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THE OSEEN-TYPE EXPANSION OF NAVIER-STOKES FLOWS WITH AN APPLICATION TO SWIMMING VELOCITY

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ABSTRACT. A linearization owing to Oseen originally is performed to study the recirculating Navier-Stokes flows at high Reynolds numbers. The procedure is generalized to produce higher order asymptotic expansion for the flow velocity. We call this the *Oseen-type expansion* of the given flow. As a concrete example, the velocity of a steady Navier-Stokes flow due to a swimming flexible sheet in two-dimensional infinite strip domain is calculated by an asymptotic expansion technic with two-parameters, the Reynolds number R and the perturbation parameter ϵ. We here expand the flow in ϵ first and then R secondly. The asymptotic result is up to second order in ϵ.

1. Introduction

High Reynolds number (R) flows have been a major topic of many pure and applied mathematicians since the birth of the boundary layer theory which dispalys the distinctive nature different from low Reynolds number flows. Briefly speaking, as the Reynolds number (R) becomes very large, a singular thin layer along the boundary (this is the *boundary layer*) emerges in the flow. Also, the flow becomes unstable in general for such large R. These two properties are notable features of large Reynolds number flows in general. However, under certain conditions (for example, if the flow is absolutely stable for any positive Reynolds number (Kim 1998)) we still have flows which stay laminar at quite large Reynolds number. These flows are subjects of current study.

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In two-dimension, such laminar flows possess an important asymptotic property in steady state. As the Reynolds number R becomes indefinitely large the vorticity of a closed nested streamline domain approaches to a constant. (Batchelor 1956) In the case of a circular domain with one-eddy configuration, the (uniform) vorticity value can be calculated by applying the matched asymptotic expansion technic to the two overlapping regions. To be more specific, we consider two regions, one is the outer flow residing in most part inside the disk and the other is the critical flow in the boundary layer between the wall and the outer flow with the $O(1/\sqrt{R})$ thickness. The calculation is successfully executed from the transcendental decay principle(Kim 1998) which guarantees that the vorticity of any order in R decays exponentially to a (nonzero) constant. The results are later confirmed by corresponding numerical computations. (Kim and Lee 1999)

Meanwhile, the above considerations suggest us a linearization of Navier-Stokes recirculating flows in which we recall Oseen's linearization (Oseen 1927) of the low Reynolds number flow. Here, the only nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ (called advection or convection term) in Navier-Stokes momentum equation is linearized by substituting the convecting velocity $\mathbf{u}$ by the given velocity $\mathbf{U}$ at the body for the flow due to a moving body. This idea is modified and then applied to the problem of high R flows. Our linearization matches with Oseen's in the leading order but is more generalized to achieve any arbitrary order asymptotic result by some successive calculations. We call this the *Oseen-type expansion* according to Oseen's first work of such expansion.

In addition, we study a concrete example of our theory in which the asymptotic velocity of a flow due to a moving thin flexible sheet is found. The geometry is supposed to be periodic along x direction in the two-dimensional plane. By the matched asymptotic expansion method with two parameters, R and ϵ(the perturbation parameter to be defined later) we successfully obtain the second order result in ϵ on the velocity. This model describes a ciliary propulsion undergoing periodic deformations. (Childress 1981, Chapter 3)

2. The Oseen-type expansion

The two-dimensional Navier-Stokes equations are

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \frac{1}{R} \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

where $\mathbf{u} = (u(x,y,t), v(x,y,t))$, $p = p(x,y,t)$ and R denote the flow velocity, the pressure and the Reynolds number respectively. (The arguments below are easily extended to three dimension, but we here focus on the two dimension case.) Taking curl to each side of the first equation, we obtain the vorticity equation for ω, the z-component of $\Omega = \nabla \times \mathbf{u}$,

$$\omega_t + \mathbf{u} \cdot \nabla\omega = \frac{1}{R}\nabla^2\omega. \tag{2}$$

