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A q-ANALOGUE OF w-BERNOULLI NUMBERS AND THEIR APPLICATIONS

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ABSTRACT. In this paper, we consider that the q-analogue of w-Bernoulli numbers $\mathcal{B}_i(w,q)$. And we calculate the sums of products of two q-analogue of w-Bernoulli numbers $\mathcal{B}_i(w,q)$ in complex cases. From this result, we obtain the Euler type formulas of the Carlitz's q-Bernoulli numbers $\beta_i(q)$ and q-Bernoulli numbers $\mathcal{B}_i(q)$. And we also calculate the p-adic Stirling type series by the definition of $\mathcal{B}_i(w,q)$ in p-adic cases.

1. Introduction

L. Carlitz [1] considered q-Bernoulli number β_n as ultra numbers of Bernoulli numbers as follow

$$\beta_0 = 1, \quad q(q\beta+1)^n - \beta_n = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1. \end{cases}$$

Also the w-Bernoulli number $B_n(w)$ ([4]) are defined by

$$e^{B(w)t} = \frac{t}{we^t - 1}.$$

Thus $B_n(w)$ can be determined inductively by

$$B_0(w) = 1, \quad w(B(w)+1)^n - B_n(w) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1. \end{cases}$$

Let $w, q \in \mathbb{C}$ with $|w| < 1$ and $|q| < 1$. Then the q-analogue of w-Bernoulli numbers $\mathcal{B}_n(w;q)$ ([7]) are defined as follow

$$\mathcal{B}_0(w;q) = 1 \qquad w(q\mathcal{B}(w;q)+1)^n - \mathcal{B}_n(w;q) = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

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with the usual convention of replacing $\mathcal{B}^i(w;q)$ by $\mathcal{B}_i(w;q)$. If $q \to 1$ then $\mathcal{B}_i(w;1) = B_i(w)$ where $B_i(w)$ is w-Bernoulli numbers.

We know that the generating function $\widetilde{G}_{w,q}$ of $\mathcal{B}_n(w;q)$ ([7]) by

$$\begin{aligned}\widetilde{G}_{w,q} &= \sum_{n=0}^{\infty} \mathcal{B}_n(w;q)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} w^n e^{[n]_q t}(1 - w - q^n t). \qquad (1)\end{aligned}$$

We find that the following holds (see Theorem 1) as a q-analogue of Lemma 5 in Section 2

$$[\widetilde{G}_{w,q}, \varphi(\widetilde{G}_{w,q})] = \widetilde{G}_{w,q} - [(\widetilde{G}_{w,q} + q\widetilde{G}'_{w,q}), \phi(t)],$$

where $[,]$, φ and ϕ are defined in Section 2. By comparing the coefficient $t^n/n!$ in the above equation for $n \geq 1$, we find that the sums of products of two q-analogue of w-Bernoulli numbers $\mathcal{B}_n(w;q)$ (see Corollary 3).

And also, we know [7] for $k \geq 2$,

$$-\frac{\mathcal{B}_k(w;q)}{k} = -\left(\frac{1}{k} - \frac{1-q}{1-w}\right) H_k(w^{-1}, q) + \frac{1}{1-w} H_{k-1}(w^{-1}, q), \qquad (2)$$

where $H_k(w^{-1}, q)$ are Carlitz's q-Euler numbers.

We apply p-adic case to the formula (2). In p-adic case, we assume that q be an element of $\mathbb{C}_p$ with $|1-q|_p < p^{\frac{1}{p-1}}$ and u be an element of $\mathbb{C}_p$ with $|1-u|_p \geq 1$ where $\mathbb{C}_p$ is the algebraic closure of $\mathbb{Q}_p$. We can calculate the p-adic Stirling type series (see Proposition 2).

2. Euler formula for $\mathcal{B}_n(w;q)$

We shall find the Euler formula of the w-Bernoulli numbers and explain a method to construct a q-analogue of power series for w-Bernoulli numbers.

