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SPLITTING MULTIPLICATIVE SETS IN DEDEKIND DOMAINS

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ABSTRACT. Let D be a Dedekind domain with divisor class group $Cl(D)$. We show that there is a correspondence between the set of splitting multiplicative subsets in D and certain subgroups of $Cl(D)$.

Let D be an integral domain. As in [14] and [3], we say that a saturated multiplicative set S of D is a *splitting multiplicative set* if for each nonzero $d \in D$, $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \bigcap aD = s'aD$ for all $s' \in S$. Then $T = \{0 \neq t \in D | sD \bigcap tD = stD$ for all $s \in S\}$ is also a splitting multiplicative set, $ST = D - \{0\}$, and $S \bigcap T = U(D)$, where $U(D)$ is the group of units of D. We call T the *m-complement set* for S.

Mott [13, Theorem 2.1] showed that there is a one-to-one correspondence between the set of saturated multiplicative subsets of D and the set of convex directed subgroups of $G(D)$, the *group of divisibility* of D, given by $S \mapsto \langle S \rangle = \{s_1/s_2U(D) | s_1, s_2 \in S\}$. Moreover, for a saturated multiplicative set S of D, the subgroup $\langle S \rangle$ of $G(D)$ is a cardinal summand of $G(D)$ if and only if S is a splitting multiplicative set of D [14, Proposition 4.1].

Let S be a splitting multiplicative subset of D with T the m-complement set for S. In [4, Corollary 3.8], it was shown that the natural group homomorphism $Cl_t(D) \to Cl_t(D_S) \times Cl_t(D_T)$ is an isomorphism, where $Cl_t(D)$, the *t-class group* of D, is the group of (fractional) t-invertible t-ideals of D modulo its subgroup of principal ideals. We recover this result (for D a Dedekind domain) in Corollary 8.

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In this paper, we show that if D is a Dedekind domain with divisor class group $Cl(D)$, then there is a correspondence between the set of splitting multiplicative sets of D and certain subgroups of $Cl(D)$. In particular, if D has no principal primes, then there exists a one-to-one correspondence between the set of splitting multiplicative subsets of D and the set of $\mathcal{A}$-summands of $Cl(D)$ (see Definition 3).

An integral domain D is *atomic* if each nonzero nonunit of D is a product of irreducible elements. Following Zaks [16], we define D to be a *half-factorial domain* (HFD) if D is atomic and for any irreducible elements $x_1, ..., x_m, y_1, ..., y_n$ of D with $x_1 \cdots x_m = y_1 \cdots y_n$, then $m = n$. Following Valenza [15], we define the *elasticity* of an atomic integral domain D as

$$\rho(D) = sup\{ \frac{m}{n} \,|x_1 \cdots x_m = y_1 \cdots y_n \text{ for irreducible } x_i, y_j \in D\}.$$

(Define $\rho(D) = 1$ if D is a field.) Notice that $1 \leq \rho(D) \leq \infty$, and $\rho(D) = 1$ if and only if D is an HFD. Thus $\rho(D)$ measures how far D is from being an HFD.

Let D be an atomic integral domain and let $\mathcal{F}$ be a nonempty subset of $\mathcal{I}(D)$, the set of irreducible elements of D. Suppose that

$$(\dagger) \qquad x_1 \cdots x_m = uy_1 \cdots y_n,$$

where $u \in U(D)$ and $x_1, ..., x_m, y_1, ..., y_n \in \mathcal{I}(D)$. Set $\mathcal{F}(x) = \{i | x_i \in \mathcal{F}\}$ and $\mathcal{F}(y) = \{j | y_j \in \mathcal{F}\}$. Following [6], we say that $\mathcal{F}$ is a *factorization set (F-set)* of D if for any equality (involving irreducibles) of the form ($\dagger$), $|\mathcal{F}(x)| \neq 0$ implies that $|\mathcal{F}(y)| \neq 0$, and that $\mathcal{F}$ is a *half-factorial set (HF-set)* if any equality of the form ($\dagger$) implies that $|\mathcal{F}(x)| = |\mathcal{F}(y)|$. Clearly an HF-set is also an F-set. F-sets and HF-sets are studied in detail in [6].