Here, from the incompressibility condition we introduce the streamfunction $\psi = \psi(x,y)$ such as

$$\mathbf{u} = (u, v) = (\psi_y, -\psi_x) = \nabla^{\perp}\psi. \tag{3}$$

We then rewrite $\omega = -\nabla^2\psi$. We linearize the vorticity equation in the similar manner of Oseen

$$\omega_t + \mathbf{u}_o \cdot \nabla\omega = \frac{1}{R}\nabla^2\omega, \tag{4}$$

where $\mathbf{u}_o$ is the linearization velocity. Let us consider a flow due to a body moving through fluid of infinite extent. Assuming that the flow relative to the body is steady, Oseen chose the steady velocity of the moving body for an appropriate $\mathbf{u}_o$. (Look at Oseen(1927) or Batchelor (1967) pp. 240–241.) In fact, this is the beginning term of the expansion of the flow velocity in R of the matched asymptotic expansion. (Lagerstrom 1988) Here we modify this idea and expand $\mathbf{u}(x,y)$ asymptotically in an arbitrary parameter ϵ (In fact, we choose ϵ as the perturbation parameter on velocity in the next section). Let us suppose the following expansions of ψ, u, v, ω in ϵ :

$$\psi(x,y;R;\epsilon) \sim \psi_0(x,y;R) + \epsilon\psi_1(x,y;R) + \epsilon^2\psi_2(x,y;R) + \cdots \tag{5}$$

$$u(x,y;R;\epsilon) \sim u_0(x,y;R) + \epsilon u_1(x,y;R) + \epsilon^2 u_2(x,y;R) + \cdots \tag{6}$$

$$v(x,y;R;\epsilon) \sim v_0(x,y;R) + \epsilon v_1(x,y;R) + \epsilon^2 v_2(x,y;R) + \cdots \tag{7}$$

$$\omega(x,y;R;\epsilon) \sim \omega_0(x,y;R) + \epsilon\omega_1(x,y;R) + \epsilon^2\omega_2(x,y;R) + \cdots \tag{8}$$

In addition, we assume that each term of the above expressions $\psi_i, u_i, v_i, \omega_i$ $(i = 0, 1, 2, \cdots)$ is expanded similarly in R. Taking the first term of the expansion of $\mathbf{u} = (u, v)$ as $\mathbf{u}_0(x,y) =$ velocity at the body, we choose the linearization velocity

$$\mathbf{u}_o = \mathbf{u} = \mathbf{u}_0 + \epsilon\mathbf{u}_1 + \epsilon^2\mathbf{u}_2 + \cdots .$$

Substituting this into (4) and gathering terms in the order of ϵ, we successively determine each order terms of the asymptotic expansion of the velocity. We here explain this procedure in detail. Substituting the chosen $\mathbf{u}_o$ into (4) and rearranging the results in powers of ϵ we obtain,

(9)
$$\left(u_0\frac{\partial}{\partial x}+v_0\frac{\partial}{\partial y}-\frac{1}{R}\nabla^2\right)\nabla^2\psi-\epsilon\left(v_1\frac{\partial}{\partial y}-u_1\frac{\partial}{\partial x}\right)\nabla^2\psi+O(\epsilon^2)=0.$$

Again substituting the expansion (5) into (9) and rewriting the result in the order of ϵ yields,

$$O(1)\ :\ \left(\mathcal{L}_0-\frac{1}{R}\nabla^2\right)\nabla^2\psi_0=0 \tag{10}$$

$$O(\epsilon)\ :\ \left(\mathcal{L}_0-\frac{1}{R}\nabla^2\right)\nabla^2\psi_1+\mathcal{L}_1\nabla^2\psi_0=0 \tag{11}$$

$$O(\epsilon^2)\ :\ \left(\mathcal{L}_0-\frac{1}{R}\nabla^2\right)\nabla^2\psi_2+\mathcal{L}_1\nabla^2\psi_1+\mathcal{L}_2\nabla^2\psi_0=0 \tag{12}$$

