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ON THE FEKETE-SZEGÖ PROBLEM AND ARGUMENT INEQUALITY FOR STRONGLY QUASI-CONVEX FUNCTIONS

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ABSTRACT. Let $\mathcal{Q}(\beta)$ be the class of normalized strongly quasi-convex functions of order β in the open unit disk. Sharp Fekete-Szegö inequalities are obtained for functions belonging to the class $\mathcal{Q}(\beta)$. We also consider the integral preserving properties in a sector.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let S be the subclass of $\mathcal{A}$ consisting of all univalent functions. We also denote by $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$ the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike, convex and close-to-convex in $\mathcal{U}$ (see, e.g., Srivastava and Owa [17]).

For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function $w(z)$ such that $w(0) = 0, |w(z)| < 1 \ \ (z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

A classical result of Fekete and Szegö [5] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter μ, for functions

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belonging to $\mathcal{S}$. There are now several results of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e.g., [1, 7, 9]).

Denote by $\mathcal{Q}(\beta)$ the class of strongly quasi-convex functions of order $\beta(\beta \geq 0)$. Thus $f \in \mathcal{Q}(\beta)$ if and only if there exists $g \in \mathcal{K}$ such that for $z \in \mathcal{U}$,

$$\left|\arg \left\{\frac{(zf'(z))'}{g'(z)}\right\}\right| \leq \frac{\pi}{2}\beta.$$

In particular, $\mathcal{Q}(1)$ is the class of quasi-convex functions introduced by Noor [13]. We also note that every quasi-convex function is close-to-convex and hence univalent in $\mathcal{U}$.

In the present paper, we derive sharp Fekete-Szegö inequalities for functions belonging to the class $\mathcal{Q}(\beta)$. Furthermore, the integral preserving properties are considered for functions in the class $\mathcal{Q}(\beta)$.

2. Results

To prove our main results, we need the following lemmas.

LEMMA 2.1. *Let p be analytic in $\mathcal{U}$ and satisfy* Re $\{p(z)\} > 0$ *for* $z \in \mathcal{U}$, *with* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. *Then*

$$|p_n| \leq 2 \quad (n \geq 1) \tag{2.1}$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}. \tag{2.2}$$

LEMMA 2.2. *Let h be convex(univalent) function in $\mathcal{U}$ and ω be an analytic function in $\mathcal{U}$ with* Re $\{\omega(z)\} \geq 0$. *If p is analytic in $\mathcal{U}$ and* $p(0) = h(0)$, *then*

$$p(z) + \omega(z)zp'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$p(z) \prec h(z) \quad (z \in \mathcal{U}).$$

LEMMA 2.3. *Let p be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\left|\arg\{p(z)\}\right| < \frac{\pi}{2}\eta \quad \textit{for } |z| < |z_0| \tag{2.3}$$

and

$$\left|\arg\{p(z_0)\}\right| = \frac{\pi}{2}\eta(0 < \eta \leq 1). \tag{2.4}$$

Then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\eta, \tag{2.5}$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \textit{ when } \arg\{p(z_0)\} = \frac{\pi}{2}\eta, \tag{2.6}$$

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \textit{ when } \arg\{p(z_0)\} = -\frac{\pi}{2}\eta, \tag{2.7}$$

and

$$\{p(z_0)\}^{\frac{1}{\eta}} = \pm ia \ (a > 0). \tag{2.8}$$

The inequality (2.1) was first proved by Carathéodory [3](also, see Duren [4, p. 41]) and the inequality (2.2) can be found in [15, p. 166]. Lemma 2.2 are the result proved by Miller and Mocanu [11], which has a number of important applications in the theory of univalent functions. Also Lemma 2.3 was proved by Nunokawa [14] as a new modification of well known Jack's Lemma [6].