Let $w \in \mathbb{C}$ with $|w| < 1$ and let

$$F_w = \frac{1}{we^t - 1} \quad \text{and} \quad G_w = F_w \cdot t$$

then G_w is the generating function of w-Bernoulli numbers $B_n(w)$, i.e.,

$$G_w = \sum_{n=0}^{\infty} B_n(w)\frac{t^n}{n!}.$$

The derivative G'_w of G_w is equal to $F_w - G_w - F_w \cdot G_w$, so we have

$$G_w^2 = G_w - (G_w + G'_w)t. \tag{3}$$

This equation is equivalent to

$$(B(w) + B(w))^n = B_n(w) - n(B_{n-1}(w) + B_n(w)) \quad \text{for} \quad n \geq 1 \tag{4}$$

where we use the usual convention of replacing $B^i(w)$ by $B_i(w)$.

For any A and $B \in \mathbb{C}[[t]]$ which have a_n and $b_n \in \mathbb{C}$ as coefficients of $t^n/n!$ respectively, the operation $[,]$ (see [6]) defined as follow

$$[A, B] = \sum_{n=0}^{\infty} A^{(n)} B^{(n)} (q-1)^n \frac{t^n}{n!}$$

where $A^{(n)}$ is the n-th derivative of A. We use the notation

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

For any a and $b \in \mathbb{C}$, we have

$$\Psi[e^{[a]_q t}, e^{[b]_q t}] = e^{[a+b]_q t}. \tag{5}$$

We define an operator φ, ϕ on $\mathbb{C}[[t]]$ similarly as in ([5]) as follows

$$\varphi = 1_{id} + (q-1)\frac{d}{dt},$$

$$\phi = 1_{id} + (w-1)\frac{d}{dt}.$$

LEMMA 1. *For any $A = e^{at}, B = e^{bt} \in \mathbb{C}[[t]]$ with $a, b \in \mathbb{C}$, we have*

$$[A, B]' = [A', B] + [A, B'] + (q-1)[A', B'].$$

Proof. This is an easy calculation by the definition of the operator φ. □

Let

$$R_w = \left\{ \sum_{n=0}^{\infty} a_n w^n X^n | a_n, w \in \mathbb{C} \text{ with } |w| < 1 \right.$$
$$\left. \text{and } \sum_{n=0}^{\infty} a_n w^n \text{ is absolutely convergent} \right\}.$$

Then we define the q-operator from $R_w[t]$ to $\mathbb{C}[[t]]$ (cf. [5]) as follow

$$\begin{array}{ccc} R_w[t] & \hookrightarrow & \mathbb{C}[[t]] \\ t \cdot w^n X^n & \longmapsto & [t, w^n e^{[n]_q t}]. \end{array}$$

For $A_w \in R[t]$, $A_{w,q}$ denotes the image of A_w under the q-operator.

LEMMA 2.
(1) $(F'_w)_q = [q + (q-1)F_{w,q}, F'_{w,q}]$.
(2) $qF'_{w,q} = -(F_{w,q} + [F_{w,q}, \varphi(F_{w,q})])$.
(3) $F_{w,q} = [q + (q-1)F_{w,q}, \varphi(F_{w,q})]$.

Proof. (1) Since $F_{w,q} = -\sum_{n=0}^{\infty} w^n e^{[n]_q t}$ by the equation (5), we have

$$\begin{aligned}
&[q + (q-1)F_{w,q}, F'_{w,q}] \\
=& -q \sum_{n=0}^{\infty} w^n [n]_q e^{[n]_q t} + (q-1)[\sum_{n=0}^{\infty} w^n e^{[n]_q t}, \sum_{n=0}^{\infty} w^n [n]_q e^{[n]_q t}] \\
=& -q \sum_{n=0}^{\infty} w^n [n]_q e^{[n]_q t} + (q-1) \sum_{n=0}^{\infty} w^n e^{[n]_q t} \sum_{m=0}^{n} [m]_q \\
=& -\sum_{n=0}^{\infty} w^n n e^{[n]_q t} \\
=& (F'_w)_q.
\end{aligned}$$