(13) etc.

where we introduce the notation

$$\mathcal{L}_i=u_i\frac{\partial}{\partial x}+v_i\frac{\partial}{\partial y},\quad i=0,1,2,\cdots. \tag{14}$$

This admits an iterative procedure to determine ψ_k if ψ_{k-1} (or u_{k-1},v_{k-1}) is known. Thus we need the zeroth order velocity u_0,v_0 to start this expansions. It is here that we adopt Oseen's idea for u_0,v_0, namely we take the velocity of the body as u_0,v_0. Of course, if more accuracy is desired, we proceed to adopt higher order terms in the expansion of $\mathbf{u}$. Present choice of $\mathbf{u}_o$ produces the first nonzero contribution at the second order term in ϵ^2 in the following example.

3. An Example: Periodic swimming sheet

The above expansion has an important application to study a swimming thin flexible sheet. The pioneering work of Taylor(1951) deals with the ciliary models and swimming microorganisms. To make the matters easy, we assume 2π-periodicity along x-axis on which the sheet is placed. The sheet is infinite and undergoing two-dimensional periodic deformations about an unperturbed position $y=0$ (Look at Figure 1 below). The velocity of the sheet is $u(x,0)=1+\epsilon f(x)$, $v(x,0)=0$ where $f(x)=\sum_{n\neq 0}c_ne^{inx}$ is the perturbing function which has zero average along the wall($0\le x\le 2\pi$). We will determine the free stream

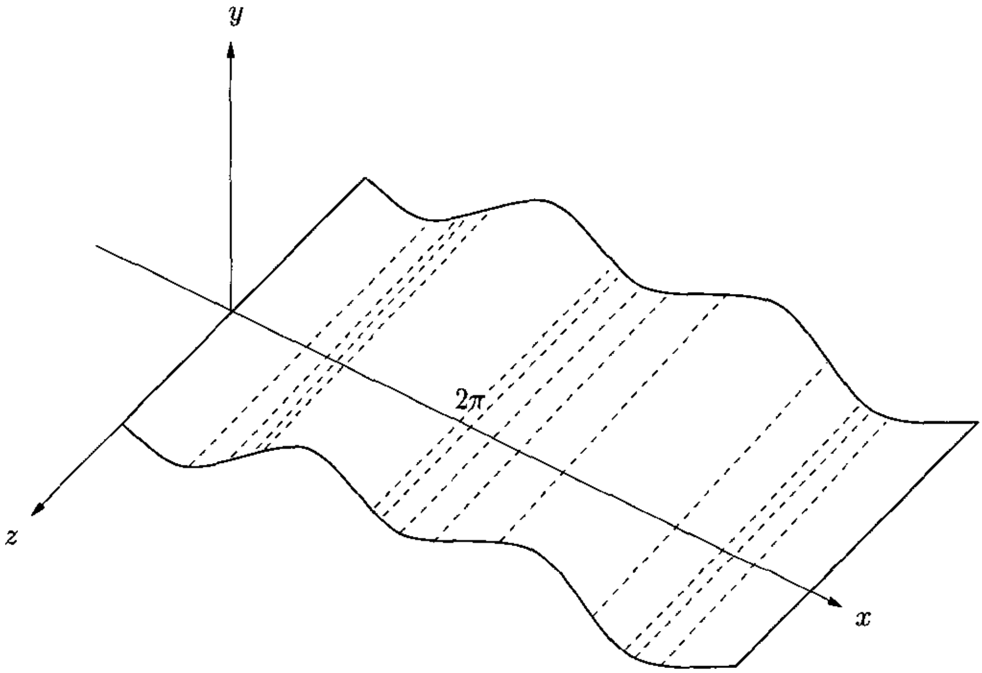

FIGURE 1. Swimming periodic sheet

velocity due to the effect of the wall value of $\mathbf{u} = (1 + \epsilon f(x), 0)$. See Figure 2.