With the help of Lemma 2.1, we now derive

THEOREM 2.1. *Let $f \in \mathcal{Q}(\beta)$ and be given by (1.1). Then for $\beta \geq 0$, we have*

$$9\left|a_3 - \mu a_2^2\right| \leq \begin{cases} 1 + \frac{(1+\beta)^2(8-9\mu)}{4} & \textit{if} \quad \mu \leq \frac{8\beta}{9(1+\beta)}, \\ 1 + 2\beta + \frac{2(8-9\mu)}{8-\beta(8-9\mu)} & \textit{if} \quad \frac{8\beta}{9(1+\beta)} \leq \mu \leq \frac{8}{9}, \\ 1 + 2\beta & \textit{if} \quad \frac{8}{9} \leq \mu \leq \frac{8(2+\beta)}{9(1+\beta)}, \\ -1 + \frac{(1+\beta)^2(9\mu-8)}{4} & \textit{if} \quad \mu \geq \frac{8(2+\beta)}{9(1+\beta)}. \end{cases}$$

For each μ, there is a function in $\mathcal{Q}(\beta)$ such that equality holds in all cases.

Proof. Let $f \in \mathcal{Q}(\beta)$. Then it follows from the definition that we may write

$$\frac{(zf'(z))'}{g'(z)} = p^{\beta}(z), \tag{2.9}$$

where g is convex and p has positive real part. Let $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and let p be given as in Lemma 2.1. Then by comparing the coefficients of both sides of (2.9), we obtain

$$4a_2 = \beta p_1 + 2b_2$$

and

$$9a_3 = \frac{\beta(\beta-1)}{2}p_1^2 + \beta p_2 + 3b_3 + 2\beta p_1 b_2.$$

So, with $x = (8-9\mu)/4$, we have

$$\begin{aligned} 9(a_3 - \mu a_2^2) &= 3\left(b_3 + \frac{1}{3}(x-2)b_2^2\right) \\ &\quad + \beta\left(p_2 + \frac{1}{4}(\beta x - 2)p_1^2\right) + \beta x p_1 b_2. \end{aligned} \tag{2.10}$$

Since rotations of f also belong to $\mathcal{Q}(\beta)$, without loss of generality, we may assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate Re $(a_3 - \mu a_2^2)$.

Since $g \in \mathcal{K}$, there exists $h(z) = 1 + k_1z + k_2z^2 + \cdots$ $(z \in \mathcal{U})$ with positive real part, such that $g'(z) + zg''(z) = g'(z)h(z)$. Hence, by equating coefficients, we get that $b_2 = k_1/2$ and $b_3 = (k_2 + k_1^2)/6$. Therefore, letting $b_2 = \rho e^{i\phi}(0 \le \rho \le 1)$ and $p_1 = 2re^{i\theta}(0 \le r \le 1)$ in (2.10), and applying Lemma 2.1, we obtain

$$\begin{aligned} 9\mathrm{Re}(a_3 - \mu a_2^2) &\le (1-\rho^2) + (x+1)\rho^2\cos 2\phi \\ &\quad + 2\beta(1-r^2) + \beta^2r^2\cos 2\theta + 2\beta x r\rho\cos(\theta+\phi) \\ &= \psi(x), \quad \text{say}. \end{aligned} \tag{2.11}$$

We consider first the case $8\beta/(9(1+\beta)) \le \mu \le 8/9$. In this case, we see that $0 \le x \le 2/(1+\beta)$. Then we obtain

$$\begin{aligned}
&\psi(x) \\
&= 1-\rho^2+(x+1)\rho^2\cos 2\phi+\beta(2(1-r^2)+\beta x r^2\cos 2\theta \\
&\quad + 2xr\rho\cos(\theta+\phi)) \\
&\le x+1+\beta(2-2r^2+\beta x r^2\cos 2\theta+2xr).
\end{aligned}$$

Since the expression $-2t^2+\beta x t^2\cos 2\theta+2xt$ is the largest when $t = x/(2-\beta x\cos 2\theta)$, we have

$$-2t^2+\beta x t^2\cos 2\theta+2xt \le \frac{x^2}{2-\beta x\cos 2\theta} \le \frac{x^2}{2-\beta x}.$$