(2) Apply the q-operator to both side of the formula $F'_w = -(F_w + F_w^2)$ and by Lemma 2-(1), we have

$$-(F_{w,q} + [F_{w,q}, F_{w,q}]) = (F'_w)_q = [q + (q-1)F_{w,q}, F'_{w,q}].$$

Thus (2) holds by the above equation.
(3) Multiply $(q-1)$ to both side of Lemma 2-(2)

$$\begin{aligned}F_{w,q} &= qF_{w,q} + (q-1)qF'_{w,q} + [(q-1)F_{w,q}, \varphi(F_{w,q})] \\ &= q\varphi(F_{w,q}) + [(q-1)F_{w,q}, \varphi(F_{w,q})] \\ &= [q + (q-1)F_{w,q}, \varphi(F_{w,q})].\end{aligned}$$

□

From the definition of $[,]$, we have

$$\begin{aligned}[[A,B],t] =& [A,B]t + (q-1)[A,B]'t \\ =& t\sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^n \frac{t^n}{n!} \\ &+ (q-1)t\left\{ \sum_{n=0}^{\infty} A^{(n+1)}B^{(n)}(q-1)^n \frac{t^n}{n!} \right. \\ &\qquad + \sum_{n=0}^{\infty} A^{(n)}B^{(n+1)}(q-1)^n \frac{t^n}{n!} \\ &\qquad \left. + \sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^n \frac{nt^{n-1}}{n!} \right\} \\ =& \sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^n \frac{t^{n+1}}{n!} \\ &+ \sum_{n=0}^{\infty} A^{(n+1)}B^{(n)}(q-1)^{n+1} \frac{t^{n+1}}{n!} \\ &+ \sum_{n=0}^{\infty} A^{(n)}B^{(n+1)}(q-1)^{n+1} \frac{t^{n+1}}{n!} \\ &+ \sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^{n+1} \frac{nt^{n}}{n!}.\end{aligned}$$

Using the equation $(Bt)^{(n)} = B^{(n)}t + nB^{(n-1)},\ n = 1, 2, \cdots$, we have

$$\begin{aligned}[A,[B,t]] &= [A, Bt + (q-1)B't] \\ &= [A, Bt] + (q-1)[A, B't] \\ &= \sum_{n=0}^{\infty} A^{(n)}(Bt)^{(n)}(q-1)^n \frac{t^n}{n!}\end{aligned}$$

$$+ (q-1)\sum_{n=0}^{\infty} A^{(n)}(B't)^{(n)}(q-1)^n \frac{t^n}{n!}$$

$$= ABt + \sum_{n=1}^{\infty} A^{(n)}(B^{(n)}t + nB^{(n-1)})(q-1)^n \frac{t^n}{n!}$$

$$+ \sum_{n=0}^{\infty} A(n)(B^{(n+1)}t + nB^{(n)})(q-1)^{n+1} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^n \frac{t^{n+1}}{n!}$$

$$+ \sum_{n=1}^{\infty} A^{(n)}B^{(n-1)}(q-1)^n \frac{t^n}{(n-1)!}$$

$$+ \sum_{n=0}^{\infty} A^{(n)}B^{(n+1)}(q-1)^{n+1} \frac{t^{n+1}}{n!}$$

$$+ \sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^{n+1} \frac{nt^n}{n!}.$$

Thus we have

$$[[A,B],t] = [A,[B,t]].$$

Using above formula, we have

$$[[A,B],t] = [A,[B,t]] = [A,[t,B]] = [[A,t],B].$$

Therefore we have the following lemma.

LEMMA 3. *For any $A, B \in C[[t]]$, we have*

(1) $[[A,B],t] = [A,[B,t]]$.
(2) $[[A,B],t] = [[A,t],B]$.