Let D be an atomic integral domain and let S be a splitting multiplicative set of D with T the m-complement for S. Then D_S and D_T are also atomic [3, Corollary 2.2], and $\rho(D) = max\{\rho(D_S), \rho(D_T)\}$ [8, Theorem 2.3]. Moreover, the values $\rho(D_S) \leq \rho(D)$ may be arbitrary [8, Theorem 2.7] (i.e., for each $r, s \in [1, \infty]$ with $r \leq s$, there exists a Dedekind domain D with a splitting multiplicative set S such that $\rho(D_S) = r \leq s = \rho(D)$). However, if S is generated by an HF-set, then $\rho(D) = \rho(D_S)$ [8, Theorem 2.11].

Throughout, we will assume that D is a Dedekind domain with $Cl(D)$ its divisor class group, $[I]$ the ideal class of I in $Cl(D)$, $U(D)$ its group of units, D^* its set of nonzero elements, $S \subseteq D^*$ a multiplicative subset of D, $X^{(1)}(D)$ its set of nonzero (maximal) prime ideals, and $\mathcal{I}(D)$ its set of irreducible elements. A multiplicative set S is generated by $C \subseteq D^*$, and written $\langle C \rangle$, if $S = \{uc_1 \cdots c_n | u \in U(D), \text{each } c_i \in C, n \geq 0\}$. For a group G and $C \subseteq G$, we also denote by $\langle C \rangle$ the subgroup of G generated by C. To avoid trivialities, we will assume that D is *not* a UFD (PID), i.e., $Cl(D) \neq \{0\}$. For general references on factorization in integral domains, see [2, 5].

If for a given abelian group G and subset $\mathcal{A} \subseteq G - \{0\}$ there exists a Dedekind domain D such that $Cl(D) = G$ and $\mathcal{A} = \{[P] | P$ is prime ideal of D and $[P] \neq 0\}$, then the pair $\{G, \mathcal{A}\}$ is called *realizable* [12], [11]. For D a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$ and S a saturated multiplicative subset of D, set

$$\mathcal{A}[S] = \{[P] \,|\, P \cap S \neq \emptyset\} \subseteq \mathcal{A}.$$

Let $G[S]$ be the subgroup of $Cl(D)$ generated by $\mathcal{A}[S]$. It is possible that $\mathcal{A}[S] = \emptyset$ (for example, if S is generated by principal primes, or if $S = U(D)$). Note that $\mathcal{A}[S] = \emptyset$ if and only if $G[S] = \{0\}$. By Nagata's Theorem [10, Corollary 7.2], $G[S] = ker\, \varphi$, where $\varphi : Cl(D) \to Cl(D_S)$ is the natural homomorphism.

For future reference, we include a result from [4, page 27].

LEMMA 1. *Let D be a Dedekind domain and let S be a splitting multiplicative set with T the m-complement for S. If P is a nonzero prime ideal of D, then either $P \bigcap S \neq \emptyset$ or $P \bigcap T \neq \emptyset$, but not both.*

Proof. We may assume that S is nontrivial. By definition, either $P \bigcap S \neq \emptyset$ or $P \bigcap T \neq \emptyset$. If both $P \bigcap S \neq \emptyset$ and $P \bigcap T \neq \emptyset$, then P contains irreducible elements $x \in S$ and $y \in T$. Then, by definition, $xD \bigcap yD = xyD$, and hence $P \supseteq (x, y) = (x, y)_v = D$, a contradiction.□

THEOREM 2. *Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$ and let S be a splitting multiplicative set with T the m-complement for S. Then*

(1) *$\mathcal{A} = \mathcal{A}[S] \bigcup \mathcal{A}[T]$ is a partition of $\mathcal{A}$.*
(2) *$Cl(D) = G[S] \bigoplus G[T]$.*

Proof. (1) follows from Lemma 1.

(2) Since $\mathcal{A}$ generates $Cl(D)$, $Cl(D) = G[S] + G[T]$ by (1). Let $g \in G[S] \bigcap G[T]$. Since $G[S]$ (resp., $G[T]$) is generated by $\mathcal{A}[S]$ (resp., $\mathcal{A}[T]$), we may write $g = \pm[P_1] \pm \cdots \pm [P_m] = \pm[Q_1] \pm \cdots \pm [Q_n]$, where each $[P_i] \in \mathcal{A}[S]$ and each $[Q_j] \in \mathcal{A}[T]$. Thus we may assume that $g = [P_1] + \cdots + [P_i] - [P_{i+1}] - \cdots - [P_m] = [Q_1] + \cdots + [Q_j] - [Q_{j+1}] - \cdots - [Q_n]$. Then $[P_1 \cdots P_i Q_{j+1} \cdots Q_n] = [P_{i+1} \cdots P_m Q_1 \cdots Q_j]$, so that