We now regard R, the Reynolds number as fixed and sufficiently large, and let the perturbation parameter for wall data $|\epsilon| << 1$. From the physical setting, it is reasonable to take $(u_0, v_0) = (1, 0)$ to start the Oseen-type expansion which is just the zeroth order perturbed velocity of the sheet.

We first substitute the velocity driven by the perturbation in the form,

$$\begin{aligned} u(x,y) &= 1 + \epsilon\psi_{1y} + \cdots \\ -v(x,y) &= 0 + \epsilon\psi_{1x} + \cdots . \end{aligned}$$

These are inserted into the vorticity equation (2) to produce,

$$\left(\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\nabla^2\psi - \epsilon\left(v_1\frac{\partial}{\partial y} - u_1\frac{\partial}{\partial x}\right)\nabla^2\psi + O(\epsilon^2) = 0. \tag{15}$$

We ignore higher-order terms than $O(\epsilon)$ and obtain the linearized problem.

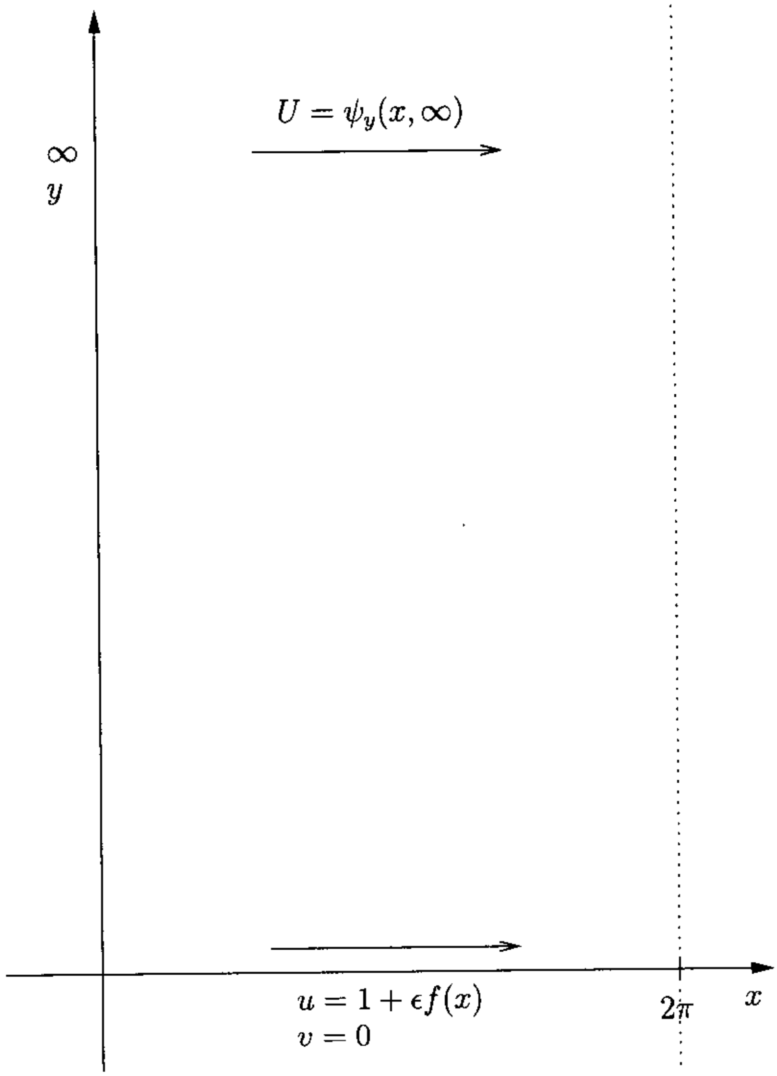

FIGURE 2. Determine U asymptotically for $R >> 1$

$$\left(\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\nabla^2\psi - \epsilon\left(v_1\frac{\partial}{\partial y} - u_1\frac{\partial}{\partial x}\right)\nabla^2\psi = 0$$