LEMMA 4.

$$(G'_w)_q = [q + (q-1)F_{w,q}, G'_{w,q}].$$

Proof. Apply the q-analogue to both side of the formula $G'_w = F'_w \cdot t + F_w$ and Lemma 2-(1), (3) and Lemma 3-(1), then we have

$$\begin{aligned}(G'_w)_q &= [(F'_w)_q, t] + F_{w,q} \\ &= [q + (q-1)F_{w,q}, [F'_{w,q}, t]] + [q + (q-1)F_{w,q}, \varphi(F_{w,q})] \\ &= [q + (q-1)F_{w,q}, [F'_{w,q}, t] + \varphi(F_{w,q})].\end{aligned}$$

And also, if we apply Lemma 1 to $G_{w,q} = [F_{w,q}, t]$, then we have

$$\begin{aligned} G'_{w,q} &= [F_{w,q}, t]' \\ &= [F'_{w,q}, t] + F_{w,q} + (q-1)F'_{w,q} \\ &= [F'_{w,q}, t] + \varphi(F_{w,q}). \end{aligned}$$

Thus Lemma 4 holds. □

PROPOSITION 1.

$$[G_{w,q}, \varphi(G_{w,q})] = G_{w,q} - [G_{w,q} + qG'_{w,q}, t].$$

Proof. Apply the q-operator to the equation (3) and Lemma 3-(2) and Lemma 4 then we have

$$\begin{aligned} [G_{w,q}, G_{w,q}] &= G_{w,q} - [(G_{w,q} + (G'_w)_q), t] \\ &= G_{w,q} - [G_{w,q} + [q + (q-1)F_{w,q}, G'_{w,q}], t] \\ &= G_{w,q} - [G_{w,q} + qG'_{w,q}, t] - (q-1)[G_{w,q}, G'_{w,q}]. \end{aligned}$$ □

From Proposition 1, we obtain the Euler type formula of $\mathcal{B}_n(q) = \mathcal{B}_n(1;q)$ as $w \to 1$ since $G_{1,q}$ is generating function of $\mathcal{B}_n(q)$ where

$$\mathcal{B}_0(q) = 1, \quad (q\mathcal{B}(q) + 1)^n - \mathcal{B}_n(q) = \begin{cases} 1 & \text{if} \quad n = 1, \\ 0 & \text{if} \quad n > 1. \end{cases}$$

COROLLARY 1. *We put* $\overline{\mathcal{B}}_k(q) = \mathcal{B}_k(q) + (q-1)\mathcal{B}_{k+1}(q)$ *for each* $k \geq 0$. *For* $n \geq 1$, *we have*

$$\begin{aligned} &\sum_{i=0}^{n} \sum_{k=0}^{i} \binom{n}{i} \binom{i}{k} (q-1)^k \mathcal{B}_i(q) \overline{\mathcal{B}}_{k+n-i}(q) \\ = \;& \mathcal{B}_n(q) - n(\mathcal{B}_{n-1}(q) + q\mathcal{B}_n(q)) - (q-1)n(\mathcal{B}_n(q) + q\mathcal{B}_{n+1}(q)). \end{aligned}$$

Proof. We apply the definition of the operation [,] and denote $m(m-1)\cdots(m+n-1)$ by $(m)_n$, then we have

$$\begin{aligned}&\sum_{n=0}^{\infty} A^{(n)}B^{(n)}(q-1)^n\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\sum_{i=0}^{\infty} a_i\frac{(i)_n}{i!}t^{i-n}\cdot\sum_{j=0}^{\infty} b_j\frac{(j)_n}{j!}t^{j-n}(q-1)^n\frac{t^n}{n!}\\ =&\sum_{\substack{n=0\\ n=i+j-k}}^{\infty}\frac{t^n}{n!}\sum_{i=0}^{n}\binom{n}{i}a_i\sum_{k=0}^{i}\binom{i}{k}(q-1)^k b_{k+n-i}.\end{aligned}$$

Compare the coefficient of both sides in Proposition 1 and $w\to 1$, then the corollary holds. □

Now we give the sums products of two q-analogue of w-Bernoulli numbers $\mathcal{B}_n(w;q)$. Let

$$\widetilde{G}_w=(w-1)F_w+G_w=F_w\cdot\phi(t), \tag{6}$$

then the q-analogue of the equation (6) as follow

$$\widetilde{G}_{w,q}=[F_{w,q},\phi(t)]=\sum_{n=0}^{\infty} w^n e^{[n]_q t}(1-w-q^n t).$$

$\widetilde{G}_{w,q}$ is generating function of $\mathcal{B}_n(w;q)$. The next lemma is an Euler type formula of $\widetilde{G}_w$.