$$bP_1 \cdots P_i Q_{j+1} \cdots Q_n = aP_{i+1} \cdots P_m Q_1 \cdots Q_j$$

for some nonzero $a, b \in D$. Since S is a splitting multiplicative set, we have $a = s_1 t_1$ and $b = s_2 t_2$ for some $s_1, s_2 \in S$, $t_1, t_2 \in T$. Let $s_1 D = I_1 \cdots I_e$, $t_1 D = J_1 \cdots J_f$, $s_2 D = I'_1 \cdots I'_l$, and $t_2 D = J'_1 \cdots J'_k$ be prime factorizations, where each $[I_c], [I'_d] \in \mathcal{A}[S]$ and $[J_c], [J'_d] \in \mathcal{A}[T]$. Thus we have

$$(\ddagger) \qquad \begin{aligned} &I'_1 \cdots I'_l J'_1 \cdots J'_k P_1 \cdots P_i Q_{j+1} \cdots Q_n \\ &= I_1 \cdots I_e J_1 \cdots J_f P_{i+1} \cdots P_m Q_1 \cdots Q_j. \end{aligned}$$

By ($\ddagger$), the unique factorization of prime ideals, and Lemma 1, we have that $s_2 P_1 \cdots P_i = s_1 P_{i+1} \cdots P_m$ and $t_2 Q_{j+1} \cdots Q_n = t_1 Q_1 \cdots Q_j$. Thus $g = [P_1] + \cdots + [P_i] - [P_{i+1}] - \cdots - [P_m] = 0$, so that $G[S] \bigcap G[T] = \{0\}$. Hence (2) holds. (Here is another proof: As mentioned earlier, $G[S] = ker\, \varphi_1 : Cl(D) \to Cl(D_S)$ and $G[T] = ker\, \varphi_2 : Cl(D) \to Cl(D_T)$ by Nagata's Theorem, where φ_1 and φ_2 are the natural homomorphisms. Then the natural map $\theta : Cl(D) \to Cl(D_S) \times Cl(D_T)$ is an isomorphism [4, Corollary 3.8], and $G[S] = \theta^{-1}(\{0\} \times Cl(D_T))$ and $G[T] = \theta^{-1}(Cl(D_S) \times \{0\})$ by the above comments. Hence $Cl(D) = G[S] \bigoplus G[T]$. □

Hence, if $Cl(D)$ is indecomposable, then we may assume that $G[T] = \{0\}$. Thus $\mathcal{A}[T] = \emptyset$, and hence either T is generated by principal primes or $T = U(D)$. If, in addition, D has no principal primes, then D has no nontrivial splitting multiplicative sets (cf., [8, Remark 2.6]).

Theorem 2 motivates the following two definitions.

DEFINITION 3. Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$. For a subgroup G of $Cl(D)$, we call G a *direct summand under $\mathcal{A}$ of $Cl(D)$*, or just an *$\mathcal{A}$-summand* of $Cl(D)$, if there exists a partition $\mathcal{A} = \mathcal{G} \bigcup \mathcal{H}$ of $\mathcal{A}$ with the property that $G = \langle \mathcal{G} \rangle$ and $Cl(D) = G \bigoplus H$, where $H = \langle \mathcal{H} \rangle$. We denote this by $Cl(D) = G \bigoplus_{\mathcal{A}} H$ or $Cl(D) = \langle \mathcal{G} \rangle \bigoplus_{\mathcal{A}} \langle \mathcal{H} \rangle$.

DEFINITION 4. Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$. For a subset $\mathcal{B}$ of $\mathcal{A}$, we say that $\mathcal{B}$ is *subrealizable* if there exists an irreducible element coming from $\mathcal{B}$, i.e., there is an irreducible $x \in D$ such that $xD = P_1 \cdots P_n$ with each $[P_i] \in \mathcal{B}$.

