THE ITERATION OF ENTIRE FUNCTION

JIANWU SUN

ABSTRACT. In this paper, we obtain the following results: Let \( f \) be a transcendental entire function with \( \log M(r, f) = O((\log r)^\alpha e^{(\log r)^\beta}) \) \((0 \leq \alpha < 1, \beta > 1)\). Then every component of \( N(f) \) is bounded. This result generalizes the result of Baker.

1. Introduction

Let \( f : \mathbb{C} \to \mathbb{C} \) denote a nonlinear entire function and \( f^n, n \in \mathbb{N} \), the \( n \)th iterate of \( f \). The set of normality, \( N(f) \), is defined to be the set of points, \( z \in \mathbb{C} \), such that the sequence \( (f^n) \) forms a normal family in some neighbourhood of \( z \). It is easy to see that \( N(f) \) is open and has the property of complete invariance under \( f \), that is \( z \in N(f) \) if and only if \( f(z) \in N(f) \). The complement \( J(f) \), of \( N(f) \) is known as the Julia set. This set is clearly closed and completely invariant under \( f \). More details of these and other basic properties of the sets \( N(f) \) and \( J(f) \) can be found in [4] and [5].

Baker [1] obtained the following result:

**Theorem A.** Let \( f(z) \) be a transcendental entire function with \( \log M(r, f) = O((\log r)^p) \), where \( 1 < p < 3 \). Then every component of \( N(f) \) is bounded.

In the paper, we generalize the condition \( \log M(r, f) = O((\log r)^p) \) \((1 < p < 3)\) to the form \( \log M(r, f) = O((\log r)^\beta e^{(\log r)^\alpha}) \) \((0 \leq \alpha < 1, \beta > 1)\) in Theorem A and obtain the following result:

**Theorem 1.** Let \( f \) be a transcendental entire function with \( \log M(r, f) = O((\log r)^\beta e^{(\log r)^\alpha}) \) \((0 \leq \alpha < 1, \beta > 1)\).

Then every component of \( N(f) \) is bounded.

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2. Some lemmas

**Lemma 1** [2]. Let $f$ be an entire function of order zero and $z = re^{i\theta}$. Then
\[ \log |f(re^{i\theta})| - N(2R) - \log |c| < 4Q(2R) \]
for all $z \in \{z : |z| < R\}$.

**Lemma 2** [2]. Let $f$ be an entire function of order zero and $z = re^{i\theta}$. Then for any $\zeta > 0$, $\eta > 0$, there exist $R_0 = R_0(\zeta, \eta)$, $k = k(\zeta, \eta)$ such that for all $R > R_0$,
\[ \log |f(re^{i\theta})| - N(2R) - \log |c| > -kQ(2R), \zeta R \leq r \leq R, \]
except in a set of circles enclosing the zeros of $f$, the sum of whose radii is at most $\eta R$. Where $Q(r) = r \int_{r}^{\infty} \frac{t(1/f)}{t^2} dt$, $N(r) = \int_{0}^{r} \frac{n(t,1/f)}{t} dt$.

Combining Lemma 1 and Lemma 2 and taking $\zeta = \eta$ to be small, we obtain:

**Lemma 3.** If $f$ is an entire function of order zero then there exist $R_0 = R_0(\zeta) > 0$ and a constant $B$ such that, for each $R > R_0$, there exists $r$ satisfying
\[ \zeta R \leq r \leq R \]
such that
\[ \log m(r,f) > \log M(r,f) - BQ(2R). \]

**Lemma 4** [5]. Suppose that $f$ is a transcendental entire function and that there exist sequences $R_n$, $\rho_n \to \infty$ and $c(n) > 1$ such that
1. $R_{n+1} = M(R_n) = M(R_n, f)$
2. $R_n \leq \rho_n \leq (R_n)^{c(n)}$
3. $m(\rho_n) = m(\rho_n, f) > (R_{n+1})^{c(n+1)}$ for all sufficiently large $n$

Then $N(f)$ contains no unbounded components.