$$\psi_y(x,0) = 1 + \epsilon f(x)$$

$$0 \le x \le 2\pi,\ 0 \le y$$

3.1. Zeroth-order term

Let the flow velocity at $y = \infty$ be $\mathbf{u} = (U, 0)$ i.e. $U = u(x, +\infty)$. Applying the momentum conservation at $y = 0$ and $y = \infty$,

$$\int_0^{2\pi} u^2(x,0)dx = 2\pi U^2 \tag{16}$$

we deduce the boundary condition at $y = +\infty$,

$$U = u(x, +\infty) = \sqrt{1 + \epsilon^2 \bar{f^2}} = 1 + \frac{1}{2}\sum_{n \neq 0} |c_n|^2 \epsilon^2 + \cdots . \tag{17}$$

where the bar($^-$) implies the average on $0 \leq x \leq 2\pi$ i.e.

$$\bar{f} = \int_0^{2\pi} \frac{1}{2\pi} f(x)dx.$$

This relation coincides with Batchelor-Wood formula(Kim 1998) which gives the core constant vorticity of a flow inside a circular cylinder.

Let us write asymptotically, for sufficiently large R,

$$U \sim U_0(R) + \epsilon U_1(R) + \epsilon^2 U_2(R) + \cdots . \tag{18}$$

Our aim is to calculate the unknown coefficients $U_0(R), U_1(R), U_2(R)$. We will calculate $U_i (i = 0, 1, 2)$ by averaging each term u_i in (6) on x from 0 to 2π and then letting $y \to \infty$. In the beginning, we easily solve the asymptotic behavior of $\psi_0(x, y)$. From the given physical situation and $R >> 1$,

$$\psi_0(x, y) \sim y \quad \text{as} \quad y \to \infty.$$

Thus it follows that $U_0(R) = u_0 = 1$.

3.2. First-order term

In the sequel, proper boundary conditions for $y \to \infty$ are derived from the expansion of $u(x, +\infty)$. Thus, for ψ_1, we come to the problem,

$$\left(\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\nabla^2\psi_1 = 0 \tag{19}$$

$$\psi_1(x, 0, R) = 0, \quad \psi_{1y}(x, 0, R) = \sum_{n \neq 0} c_n e^{inx} \tag{20}$$

$$\psi_{1y}(x, y, R) = 0 \quad \text{as} \quad y \to \infty \tag{21}$$

We obtain the exact solution as,

$$\psi_1(x,y) = \sum_{n\neq 0} \frac{c_n}{\sqrt{n^2+inR}-|n|} e^{inx}\left(e^{-|n|y} - e^{-\sqrt{n^2+inR}y}\right) \tag{22}$$

$$u_1(x,y) = \sum_{n\neq 0} \frac{c_n}{\sqrt{n^2+inR}-|n|} e^{inx}\left(-|n|e^{-|n|y} + \sqrt{n^2+inR}e^{-\sqrt{n^2+inR}y}\right) \tag{23}$$

$$v_1(x,y) = \sum_{n\neq 0} \frac{inc_n}{\sqrt{n^2+inR}-|n|} e^{inx}\left(e^{-|n|y} - e^{-\sqrt{n^2+inR}y}\right) \tag{24}$$

Note that we adopt the following convention :

$$\sqrt{in} = \begin{cases} \sqrt{n}\frac{1+i}{\sqrt{2}} & \text{if } n > 0 \\ \sqrt{-n}\frac{1-i}{\sqrt{2}} & \text{if } n < 0. \end{cases}$$

Averaging (24) on x from 0 to 2π yields $\overline{\psi_{1y}}(x,\infty) = U_1(R) = 0$.