If $\mathcal{B}$ is subrealizable, set

$$S[\mathcal{B}] = \langle \ \{x \in \mathcal{I}(D) \,|\, xD = P_1 \cdots P_n \text{ and each } [P_i] \in \mathcal{B}\} \ \rangle,$$

the multiplicative set generated by the set of all irreducibles coming from the classes in $\mathcal{B}$. For completeness, we define $S[\emptyset] = U(D)$ and accept $\emptyset$ as subrealizable. Clearly $S[\mathcal{A}]$ is the multiplicative subset of D generated by the set of all irreducible, but not prime, elements of D. Therefore, if D has no principal primes, then $S[\emptyset] = U(D)$ and $S[\mathcal{A}] = D^*$. Let $G = \langle \mathcal{G} \rangle$ be an $\mathcal{A}$-summand of $Cl(D)$. Then $\mathcal{G}$ is subrealizable (cf., Theorem 5). If $Cl(D)$ is torsion, then each $\mathcal{B} \subseteq \mathcal{A}$ is subrealizable. In general, if $\mathcal{B} = \{[P]\}$, then $\mathcal{B}$ is subrealizable if and only if P is nonprincipal with $|[P]| < \infty$. In this case, we denote $S[\mathcal{B}]$ by $S[P]$. Let P be a nonzero prime ideal of D. Following [6], we set $\mathcal{H}_P = P \cap \mathcal{I}(D)$ and $\mathcal{H}_{[P]} = \cup\{\mathcal{H}_Q | Q \in X^{(1)}(D) \text{ and } [Q] = [P]\}$. Then $\mathcal{H}_{[P]}$ is an F-set [6, Theorem 1.7]. As in [1], we say that a saturated multiplicative set $S \neq U(D)$ of D is a *GCD-set* if each pair of elements $x, y \in S$ has a *gcd*(x, y) in D (and hence in S). It is known that if $\mathcal{B} = \{[P]\}$ is subrealizable, then $S[P]$ is a GCD-set if and only if $[P]$ contains exactly one prime ideal, if and only if $S[P] = \langle x \rangle$, where $xD = P^n, n = |[P]|$ [1, Proposition 3.2]. On the other hand, if $Cl(D)$ is torsion, then $S[P]$ is a splitting multiplicative set of D if and only if $\mathcal{H}_{[P]}$ is an HF-set [9, Theorem 3.8], if and only if $S[P] \bigcap \mathcal{I}(D) = \mathcal{H}_{[P]}$ [6, Corollary 3.10].

Let $\mathcal{A} = \mathcal{G} \bigcup \mathcal{H}$ be a partition of $\mathcal{A}$ and let G (resp., H) be the subgroup of $Cl(D)$ generated by $\mathcal{G}$ (resp., $\mathcal{H}$). We next show that if D has no principal primes and $Cl(D) = G \bigoplus H$, then $S[\mathcal{G}]$ is a splitting multiplicative set and $S[\mathcal{H}]$ is the m-complement set for $S[\mathcal{G}]$. Also, Theorem 5 generalizes [8, Theorem 3.1].

THEOREM 5. *Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$ and let $\mathcal{G}\bigcup\mathcal{H}$ be a partition of $\mathcal{A}$. Suppose that $Cl(D) = \langle\mathcal{G}\rangle \bigoplus_{\mathcal{A}}\langle\mathcal{H}\rangle$. Then*

(1) *$\mathcal{G}$ and $\mathcal{H}$ are each subrealizable.*
(2) *$S[\mathcal{G}]$ and $S[\mathcal{H}]$ are each splitting multiplicative subsets of D.*
(3) *$\rho(D) = max\{\rho(D_{S[\mathcal{G}]}), \rho(D_{S[\mathcal{H}]})\}$.*

In particular, if D has no principal primes, then $S[\mathcal{H}]$ is the m-complement for $S[\mathcal{G}]$.

Proof. (1) Assume that $\mathcal{G}, \mathcal{H} \neq \emptyset$. (Note that if $\mathcal{G} = \emptyset$, then $\mathcal{H} = \mathcal{A}$ and $\emptyset$ and $\mathcal{A}$ are each subrealizable.) Suppose that $\mathcal{G}$ is not subrealizable. Then there exists an irreducible $x \in D$ having $xD = P_1 \cdots P_m Q_1 \cdots Q_n$, where each $[P_i] \in \mathcal{G}$, each $[Q_j] \in \mathcal{H}$, and $m, n \geq 1$. Then $0 \neq [P_1] + \cdots + [P_m] = -[Q_1] - \cdots - [Q_n] \in \langle\mathcal{G}\rangle \bigcap \langle\mathcal{H}\rangle$, a contradiction. Thus $\mathcal{G}$ and $\mathcal{H}$ are each subrealizable.