**Lemma 5.** Let $\phi(r)$, $H(r)$ be two positive functions tending to $\infty$ as $r \to \infty$, and $A = A(r) > 1$, and $\phi(Ar)/\phi(r) \to c(r \to \infty), (c \geq 1)$. If $H(r) = O(\phi(r))$, then there exists $r_0 > 1$, $H(\rho) / H(r)$ is upper bounded in $[r_0, +\infty)$. 
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Proof. Suppose that \( H(Ar)/H(r) \) is not bounded from above in \([r_0, +\infty)\), then there exists a sequence \( \{r_n\}, r_n \to \infty (n \to \infty) \), for arbitrary \( G > 0 \), there exists a natural number \( n_0 \) such that for \( n > n_0 \), we have

\[
(1) \quad H(Ar_n)/H(r_n) > G.
\]

Since \( \lim_{r \to \infty} H(r)/\phi(r) = k \quad (0 \leq k < +\infty) \), we see that

1. If \( k \neq 0 \), then by (1) we obtain \( H(Ar_n)/\phi(Ar_n) > G \frac{H(r_n)}{\phi(r_n)} \phi(Ar_n) \), take \( G = 2c \), thus

\[
\lim_{n \to \infty} \frac{H(Ar_n)/\phi(Ar_n)}{\phi(Ar_n)} \geq G \lim_{n \to \infty} \frac{H(r_n)}{\phi(r_n)} \lim_{n \to \infty} \frac{\phi(r_n)}{\phi(Ar_n)},
\]

i.e., \( k \geq 2c(1/c)^2 k = 2k \), this is a contradiction.

2. If \( k = 0 \), then \( \lim_{r \to \infty} H(r)/\phi(r) = 0 \) take \( G = 4c \), when \( n > n_0 \), by (1) we get \( H(Ar_n)/H(r_n) > 4c \), so, when \( n > n_0 \), for arbitrary natural number \( m \), we have

\[
(2) \quad H(A^m r_n) > 4cH(A^{m-1} r_n) > \cdots > (4c)^m H(r_n).
\]

Since \( \lim_{n \to \infty} \phi(Ar_n)/\phi(r_n) = c \), take \( \epsilon_0 = c > 0 \), there exists a \( n_1 > n_0 \), such that for \( n > n_1 \) we obtain

\[
| \phi(Ar_n)/\phi(r_n) - c | < \epsilon_0 = c, \quad \text{i.e., } \phi(Ar_n) < 2c\phi(r_n),
\]

thus, for arbitrary natural number \( m \), we get

\[
(3) \quad \phi(A^m r_n) < 2c\phi(A^{m-1} r_n) < \cdots < (2c)^m \phi(r_n),
\]

take \( n = n_2 > n_1 > n_0 \), by (2) and (3) we have

\[
(4) \quad H(A^m r_{n_2})/\phi(A^m r_{n_2}) > \frac{(4c)^m H(r_{n_2})}{(2c)^m \phi(r_{n_2})} = 2^m \frac{H(r_{n_2})}{\phi(r_{n_2})} \to \infty (m \to \infty).
\]

As \( m \to \infty \), we obtain \( A^m r_{n_2} \to \infty \), so, (4) is a contradiction with \( \lim_{r \to \infty} H(r)/\phi(r) = 0 \).

This completes the proof of the Lemma 5. \( \square \)

Lemma 6. Let \( f \) be a transcendental entire function with \( \log M(r, f) = O((\log r)^\alpha e^{(\log r)^\beta}) \quad (0 \leq \alpha < 1, \beta > 1) \). Then

\[
\log M(\delta r, f) \sim \log M(r, f) \quad (r \to \infty, \delta \geq 2).
\]

Proof. (i) We may assume \( f(0) = 1 \), otherwise, we only need to make a transformation

\[
F(z) = f(z) - f(0) + 1.
\]
By Jensen's theorem

\[
N(r, 1/f) = \int_0^r \frac{n(t, 1/f)}{t} dt
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \log | f(re^{i\theta}) | d\theta \leq \log M(r, f),
\]

for \( r > 1 \) and \( A > 1 \), by (5) we have

\[
n(r, 1/f) \log A \leq \int_r^{Ar} \frac{n(t, 1/f)}{t} dt \leq N(Ar, 1/f) \leq \log M(Ar, f).
\]

So

\[
n(r, 1/f) \leq \frac{\log M(Ar, f)}{\log A}. \tag{6}
\]

Since

\[
\log M(r, f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}}). \tag{7}
\]

Take \( A = r^{\sigma(r)} \) and \( \sigma(r) = \frac{1}{(\log r)^{\alpha}} \). By (6) we have

\[
n(r, 1/f) \leq \frac{\log M(r^{1+\sigma(r)}, f)}{\sigma(r) \log r}. \tag{8}
\]