Thus

$$\begin{aligned}
\psi(x) &\le x+1+\beta\Big(2+\frac{x^2}{(2-\beta x)}\Big) \\
&= 1+2\beta+\frac{2(8-9\mu)}{8-\beta(8-9\mu)},
\end{aligned}$$

and from (2.11), we obtain the second inequality of the theorem. Equality occurs only if

$$p_1 = \frac{2(8-9\mu}{8-\beta(8-9\mu)}, \quad p_2 = 2, \quad b_2 = b_3 = 1,$$

and the corresponding function f is defined by

$$(zf'(z))' = \frac{1}{(1-z)^2}\left(\lambda\frac{1+z}{1-z}+(1-\lambda)\frac{1-z}{1+z}\right)^\beta, \quad f(0)=0,$$

where

$$\lambda = \frac{8+(1-\beta)(8-9\mu)}{16-2\beta(8-9\mu)}.$$

We now prove the first inequality. Let $\mu \le 8\beta/(9(1+\beta))$. Then we obtain that $x \ge 2/(1+\beta) = x_0$, and

$$\begin{aligned}
\psi(x) &= \psi(x_0)+(x-x_0)(\rho^2\cos 2\phi+\beta^2 r^2\cos 2\theta+2\beta\rho r\cos(\theta+\phi)) \\
&\le \psi(x_0)+(x-x_0)(1+\beta)^2 \\
&\le 1+\frac{(1+\beta)^2(8-9\mu)}{4},
\end{aligned}$$

as required. Equality occurs only if $c_1 = c_2 = 2,\ b_2 = b_3 = 1$, and the corresponding function f is defined by

$$(zf'(z))' = \frac{1}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\beta}, \quad f(0) = 0.$$

Let $x_1 = -2/(1+\beta)$. At first, we will show that $\psi(x_1) \leq 1+2\beta$. Then the remaining inequalities follow easily from this one. We have

$$(-2+\beta x_1 \cos 2\theta)t^2 + 2x_1 t\rho\cos(\theta+\phi) \leq \frac{x_1^2\rho^2\cos^2(\theta+\phi)}{2-\beta x_1\cos 2\theta}$$

for all real t. Hence we obtain

$$\begin{aligned}&\psi(x_1) - (1+2\beta)\\ &\leq \rho^2\left(-1+(x_1+1)\cos 2\phi + \frac{\beta x_1^2(1+\cos 2(\theta+\phi))}{2(2-\beta x_1\cos 2\theta)}\right).\end{aligned}$$

Thus we consider the inequality

$$\beta x_1^2(1+\cos 2(\theta+\phi)) + 2(2-\beta x_1\cos 2\theta)(-1+(x_1+1)\cos 2\phi) \leq 0,$$

which is true if

$$2\beta^2\cos^2\theta\sin^2\phi + 2\beta\cos\theta\sin\theta\cos\phi\sin\phi + \cos^2\phi \geq 0. \tag{2.12}$$

Now, for all real t,

$$2t^2 + 2t\sin\theta\cos\phi + \cos^2\phi \geq 0,$$

so, by taking $t = \beta\cos\theta\sin\phi$, we obtain (2.12). Thus $\psi(x_1) \leq 1+2\beta$.