(2) We may assume that D has no principal primes. Write $\langle\mathcal{G}\rangle = G, \langle\mathcal{H}\rangle = H$. Since $G \bigcap H = \{0\}$, there is no irreducible element having prime divisors from both $\mathcal{G}$ and $\mathcal{H}$. Thus $S[\mathcal{G}]$ and $S[\mathcal{H}]$ are saturated and for each nonzero nonunit d of D, $d = st$, where $s \in S[\mathcal{G}], t \in S[\mathcal{H}]$. We claim that $tD \bigcap s'D = ts'D$ for all $s' \in S[\mathcal{G}]$. To see this, select $w \in tD \bigcap s'D$. Thus $w = tr_1 = s'r_2$ for some $r_1, r_2 \in D$. Let $tD = Q_1^{k_1} \cdots Q_n^{k_n}$, $s'D = P_1^{l_1} \cdots P_m^{l_m}$, where each $[Q_j] \in \mathcal{H}$ and each $[P_i] \in \mathcal{G}$. Then $Q_1^{k_1} \cdots Q_n^{k_n}(r_1 D) = P_1^{l_1} \cdots P_m^{l_m}(r_2 D)$. Since $\mathcal{G} \bigcap \mathcal{H} = \emptyset$, $r_1 D \subseteq P_1^{l_1} \cdots P_m^{l_m} = s'D$ by the unique factorization of prime ideals in the Dedekind domain D. Thus $w \in ts'D$; so we have equality. Hence $S[\mathcal{G}]$ is a splitting multiplicative set of D.

(3) Since principal primes play no role in determining $\rho(D)$ [8, Theorem 4.1], we have $\rho(D) = max\{\rho(D_{S[\mathcal{G}]}), \rho(D_{S[\mathcal{H}]})\}$ by [8, Theorem 2.3(2)].

The "in particular" statement follows since $G \bigcap H = \{0\}$ implies that $\langle\mathcal{I}(D) - S[\mathcal{G}]\rangle = S[\mathcal{H}]$. Thus $S[\mathcal{H}]$ is the m-complement splitting set for $S[\mathcal{G}]$. □

Let G be an abelian group and $C \subseteq G$. We say that C is an *independent set* in G if $n_1 c_1 + \cdots + n_l c_l = 0$, $n_i \in \mathbb{Z}$, distinct $c_i \in C$, implies that each $n_i c_i = 0$.

Corollary 6. *Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$. If $\mathcal{G} \subseteq \mathcal{A}$ is independent and each element of $\mathcal{G}$ is torsionfree, then $\mathcal{G}$ is not subrealizable; so $G = \langle \mathcal{G} \rangle$ is not an $\mathcal{A}$-summand of $Cl(D)$.*

Proof. Let $\mathcal{G}$ be an independent set of $\mathcal{A}$. Suppose that $\mathcal{G}$ is subrealizable. Then there exists $x \in \mathcal{I}(D)$ with $xD = P_1^{k_1} \cdots P_n^{k_n}$, where $[P_1], ..., [P_n] \in \mathcal{G}$ are distinct; so $k_1[P_1] + \cdots + k_n[P_n] = 0$. Thus $k_1 = \cdots = k_n = 0$, a contradiction. Hence there are no irreducible elements coming from $\mathcal{G}$. Thus, by Theorem 5(1), $G = \langle \mathcal{G} \rangle$ is not an $\mathcal{A}$-summand. □

Combining Theorem 2 and Theorem 5, we have

Theorem 7. *Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$ and let S be a splitting multiplicative subset of D. Then there is a partition $\mathcal{A} = \mathcal{G} \bigcup \mathcal{H}$ of $\mathcal{A}$ such that S is generated by $S[\mathcal{G}]$ with some set of principal primes, and the m-complement set T for S is generated by $S[\mathcal{H}]$ with the set of all principal primes not in S. In particular, if D has no principal primes, then there is a one-to-one correspondence between the set of splitting multiplicative subsets of D and the set of $\mathcal{A}$-direct summands of $Cl(D)$.*

Corollary 8. *([4, Corollary 3.8]) Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$. Let G(resp., H) be a subgroup of $Cl(D)$ generated by a subset $\mathcal{G}$(resp., $\mathcal{H}$) of $\mathcal{A}$ and let $Cl(D) = G \bigoplus_{\mathcal{A}} H$. Then*

(1) $Cl(D)/G \cong Cl(D_{S[\mathcal{G}]})$ *and* $Cl(D)/H \cong Cl(D_{S[\mathcal{H}]})$.
(2) $Cl(D) \cong Cl(D_{S[\mathcal{G}]}) \bigoplus Cl(D_{S[\mathcal{H}]})$.