LEMMA 5.
$$(\widetilde{G}_w)^2=\widetilde{G}_w-(\widetilde{G}_w+\widetilde{G}'_w)\cdot\phi(t).$$

Proof. Using the equation

$$\widetilde{G}'_w=(w-1)F'_w+G'_w=(w-1)F'_w+F'_w t+F_w, \tag{7}$$

we have

$$\begin{aligned}(\widetilde{G}_w)^2 =&(F_w\cdot\phi(t))^2\\ =&(w-1)^2F_w^2+2(w-1)F_w\cdot G_w+G_w^2\\ =&(w-1)^2(-F_w-F'_w)+2(w-1)(-G_w-F'_w t)\\ &+(-G_w t-F'_w t^2)\\ =&\widetilde{G}_w-(w-1)\widetilde{G}_w-(w-1)\widetilde{G}'_w-\widetilde{G}_w t-\widetilde{G}'_w t.\end{aligned}$$ □

The next lemma is the q- analogue of the equation (7).

LEMMA 6.
$$(\widetilde{G}'_w)_q = [q + (q-1)F_{w,q}, \widetilde{G}'_{w,q}].$$

Proof. Apply the q-operator to the equation (7) and Lemma 2-(1), (3), then we have

$$\begin{aligned}(\widetilde{G}'_w)_q &= (w-1)(F'_w)_q + (G'_w)_q \\ &= (w-1)[q + (q-1)F_{w,q}, F'_{w,q}] + [q + (q-1)F_{w,q}, G'_{w,q}]. \end{aligned}$$ □

THEOREM 1.

$$[\widetilde{G}_{w,q}, \varphi(\widetilde{G}_{w,q})] = \widetilde{G}_{w,q} - [\widetilde{G}_{w,q} + q\widetilde{G}'_{w,q}, \phi(t)].$$

Proof. Apply the q-operator to the Lemma 5 and Lemma 3-(2) and Lemma 6, then we have

$$\begin{aligned}[\widetilde{G}_{w,q}, \widetilde{G}_{w,q}] &= \widetilde{G}_{w,q} - [\widetilde{G}_{w,q} + (\widetilde{G}'_w)_q, \phi(t)] \\ &= \widetilde{G}_{w,q} - [\widetilde{G}_{w,q} + q\widetilde{G}'_{w,q}, \phi(t)] - (q-1)[\widetilde{G}_{w,q}, \widetilde{G}'_{w,q}]. \end{aligned}$$ □

We obtain some corollaries from Theorem 1.

COROLLARY 2. (see [5])

$$\Psi[\overline{G}_q, \varphi(\overline{G}_q)] = \overline{G}_q - \Psi[\overline{G}_q + q\overline{G}'_q, \varphi(t)],$$

where $\overline{G}_q$ is the generating function of ordinary Carlitz's q-Bernoulli numbers.

Proof. Let $w = q$ in Theorem 1, then the corollary holds. □

We have the sums products of two q-analogue of w-Bernoulli numbers $\mathcal{B}_n(w;q)$ as follow.

COROLLARY 3. *We put* $\overline{\mathcal{B}}_k(w;q) = \mathcal{B}_k(w;q) + (q-1)\mathcal{B}_{k+1}(w;q)$ *for each* $k \geq 0$. *For* $n \geq 1$, *we have*

$$\begin{aligned}&\sum_{i=0}^{n}\sum_{k=0}^{i}\binom{n}{i}\binom{i}{k}(q-1)^k\mathcal{B}_i(w;q)\overline{\mathcal{B}}_{k+n-i}(w;q)\\ &= \mathcal{B}_n(w;q) - n(\mathcal{B}_{n-1}(w;q) + q\mathcal{B}_n(w;q))\\ &\quad - (q-1)n(\mathcal{B}_n(w;q) + q\mathcal{B}_{n+1}(w;q))\\ &\quad - (w-1)(\mathcal{B}_n(w;q) + q\mathcal{B}_{n+1}(w;q)).\end{aligned}$$

Proof. It is an easy calculation by the same method of Corollary 1.□

From Corollary 3, we have the Euler type formula of ordinary w-Bernoulli numbers $B_i(w)$ if $q \to 1$ (Carlitz's q-Bernoulli numbers $\beta_i(q)$ if $w = q$).