Next, we consider two possibilities. We suppose that $x_1 \leq x \leq 0$, that is, $8/9 \leq \mu \leq 8(2+\beta)/(9(1+\beta))$. Note that for $0 \leq \lambda \leq 1$,

$$\psi(\lambda x_1) = \lambda\psi(x_1) + (1-\lambda)\psi(0) = 1+2\beta.$$

Hence we have $\psi(x) \leq 1+2\beta$ and this proves the third inequality of the theorem. Equality occurs only if $p_1 = b_2 = 0,\ \ p_2 = 2,\ \ b_3 = 1/3$, and the corresponding function f is defined by

$$(zf'(z))' = \frac{(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}, \quad f(0) = 0.$$

Secondly, we suppose that $x \le x_1$, that is, $\mu \ge (8(2+\beta))/(9(1+\beta))$. Then we have

$$\begin{aligned}\psi(x_0) &= \psi(x_1) + (x - x_1)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\beta\rho r \cos(\theta+\phi)) \\ &\le \psi(x_1) + (x_1 - x)(1+\beta)^2 \\ &\le -1 + \frac{(1+\beta)^2(9\mu - 8)}{4},\end{aligned}$$

and this is the last inequality of the theorem. Equality occurs only if $p_1 = 2i,\ p_2 = -2,\ b_2 = i,\ b_3 = -1$, and the corresponding function f is defined by

$$(zf'(z))' = \frac{1}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^{\beta}, \quad f(0) = 0.$$

Therefore we complete the proof of Theorem 2.1.

For a function f belonging to the class $\mathcal{A}$, we define the integral operator F_γ as follows :

$$F_\gamma(f) := F_\gamma(f)(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f(t)dt \quad (\gamma \ge 0\ ;\ z \in \mathcal{U}). \tag{2.13}$$

Many authors have studied the integral operator of the form (2.13) where γ is a real constant and f belongs to some favored classes of functions. Various interesting developments involving the operator (2.13), for examples, can be found in [2, 8, 10]. We also denote the class $\mathcal{K}[A,B]$ by

$$\mathcal{K}[A,B] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz}\ (z \in \mathcal{U}\ ;\ -1 \le B < A \le 1)\right\}.$$

Next, we prove

THEOREM 2.2. *Let $f \in \mathcal{A}$. If*

$$\left|\arg\left\{\frac{(zf'(z))'}{g'(z)}\right\}\right| < \frac{\pi}{2}\delta \ \ (0 < \delta \le 1\ ;\ z \in \mathcal{U})$$

for some $g \in \mathcal{K}[A,B]$, then

$$\left|\arg\left\{\frac{(zF_\gamma'(f))'}{F_\gamma'(g)}\right\}\right| < \frac{\pi}{2}\eta,$$

where F_γ is given by (2.13) and $\eta(0 < \eta \leq 1)$ is the solution of the equation:
(2.14)

$$\delta = \begin{cases} \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta \sin\frac{\pi}{2}(1-t(A,B,c))}{\left(\frac{1+A}{1+B}+c\right)+\eta\cos\frac{\pi}{2}(1-t(A,B,c))}\right) & \text{for } B \neq -1, \\ \eta & \text{for } B = -1, \end{cases}$$

when

$$t(A,B,c) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB+c(1-B^2)}\right). \tag{2.15}$$

Proof. Let

$$p(z) = \frac{z(F'_\gamma(f))'}{F'_\gamma(g)} \quad \text{and} \quad q(z) = 1 + \frac{zF''_\gamma(g)}{F'_\gamma(g)}.$$

From the assumption for g and an application of Briot-Bouquet differential subordination [12, p. 81], we see that $F_\gamma(g) \in \mathcal{K}[A,B]$. Using the equation

$$zF'_\gamma(f)(z) + \gamma F_\gamma(f)(z) = (1+\gamma)f(z)$$

and simplifing, we obtain

$$\frac{(zf'(z))'}{g'(z)} = p(z) + \frac{zp'(z)}{q(z)+c}.$$

Since $q \in \mathcal{K}[A,B]$, we note [16] that

$$\left|\frac{(zF'_\gamma(g))'}{F'_\gamma(g)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (z \in \mathcal{U}\ ;\ B \neq -1) \tag{2.16}$$

and

$$\mathrm{Re}\left\{\frac{(zF'_\gamma(g))'}{F'_\gamma(g)}\right\} > \frac{1-A}{2} \qquad (z \in \mathcal{U}\ ;\ B = -1). \tag{2.17}$$