Therefore, there exists \( r_0 > 1 \) and \( k > 0 \), such that for \( r > r_0 \), and put \( r = e^u \), by (7) we obtain

\[
\frac{\log M(r^{1+\sigma(r)}, f)}{r^{1/2} \sigma(r) \log r} \leq \frac{k(\log r^{1+\sigma(r)})^\beta e^{(\log r^{1+\sigma(r)})^\alpha}}{r^{1/2} \sigma(r) \log r}
\]
\[
= \frac{k(1 + (\log r)^{\alpha})^\beta (\log r)^\beta e^{(1 + (\log r)^{\alpha})^\alpha} (\log r)^\alpha}{r^{1/2} (\log r)^{1-\alpha}}
\]
\[
= \frac{k(1 + 1/u^{\alpha})^\beta u^{\beta} e^{(1+1/u^{\alpha})^\alpha} u^\alpha}{(e^u)^{1/2} u^{1-\alpha}} \leq \frac{k2^\beta u^{\beta+\alpha-1} e^{2^\alpha u^\alpha}}{e^{u/2}}.
\]
Since $\alpha < 1$, \( \log M(r^{1+\sigma(r)}, f) \) decreases when $r > r_1 > r_0$, and by (8) we have

\[
Q(r) = r \int_{\sigma(r) \log r}^{+\infty} \frac{n(t, 1/f)}{t^2} \, dt \\
\leq r \int_{\sigma(r) \log r}^{+\infty} \frac{\log M(t^{1+\sigma(t)}, f)}{t^2 \sigma(t) \log t} \, dt \\
= \lim_{b \to +\infty} \frac{\log M(r^{1+\sigma(r)}, f)}{\sigma(r) \log r} \int_{\sigma(r) \log r}^{+\infty} t^{-3/2} \, dt \\
\leq \frac{r^{1/2} \log M(r^{1+\sigma(r)}, f)}{\sigma(r) \log r} \int_{\sigma(r) \log r}^{+\infty} t^{-3/2} \, dt \\
= \frac{2 \log M(r^{1+\sigma(r)}, f)}{\sigma(r) \log r}.
\]

Let \( \phi(r) = (\log r)^\beta e^{(\log r)^\alpha} \) \((0 \leq \alpha < 1, \beta > 1)\), \( A = r^{\sigma(r)} \) and \( \sigma(r) = \frac{1}{(\log r)^\alpha} \). Then

\[
\phi(Ar)/\phi(r) = \frac{(\log r^{1+\sigma(r)})^\beta e^{(\log r^{1+\sigma(r)})^\alpha}}{(\log r)^\beta e^{(\log r)^\alpha}} \\
= (1 + \sigma(r))^\beta e^{(\log r)^\alpha [1+\sigma(r)]^\alpha - 1} \\
= (1 + \sigma(r))^\beta e^{(\log r)^\alpha \alpha \sigma(r)(1+o(1))} \\
= (1 + \frac{1}{(\log r)^\alpha})^\beta e^{(\log r)^\alpha \frac{1}{(\log r)^\alpha} [1+o(1)]} \to e^\alpha (\geq 1) \quad (r \to \infty).
\]

So, from (7) \( \log M(r, f) = O(\phi(r)) \).

On the other hand, from (9), (10), and Lemma 5, there exists \( L > 0 \), for \( r > r_1 > r_0 \) we have

\[
\frac{Q(r)}{\log M(r, f)} \leq \frac{2 \log M(r^{1+\sigma(r)}, f)}{\sigma(r) \log r \log M(r, f)} \\
= \frac{\log M(Ar, f)}{\log M(r, f)} \frac{2}{\sigma(r) \log r} \\
\leq \frac{2L}{\sigma(r) \log r} = \frac{2L}{(\log r)^{1-\alpha}} \to 0 (r \to \infty).
\]