3. Applications of $\mathcal{B}_n(w;q)$

In p-adic case, we assume that q be an element of $\mathbb{C}_p$ with $|1-q|_p < p^{\frac{1}{p-1}}$ and u be an element of $\mathbb{C}_p$ with $|1-u|_p \geq 1$. Let $a, N \in \mathbb{Z}$ with $0 \leq a \leq p^N - 1$ and $N \geq 0$. Then the measure is defined on $\mathbb{Z}_p$ by

$$E_u(a + p^N\mathbb{Z}_p) = \frac{u^{p^N - a}}{1 - u^{p^N}},$$

which is E_u is Koblitz's measure ([2], [3]). We have in (see [3]) that

$$\int_{\mathbb{Z}_p}[a]_q^k dE_u(a) = \frac{u}{1-u}H_k(u,q).$$

We apply p-adic case to the formula (2). We define a (u,q)-Bernoulli number $\mathcal{B}_m(u^{-1};q) \in \mathbb{C}_p$ by making use of this integral:

$$\begin{aligned}\mathcal{B}_0(u^{-1};q) &= \int_{\mathbb{Z}_p}(u^{-1}-1)dE_u(a) = 1,\\ \mathcal{B}_m(u^{-1};q) &= \int_{\mathbb{Z}_p}\{[a]_q^m(u^{-1}-1) + mq^a[a]_q^{m-1}\}dE_u(a),\end{aligned}$$

where $a, N \in \mathbb{Z}$ with $0 \le a \le p^N - 1$ and $N \ge 0$. The generating function $\widetilde{G}_{u^{-1},q}$ of $\mathcal{B}_m(u^{-1};q)$

$$\widetilde{G}_{u^{-1},q} = \sum_{m=0}^{\infty} \mathcal{B}_m(u^{-1};q)\frac{t^m}{m!}$$

is given by

$$\widetilde{G}_{u^{-1},q}(t) = -\lim_{N\to\infty} \frac{u^{p^N}}{1-u^{p^N}} \sum_{a=0}^{p^N-1} u^{-a} e^{[a]_q t}(1 - u^{-1} - q^a t),$$

which satisfies the (u,q)-difference equation

$$\widetilde{G}_{u^{-1},q}(t) = u^{-1}e^t \widetilde{G}_{u^{-1},q}(qt) + 1 - u^{-1} - t.$$

The (u,q)-Bernoulli polynomials in the variable $z \in \mathbb{Z}_p$, we define

$$\mathcal{B}_m(z;u^{-1},q) = \int_{\mathbb{Z}_p} \{(u^{-1}-1)[a+z]_q^m + mq^{a+z}[a+z]_q^{m-1}\} dE_u(a).$$

These can be written as

$$\mathcal{B}_m(z;u^{-1},q) = (q^z\mathcal{B}(u^{-1};q) + [z]_q)^m.$$

Indeed, we have

$$\begin{aligned}
\mathcal{B}_m(z;u^{-1},q) &= \int_{\mathbb{Z}_p} \left\{(u^{-1}-1)[a+z]_q^m + mq^{a+z}[a+z]_q^{m-1}\right\} dE_u(a)\\
&= \sum_{k=0}^{m} \binom{m}{k} [z]_q^{m-k} q^{zk} \int_{\mathbb{Z}_p} [a]_q^k (u^{-1}-1) dE_u(a)\\
&\quad + \sum_{k=0}^{m-1} \binom{m-1}{k} [z]_q^{m-1-k} q^{(k+1)z} m \int_{\mathbb{Z}_p} q^a [a]_q^k dE_u(a)\\
&= \sum_{k=0}^{m} \binom{m}{k} [z]_q^{m-k} q^{kz} \int_{\mathbb{Z}_p} \{[a]_q^k (u^{-1}-1)\\
&\quad + kq^a[a]_q^{k-1}\} dE_u(a)\\
&= \sum_{k=0}^{m} \binom{m}{k} [z]_q^{m-k} q^{kz} \mathcal{B}_k(u^{-1};q)\\
&= (q^z\mathcal{B}(u^{-1};q) + [z]_q)^m.
\end{aligned}$$