Then, from (2.16) and (2.17), we have

$$q(z) + c = \rho e^{i\frac{\pi}{2}\phi},$$

where
$$\begin{cases} \frac{1-A}{1-B}+c \ < \ \rho \ < \ \frac{1+A}{1+B}+c \\ -t(A,B,c) \ < \ \phi \ < \ t(A,B,c) \ \text{ for } B \neq -1, \end{cases}$$

where $t(A,B,c)$ is given by (2.15), and

$$\begin{cases} \frac{1-A}{2}+c \ < \ \rho \ < \ \infty \\ -1 \ < \ \phi \ < \ 1 \ \text{ for } B = -1. \end{cases}$$

Here, we note that p is analytic in $\mathcal{U}$ with $p(0)=1$ and $\operatorname{Re} p(z) > 0$ in $\mathcal{U}$ by applying the assumption and Lemma 2.2 with $\omega(z) = 1/(q(z)+c)$. Hence $p(z) \neq 0$ in $\mathcal{U}$.

If there exists a point $z_0 \in \mathcal{U}$ such that the conditions (2.3) and (2.4) are satisfied, then (by Lemma 2.3) we obtain (2.5) under the restrictions (2.6-8).

At first, we suppose that

$$\{p(z_0)\}^{\frac{1}{\eta}} = ia \qquad (a>0).$$

For the case $B \neq -1$, we then obtain

$$\begin{aligned} &\arg\left\{\frac{(z_0f'(z_0))'}{g'(z_0)}\right\} \\ &= \arg\left\{p(z_0)\left(1+\frac{1}{q(z_0)+c}\frac{z_0p'(z_0)}{p(z_0)}\right)\right\} \\ &= \arg\{p(z_0)\} + \arg\ \left\{1+i\eta k(\rho e^{i\frac{\pi}{2}\phi})^{-1}\right\} \\ &= \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\eta k \sin[\frac{\pi}{2}(1-\phi)]}{\rho+\eta k\cos[\frac{\pi}{2}(1-\phi)]}\right) \\ &\geq \frac{\pi}{2}\eta + \tan^{-1}\left(\frac{\eta \sin[\frac{\pi}{2}\{1-t(A,B)\}]}{\left(\frac{1+A}{1+B}+c\right)+\eta\cos[\frac{\pi}{2}\{1-t(A,B)\}]}\right) \\ &= \frac{\pi}{2}\delta, \end{aligned}$$

where δ and $t(A,B)$ are given by (2.14) and (2.15), respectively. Similarly, for the case $B=-1$, we have

$$\arg\left\{\frac{(z_0f'(z_0))'}{g'(z_0)}\right\} \ \geq \ \frac{\pi}{2}\eta = \frac{\pi}{2}\delta.$$

These evidently contradict the assumption of the theorem.

Next, in the case $p(z_0)^{\frac{1}{\eta}} = -ia$ $(a > 0)$, applying the same method as the above, we also can prove the theorem easily. Therefore we complete the proof of Theorem 2.2.

REMARK. From Theorem 2.2, we see easily that every function in $\mathcal{Q}(\delta)$ $(0 < \delta \leq 1)$ preserves the angles under the integral operator defined by (2.13).

By letting $g(z) = z$ and $B \to A$ $(A < 1)$ in Theorem 2.2, we have

COROLLARY. *If $f \in \mathcal{A}$ and*

$$|\arg\{(zf'(z))'\}| < \frac{\pi}{2}\delta \quad (0 < \delta \leq 1\,; \quad z \in \mathcal{U})$$

then

$$\left|\arg\left\{(zF_\gamma'(f))'\right\}\right| < \frac{\pi}{2}\eta$$

where F_γ is given by (2.13) and $\eta (0 < \eta \leq 1)$ is the solution of the equation:

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left(\frac{\eta}{1+c}\right).$$

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