Proof. (1) Note that $ker(Cl(D) \to Cl(D_{S[\mathcal{G}]})) = G$ and $ker(Cl(D) \to Cl(D_{S[\mathcal{H}]})) = H$ by Nagata's Theorem [10, Corollary 7.2]; so (1) holds.
(2) This is clear from $Cl(D)/G \cong H$ and $Cl(D)/H \cong G$. □

Remark 9. Let D_i be a Dedekind domain with realizable pair $\{G_i, \mathcal{A}_i\}$, $i = 1, 2$, and let $0 \to G_1 \to G \to G_2 \to 0$ be a split short exact sequence. Then there exists a Dedekind domain D such that $Cl(D) = G$ (this is also known by Claborn's Theorem [10, Theorem 14.10]). To see this, let $\mathcal{A} = \{(\pm g, 0), (0, \pm h) | g \in \mathcal{A}_1, h \in \mathcal{A}_2\}$. By [12, Theorem 1.4], $\{G, \mathcal{A}\}$ is a realizable pair. Thus there exists a

Dedekind domain D such that $Cl(D) = G$. Let $\mathcal{G}_1 = \{(\pm g, 0) | g \in \mathcal{A}_1\}$, $\mathcal{G}_2 = \{(0, \pm h) | h \in \mathcal{A}_2\}$. Then $\mathcal{A} = \mathcal{G}_1 \bigcup \mathcal{G}_2$ is a partition of $\mathcal{A}$. By Theorem 5, $S[\mathcal{G}_1]$ and $S[\mathcal{G}_2]$ are each splitting multiplicative subsets of D, and if D has no principal primes, then $S[\mathcal{G}_2]$ is the m-complement set for $S[\mathcal{G}_1]$. In particular, if G_1 and G_2 are torsion abelian groups, then we may take $\mathcal{A} = \{(g,0),(0,h) | g \in \mathcal{A}_1, h \in \mathcal{A}_2\}$ since $\{G, \mathcal{A}\}$ is realizable if and only if $\mathcal{A}$ generates G [12, Corollary 1.5]. In this case, $\rho(D) = max\{\rho(D_1), \rho(D_2)\}$ (cf., [8, Theorem 3.1]).

The following corollary generalizes [7, Theorem 2.5].

COROLLARY 10. *Let D be a Dedekind domain with realizable pair $\{Cl(D), \mathcal{A}\}$. Suppose that $Cl(D) = \langle \mathcal{G} \rangle \bigoplus_{\mathcal{A}} \langle \mathcal{H} \rangle$ and $\mathcal{G}$ is an independent subset of $\mathcal{A}$. Then*

(1) *$\langle \mathcal{G} \rangle$ is a torsion subgroup of $Cl(D)$.*
(2) *$S[\mathcal{G}]$ is a splitting multiplicative set generated by an HF-set.*
(3) *$\rho(D) = \rho(D_{S[\mathcal{G}]})$.*

In particular, if $\mathcal{A}$ is independent, then $Cl(D)$ is torsion and D is an HFD.

Proof. (1) Let $g = [P]$ be a torsion free element in $\mathcal{G}$. For irreducible $x \in P$, we write $xD = P^n Q_1^{n_1} \cdots Q_k^{n_k}$, where $[P], [Q_1], ..., [Q_k]$ are distinct and $n_i \in \mathbb{Z}^+$. Assume that each $[Q_1], ..., [Q_j] \in \mathcal{G}$ and each $[Q_{j+1}], ..., [Q_k] \in \mathcal{H}$. Now, $n[P] + n_1[Q_1] + \cdots + n_j[Q_j] = -n_{j+1}[Q_{j+1}] - \cdots - n_k[Q_k] = 0$ since $\langle \mathcal{G} \rangle \bigcap \langle \mathcal{H} \rangle = \{0\}$. Since $\mathcal{G}$ is independent, $n[P] = 0$, and hence $n = 0$, a contradiction. Thus $\langle \mathcal{G} \rangle$ is a torsion subgroup of $Cl(D)$.

