$\text{(3.2)}\qquad \begin{aligned}\langle T,\nabla_S S\rangle = \sin^2\tfrac{\pi t}{l}\Big\{&\frac{d\alpha^i}{dt}\left(\frac{\partial}{\partial x^i}\frac{d\alpha^j}{dt}\right)\mathrm{Re}\, g\left(\tfrac{\partial}{\partial z^j},\phi(T)\right)\\ &+\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\Gamma^k_{ij}\mathrm{Re}\, g\left(\tfrac{\partial}{\partial z^k},\phi(T)\right)\Big\}.\end{aligned}$$

But since

$$\begin{aligned}g\left(\tfrac{\partial}{\partial z^j},\phi(T)\right) &= G_{j\bar{k}}(c_o(t),\dot{c}_o(t))\overline{\phi(T)}^k\\ \Gamma^k_{ij} &= G^{k\bar{m}}(c_o(t),\dot{c}_o(t))\frac{\partial G_{j\bar{m}}}{\partial z^i}(c_o(t),\dot{c}_o(t))\frac{\partial}{\partial z^k},\end{aligned}$$

the right hand side of (3.2) can be continuously extended to $[0,l]$. In particular,

$$\frac{d\alpha^i}{dt}\left(\frac{\partial}{\partial x^i}\frac{d\alpha^j}{dt}\right)\mathrm{Re}\, g\left(\tfrac{\partial}{\partial z^j},\phi(T)\right)+\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\Gamma^k_{ij}\mathrm{Re}\, g\left(\tfrac{\partial}{\partial z^k},\phi(T)\right)$$

is bounded. Therefore, we get

$$\lim_{\delta\to 0}\langle\nabla_S S,T\rangle\Big|_{\delta}^{l-\delta}=0.$$

Finally, since $\langle T,S\rangle=0$ at $s=0$, we get

$$\begin{aligned}0\le\frac{1}{2}\frac{d^2}{ds^2}\Big|_{s=0}L_F(c_s) &= -\int_0^l\langle S,\nabla_T\nabla_T S+\mathcal{R}(T,S)T\rangle dt\\ &\le\int_0^l\sin^2\tfrac{\pi t}{l}\left(\tfrac{\pi^2}{l^2}-c^2\right)dt\end{aligned}$$

and hence we get $l\le\frac{\pi}{c}$. □

References

[1] M. Abate, G. Patrizio, *Finsler metric–A global approach*, Lecture Notes in Mathematics, vol. **1591**, Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[2] D. Bao, S. S. Chern, and Z. Shen, *An Introduction to Riemannian–Finsler geometry*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 2000.

[3] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1978.

[4] M. Gromov, *Metric structures for Riemannian and non-Riemannian Spaces*, Birkhäuser, Boston, Basel, Berlin, 1999.

[5] S. Kobayashi, *Negative vector bundles and complex Finsler structures*, Nagoya Math. J. **57** (1975), 153–166.

[6] S. Kobayashi, *Hyperbolic complex spaces*, Die Grundlehren der Mathematischen Wissenschaften, vol. **318**, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1998.

[7] S. Kobayashi and H. Wu, *Complex differential geometry*, DMV Seminar, vol. 3, Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.

[8] M. Matsumoto, *Foundations of Finsler geometry and special Finsler spaces*, Kaiseisha, Shigaken, Japan, 1986.

[9] W. Rinow, *Die innere Geometrie der metrischen Räume*, Die Grundlehren der Mathematischen Wissenschaften, vol. **105**, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1961.

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