Now, we define the locally holomorphic function $\mathcal{G}_q(x,z)$ for $z \in \mathbb{Z}_p$ and $x \in \mathbb{C}_p$ with $|[x]_p| > 1$ by

$$\mathcal{G}_q(x,z) = ([x]_q + [z]_q)\log([x]_q + [z]_q) - ([x]_q + [z]_q).$$

We have

$$\mathcal{G}_q(x,z) = -[x]_q + [x]_q \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)} \frac{[z]_q^{n+1}}{[x]_q^{n+1}} + ([x]_q + [z]_q)\log[x]_q$$

and

$$\begin{aligned}\mathcal{G}_q'(x,z) &= \frac{\partial}{\partial x}\mathcal{G}_q(x,z) \\ &= -\frac{q^x}{1-q}\log q \left\{ \log[x]_q + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{[z]_q^n}{[x]_q^n} \right\}.\end{aligned}$$

We consider the function $\mathcal{G}_{p,q,u}(x)$ as follow

$$\begin{aligned}\mathcal{G}_{p,q,u}(x) = -\int_{\mathbb{Z}_p} \Big\{ &(q[x]_q + [x]_q - 1)\mathcal{G}_q'(x,z) \\ &- \frac{q^x}{1-q}\log q \cdot (2 - q - u^{-1})\mathcal{G}_q(x,z) \\ &- q^x \log q \cdot ([x]_q + [z]_q) \Big\} dE_u(z).\end{aligned}$$

Thus we can find the formula of p-adic Stirling type series for $\mathcal{B}_n(u^{-1};q)$.

$$\begin{aligned}&(q[x]_q + [x]_q - 1)\mathcal{G}_q'(x,z) \\ &- \frac{q^x}{1-q}\log q \cdot (2 - q - u^{-1})\mathcal{G}_q(x,z) - q^x \log q \cdot ([x]_q + [z]_q) \\ =& -(q[x]_q + [x]_q - 1)\frac{q^x}{1-q}\log q \left\{ \log[x]_q + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{[z]_q^n}{[x]_q^n} \right\} \\ &- q^x \log q \cdot ([x]_q + [z]_q)\end{aligned}$$

$$
\begin{aligned}
&-\frac{q^x}{1-q}\log q\cdot(2-q-u^{-1})\Big\{-[x]_q+([x]_q+[z]_q)\log[x]_q \\
&\qquad\qquad +[x]_q\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\frac{[z]_q^{n+1}}{[x]_q^{n+1}} \\
&\qquad\qquad +([x]_q+[z]_q)\log[x]_q\Big\} \\
=&\ \frac{q^x}{1-q}\log q\Big\{\{(u^{-1}-1)[z]_q+q^z\}\log[x]_q \\
&\qquad +(u^{-1}-1)[x]_q\log[x]_q-(u^{-1}-1)[x]_q \\
&\qquad +\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\{(u^{-1}-1)[z]_q^{n+1} \\
&\qquad +q^z(n+1)[z]_q^n\}\frac{1}{[x]_q^n}\Big\}.
\end{aligned}
$$

Therefore we obtain the following

Proposition 2. *For $x\in\mathbb{C}_p$ with $|[x]_q|_p>1$, we have*

$$
\begin{aligned}
\mathcal{G}_{p,q,u}(x)=\frac{q^x}{1-q}\log q\Big\{&([x]_q+\mathcal{B}_1(u^{-1};q))\log[x]_q-[x]_q \\
&+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\mathcal{B}_{n+1}(u^{-1};q)\frac{1}{[x]_q^n}\Big\}.
\end{aligned}
$$

Remark 1. This formula resembles the formula of $G_{p,q,u}(x)$ ([3]), that is,

$$
\begin{aligned}
G_{p,q,u}(x)=\frac{u}{1-u}\Big\{&(x+H_1(u,q))\log x-x \\
&+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}H_{n+1}(u,q)\frac{1}{x^n}\Big\}.
\end{aligned}
$$

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