(2) By Theorem 5(2), $S[\mathcal{G}]$ is a splitting multiplicative set of D. Let $x_1 \cdots x_m = y_1 \cdots y_n$ with each $x_i, y_j \in D$ irreducible, but not prime. If the splitting multiplicative set $S[\mathcal{G}]$ contains exactly $x_1, ..., x_i$ and $y_1, ..., y_j$, then $x_1 \cdots x_i D = y_1 \cdots y_j D$ by [3, Corollary 1.4]. Since $\mathcal{G}$ is independent, each irreducible r coming from the ideal classes in $\mathcal{G}$ is of the form $rD = P_1 \cdots P_n$, where $[P_1] = \cdots = [P_n]$ and $n = |[P_i]|$ by (1). Hence $\{x \in \mathcal{I}(D) | xD = P_1 \cdots P_n, \text{each } [P_i] \in \mathcal{G}\}$ is an HF-set (cf., [7, Theorem 2.5]).

(3) It follows directly from [8, Theorem 2.10].

The "in particular" statement now follows since $\mathcal{A} = \mathcal{A} \bigcup \emptyset$ and $S[\mathcal{A}]$ is the multiplicative set generated by the set of all irreducible, but not prime, elements of D. □

We close with a corollary.

COROLLARY 11. ([9, Theorem 3.8]) *Let D be a Dedekind domain with torsion divisor class group $\{Cl(D), \mathcal{A}\}$. Then the following statements are equivalent.*

(1) *$\mathcal{A}$ is independent.*
(2) *For each subset $\mathcal{B}$ of $\mathcal{A}$, $S[\mathcal{B}]$ is a splitting multiplicative set.*
(3) *For each nonprincipal prime ideal P of D, $S[P]$ is a splitting multiplicative set.*
(4) *For each nonprincipal prime ideal P of D, $\mathcal{H}_{[P]}$ is an HF-set.*
(5) *For each $\emptyset \neq \mathcal{C} \subseteq \mathcal{A}$, $\bigcup_{[P]\in\mathcal{C}} \mathcal{H}_{[P]}$ is an HF-set.*
(6) *For each nonprincipal prime ideal P of D, $\mathcal{H}_{[P]} = S[P] \bigcap \mathcal{I}(D)$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $\mathcal{A}$ is independent. Since $Cl(D)$ is torsion, each subset $\mathcal{B}$ of $\mathcal{A}$ is subrealizable and $\mathcal{A} = \mathcal{B} \bigcup (\mathcal{A} - \mathcal{B})$ is a partition of $\mathcal{A}$. Now, we may assume that $\emptyset \neq \mathcal{A} - \mathcal{B} \subset \mathcal{A}$. Let G (resp., H) be a subgroup of $Cl(D)$ generated by $\mathcal{B}$ (resp., $\mathcal{A} - \mathcal{B}$). We show that $Cl(D) = G \bigoplus H$. Select $g \in Cl(D)$. Since $Cl(D)$ is a torsion abelian group, $\mathcal{A}$ generates $Cl(D)$ as a monoid; so $g = [P_1] + \cdots + [P_n]$, where each $[P_i] \in \mathcal{A}$. Since $\mathcal{B} \bigcup (\mathcal{A} - \mathcal{B})$ is a partition of $\mathcal{A}$, we have $g \in G + H$. Let $g \in G \bigcap H$. Thus $g = [P_1] + \cdots + [P_k] = [Q_1] + \cdots + [Q_l]$, where each $[P_i] \in \mathcal{B}$ and each $[Q_j] \in \mathcal{A} - \mathcal{B}$. This implies that $[P_1] + \cdots + [P_k] - [Q_1] - \cdots - [Q_l] = 0$. Since $\mathcal{A}$ is independent, we have that $g = 0$. By Theorem 5, $S[\mathcal{B}]$ is a splitting multiplicative set. (2) $\Rightarrow$ (3) and (5) $\Rightarrow$ (4) are both clear. (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (1), and (4) $\Leftrightarrow$ (6) follow from [9, Theorem 3.8], while (4) $\Rightarrow$ (5) follows from [6, Corollary 3.11]. □

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