So,

\[
Q(r) = o(\log M(r, f)).
\]
(ii) Since \( \log M(r, f) = O((\log r)^{\beta} e^{(\log r)^{\alpha}}) \), the order \( \rho \) of \( f \) is equal to zero, and \( n(r, 1/f) = o(r) \) and

\[
\log M(r, f) \leq \log \prod_{n=1}^{+\infty} (1 + r/r_n)
= \int_{0}^{+\infty} \log(1 + r/t)dn(t, 1/f)
\leq \int_{0}^{+\infty} \frac{r}{t}dn(t, 1/f)
= r \int_{0}^{+\infty} \frac{n(t, 1/f)}{t(t+r)} dt
= r \left( \int_{0}^{r} + \int_{r}^{+\infty} \right) \frac{n(t, 1/f)}{t(t+r)} dt
\leq \frac{1}{r} \int_{0}^{r} \frac{n(t, 1/f)}{t} dt + r \int_{r}^{+\infty} \frac{n(t, 1/f)}{t^2} dt
= N(r) + Q(r)
= N(r, 1/f) + Q(r, 1/f).
\]

So, from Lemma 2, (11), and (12), we obtain

\[
* \log |f(re^{i\theta})| > N(2R) - kQ(2R) \quad (\zeta R \leq r \leq R)
= N(2R) + Q(2R) - (k+1)Q(2R)
\geq \log M(2R, f) - (k+1) o(\log M(2R, f))
\]

(13) \quad = \log M(2R, f)(1 - o(1))

(14) \quad \geq \log M(r, f)(1 - o(1)).

On the other hand,

(15) \quad \log |f(z)| \leq \log M(r, f) \leq \log M(\delta r, f) \quad (|z| = r, \delta \geq 2).

In (13), let \( 2R = \delta r, \delta \geq 2 \). Then from (13), (14), and (15), we get

(16) \quad \log |f(z)| \sim \log M(\delta r, f) \quad (r \to \infty)

(17) \quad \log |f(z)| \sim \log M(r, f) \quad (r \to \infty).

By (16) and (17), we get

(18) \quad \log M(\delta r, f) \sim \log M(r, f) \quad (r \to \infty, \delta \geq 2).

So, from (18), Lemma 6 is proved.
3. Proof of Theorem 1

By Lemma 6, we have

\( \log M(\delta r, f) \sim \log M(r, f) = O((\log r)^{\alpha} e^{(\log r)^{\alpha}}) \quad (r \to \infty, \delta \geq 2), \)

thus, for \( \epsilon \in (0, 1) \) and \( \delta \geq 2 \) we get

\[
(1 - \epsilon) \log M(r, f) < \log M(\delta r, f) < (1 + \epsilon) \log M(r, f) < 2 \log M(r, f).
\]

(20)

By (6) and (20), for \( r > 1 \) and \( A \geq 2 \), we obtain

\[
Q(r) = r \int_r^{+\infty} \frac{n(t, 1/f)}{t^2} dt \\
\leq r \int_r^{+\infty} \frac{\log M(At, f)}{t^2 \log A} dt \\
< \frac{2r}{\log A} \int_r^{+\infty} \frac{\log M(t, f)}{t^2} dt.
\]

(21)

Therefore, by (19) there exists \( r_0 > 1 \) and \( K > 0 \), such that for \( r > r_0 \), and put \( r = e^\mu \) we have

\[
\frac{\log M(r, f)}{e^{1/2}} \leq \frac{K(\log r)^{\beta} e^{(\log r)^{\alpha}}}{e^{1/2}} = \frac{K \mu^\beta e^{(\mu^\alpha)}}{e^{1/2}}.
\]

Since \( \alpha < 1 \), \( \frac{\log M(r, f)}{e^{1/2}} \) decreases when \( r > r_1 > r_0 \), by (21) we get

\[
Q(r) < \frac{2r}{\log A} \int_r^{+\infty} \frac{\log M(t, f)}{t^2} dt \\
< \frac{2r^{1/2}}{\log A} \log M(r, f) \int_r^{+\infty} t^{-3/2} dt \\
= \frac{4 \log M(r, f)}{\log A}.
\]

(22)

Thus, by Lemma 3 and (22), there exists \( R_0 \) and a constant \( c > 1 \), such that, for each \( R > R_0 \) and each \( A \geq 2 \), there exists \( r \) satisfying

\[ R/\delta \leq r \leq R \]
with
\[ \log m(r, f) > \log M(r, f) - c \frac{\log M(2R, f)}{\log A}. \] (23)

On the other hand, since \( f \) is transcendental there exists \( r_2 \) such that for all \( r > r_2 > r_1, \ M(r, f) \geq r^2 \), take
\[ R_1 > \max(e^2, r_2) \quad \text{and} \quad R_{n+1} = M(R_n, f). \] (24)