3.3. Second-order term

Similarly we go to the next order terms and obtain,

$$\left(\frac{\partial}{\partial x} - \frac{1}{R}\nabla^2\right)\nabla^2\psi_2 = \left(-u_1\frac{\partial}{\partial x} + v_1\frac{\partial}{\partial y}\right)\nabla^2\psi_1$$
$$\psi_2(x,0,R) = 0, \quad \psi_{2y}(x,0,R) = 0$$
$$\psi_{2y}(x,\infty,R) = \frac{1}{2}\sum_{n\neq 0}|c_n|^2 \text{ as } R\to\infty.$$

It seems to be very messy to solve this problem directly. Instead, by averaging on x the above system is transformed into a simple problem for the averaged $\overline{\psi_2}$,

$$-\frac{1}{R}\frac{d^4}{dy^4}\overline{\psi_2} = \frac{d}{dy}f(y)$$
$$\overline{\psi_2}(0) = 0, \quad \overline{\psi_{2y}}(0) = 0$$
$$\overline{\psi_{2y}}(\infty) = \frac{1}{2}\sum_{n\neq 0}|c_n|^2$$

where

$$
\begin{aligned}
f(y) = \sum_{n\neq 0} |c_n|^2 & \frac{in}{(\sqrt{n^2+inR}-|n|)^2} \\
& \times \Big\{ n^2 e^{-2|n|y} - n^2 e^{-(|n|+\sqrt{n^2+inR})y} \\
& - (n^2-inR)e^{-(|n|+\sqrt{n^2-inR})y} \\
& + (n^2-inR)e^{-(\sqrt{n^2+inR}+\sqrt{n^2-inR})y} \Big\}.
\end{aligned}
$$

Solving this equation for $\overline{\psi_2}$ we obtain, as $y \to \infty$,

$$
\begin{aligned}
\overline{\psi_2}_y(y) &= U_2(R) \\
&= \sum_{n\neq 0} |c_n|^2 \frac{in}{(\sqrt{n^2+inR}-|n|)^2} \Big\{ \frac{n^2}{(\sqrt{n^2+inR}+|n|)^2} \\
&+ \frac{n^2-inR}{(\sqrt{n^2-inR}+|n|)^2} - \frac{n^2-inR}{(\sqrt{n^2+inR}+\sqrt{n^2-inR})^2} \Big\}.
\end{aligned}
$$

3.4. Result of the expansion

We combine the above calculations to conclude,

$$
\begin{aligned}
U(R) \sim\ & 1 + \sum_{n\neq 0} |c_n|^2 \frac{in}{(\sqrt{n^2+inR}-|n|)^2} \left\{ \frac{n^2}{(\sqrt{n^2+inR}+|n|)^2} \right. \\
+\ & \left. \frac{n^2-inR}{(\sqrt{n^2-inR}+|n|)^2} - \frac{n^2-inR}{(\sqrt{n^2+inR}+\sqrt{n^2-inR})^2} \right\}
\end{aligned}
$$

For a special case $f(x) = \sin x$, this yields explicitly,

$$
U(R) \sim 1 + \left(\frac{1}{4} - \frac{\sqrt{2}}{4\sqrt{R}} + O(\frac{1}{R}) \right) \epsilon^2 + O(\epsilon^3) \tag{25}
$$

which shows the absence of the first order contribution in this series.

4. Concluding remarks

We first note that the condition of momentum conservation (16) recovers the Batchelor-Wood formula(Kim 1998) giving the core constant limit vorticity in the circular domain. Here, we derived the same equation under a different situation which then predicts the velocity as $y \to \infty$ in a periodic strip domain.

For practical application, certain condition on the orders of magnitude of two parameters ϵ and R are necessary. In fact, the calculated expansions above are asymptotic and valid under some restrictions. (e.g. $\epsilon^3 \ll \frac{\epsilon^2}{\sqrt{R}}$ in (25)) For more details we refer the book by Lagerstrom(1988).

The linearization we study in this paper may be generalized to three dimensions. Then, it would be better to use the velocity equation (1) instead of the vorticity equation (2). In two dimension, the vorticity equation is a scalar equation, which greatly simplifies the matter.

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