We now separate two cases:

Case 1. Suppose that
\[ \log M((R_n)^{2+1/n}/8, f) \geq 4 \log M(R_n, f). \] (25)

Put \( A = e^{8c} \), by (23) for sufficiently large \( n \) there exists \( \rho_n \) satisfying
\[ R_n < \frac{1}{8} (R_n)^{2+1/n} \leq \rho_n \leq (R_n)^{2+1/n} \] (26)
and
\[ \log m(\rho_n, f) > \log M(\rho_n, f) - c \frac{\log M(2(R_n)^{2+1/n}, f)}{\log e^{8c}}. \]

By (20), for sufficiently large \( n \) we have
\[ \log M(2(R_n)^{2+1/n}, f) = \log M(16(R_n)^{2+1/n}/8, f) < 2 \log M((R_n)^{2+1/n}/8, f). \]

So
\[ \log m(\rho_n, f) > \log M(\rho_n, f) - \frac{1}{4} \log M((R_n)^{2+1/n}/8, f) \]
\[ > \frac{3}{4} \log M((R_n)^{2+1/n}/8, f). \]

By (25), we get
\[ \log m(\rho_n, f) > 3 \log M(R_n, f). \]

Thus
\[ m(\rho_n, f) > (M(R_n, f))^3 = (R_{n+1})^3 > (R_{n+1})^{2+\frac{1}{n+1}}. \] (27)

Case 2. Suppose that
\[ \log M((R_n)^{2+1/n}/8, f) < 4 \log M(R_n, f). \] (28)

Put \( A = e^{cn^3} \), by (23) for sufficiently large \( n \), there exists \( \rho_n \) satisfying
\[ R_n < (R_n)^{2+1/n-1/n^3} < \frac{1}{64} (R_n)^{2+1/n} \leq \rho_n < \frac{1}{8} (R_n)^{2+1/n}. \] (29)
and
\[
\log m(\rho_n, f) > \log M(\rho_n, f) - c \frac{\log M(2(R_n)^{2+1/n}/8, f)}{\log e^{\alpha n^3}}.
\]

By (20) and (28), for sufficiently large \(n\), we obtain
\[
\log m(\rho_n, f) > \log M(\rho_n, f) - \frac{2 \log M(\frac{1}{8}(R_n)^{2+1/n}, f)}{n^3}
\]
(30)
\[
> \log M(\rho_n, f) - \frac{8 \log M(R_n, f)}{n^3}.
\]

By (29), \(\rho_n > (R_n)^{2+1/n-1/n^3}\) and \(\log M(r, f)\) is a convex function of \(\log r\), for sufficiently large \(n\), we have
\[
\log M(\rho_n, f) > (2+1/n-1/n^3)[\log M(R_n, f) - \log M(1, f)] + \log M(1, f)
\]
(31)
\[
> (2+1/n-1/n^3) \log M(R_n, f).
\]

Thus, by (30) and (31), for sufficiently large \(n\), we get
\[
\log m(\rho_n, f) > (2+1/n - 2/n^3 - 8/n^3) \log M(R_n, f)
\]
\[
> (2 + \frac{1}{n + 1}) \log M(R_n, f)
\]
so
(32)
\[
m(\rho_n, f) > (R_{n+1})^{2+\frac{1}{n+1}}.
\]

Hence, by (24), (26), and (27) or (24), (29), and (32), we obtain the following result:

If \(f\) is a transcendental entire function with \(\log M(r, f) = O((\log r)^\beta e^{(\log r)^\alpha})\) \((0 \leq \alpha < 1, \beta > 1)\), then there exists sequences \(R_n, \rho_n \to \infty\) such that

1. \(R_{n+1} = M(R_n, f)\),
2. for sufficiently large \(n\),

\[
R_n \leq \rho_n \leq (R_n)^{2+1/n} \quad \text{and} \quad m(\rho_n, f) > (R_{n+1})^{2+\frac{1}{n+1}}.
\]

Put \(c(n) = 2 + 1/n > 1\), by Lemma 4, we get every component of \(N(f)\) is bounded.

Theorem 1 is proved. \(\square\)
References


Department of Mathematics, Huaiyin Teacher’s College, Jiangsu 223001, P. R. China