ON THE MINKOWSKI UNITS OF 2-PERIODIC KNOTS

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ABSTRACT. In this paper we give a relationship among the Minkowski units, for all odd prime number including \( \infty \), of 2-periodic knot in \( S^3 \), its factor knot, and the 2-component link consisting of the factor knot and the set of fixed points of the periodic action.

1. Introduction

A knot \( k \) in \( S^3 \) is called an \( n\)-periodic knot (\( n \geq 2 \)) if there exists a \( \mathbb{Z}_n \)-action on the pair \( (S^3, k) \) such that the fixed point set \( f \) of the action is homeomorphic to a 1-sphere in \( S^3 \) disjoint from the knot \( k \). It is well known that \( f \) is unknotted. Hence the quotient map \( p : S^3 \rightarrow S^3 / \mathbb{Z}_n \) is an \( n \)-fold cyclic branched covering branched over \( p(f) = f_* \) and \( p(k) = k_* \) is also a knot in the orbit space \( S^3 / \mathbb{Z}_n \cong S^3 \), which is called the factor knot of \( k \). Several relationships among the invariants of \( n \)-periodic knot \( k \), its factor knot \( k_* \), and the 2-component link \( \ell = k_* \cup f_* \) have been studied by many authors [2, 6, 7, 9, 10, 12].

The Minkowski unit for a tame knot was first defined by Goeritz for odd prime integers [1]. Such Minkowski units derived from knot diagrams are invariants of the linking pairing on the 2-fold branched covering space. In [11], Murasugi defined the Minkowski unit \( C_p(\ell) \) for an oriented tame link \( \ell \) by using his symmetric link matrix \( M \) [8] of \( \ell \) for any prime integer \( p \), including \( p = \infty \), which is a generalization of Goeritz’s, although the underlying quadratic form is quite different from the one used by Goeritz.

In section 2, we show that for any prime integer \( p \), including \( p = \infty \), the Minkowski unit \( C_p(H(L)) \) of the modified Goeritz matrix \( H(L) \) [13] associated to a regular diagram \( L \) of an oriented tame link \( \ell \) is also an

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invariant of the link type \( \ell \) and it is equal to the Minkowski unit \( C_p(\ell) \) of the link \( \ell \), as defined by Murasugi.

In section 3, for any odd prime integer \( p \), including \( \infty \), we give a relationship among the Minkowski units \( C_p(k) \) of a 2-periodic knot \( k \), its factor knot \( k_* \), and the link \( \ell = k_* \cup f_* \) together with \( |\Delta_{k_*}(-1)| \) and \( |\Delta_{\ell}(-1, -1)| \), where \( \Delta_{k_*}(t) \) and \( \Delta_{\ell}(t_1, t_2) \) are the Alexander polynomials of \( k_* \) and the 2-component link \( \ell = k_* \cup f_* \), respectively.

2. The Minkowski units of the modified Goeritz matrices

Let \( \ell \) be an oriented link in \( S^3 \) and let \( L \) be its oriented link diagram in the plane \( \mathbb{R}^2 \subset \mathbb{R}^3 = S^3 - \{\infty\} \). Colour the regions of \( \mathbb{R}^2 - L \) alternately black and white. Denote the white regions by \( X_0, X_1, \ldots, X_w \) (We always take the unbounded region to be white and denote it by \( X_0 \)). Let \( C(L) \) be the set of all crossings of \( L \). Assign an incidence number \( \eta(c) = \pm 1 \) to each crossing \( c \in C(L) \) and define a crossing \( c \in C(L) \) to be of type I or type II as indicated in Figure 1.

\[
\begin{align*}
\text{Figure 1} \\
\begin{array}{cccc}
\eta(c) = +1 & \eta(c) = -1 & \text{Type I} & \text{Type II} \\
\end{array}
\end{align*}
\]

Let \( S(L) \) denote the compact surface with boundary \( L \), which is built up out of discs and bands. Each disc lies in \( S^2 = \mathbb{R}^2 \cup \{\infty\} \) and is a closed black region less a small neighbourhood of each crossing. Each crossing gives a small half-twisted band. Let \( b_0(L) \) denote the number of the connected components of the surface \( S(L) \).

Let \( G(L) = (g_{ij})_{1 \leq i, j \leq w} \), where \( g_{ij} = - \sum_{c \in C_L(X_i, X_j)} \eta(c) \) for \( i \neq j \) and

\[
g_{ii} = \sum_{c \in C_L(X_i)} \eta(c), \text{ where } C_L(X_i, X_j) = \{ c \in C(L) | c \text{ is incident to} \}
\]

\( X_i \)
both $X_i$ and $X_j$} and $G_L(X_i) = \{c \in C(L) | c \text{ is incident to } X_i \}$. The symmetric integral matrix $G(L)$ is called Goeritz matrix of $\ell$ associated to $L[1,3]$.

Let $C_{II}(L) = \{c_1, c_2, \cdots, c_d \}$ denote the set of all crossings of type II in $L$ and let $A(L) = \text{diag}(-\eta(c_1), -\eta(c_2), \cdots, -\eta(c_d))$, the $d \times d$ diagonal matrix. Then Traldi [13] defined the modified Goeritz matrix $H(L)$ of $\ell$ associated to $L$ by $H(L) = G(L) \oplus A(L) \oplus B(L)$, where $B(L)$ denotes the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix.

Two integral matrices $H_1$ and $H_2$ are said to be equivalent if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

$T_1 : H \to UHU^t$, where $U$ is a unimodular integral matrix,

$T_2 : H \to H \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

If $L_1$ and $L_2$ are link diagrams of ambient isotopic oriented links, then the modified Goeritz matrices $H(L_1)$ and $H(L_2)$ are equivalent. The signature $\sigma(\ell)$ and the nullity $n(\ell)$ of an oriented link $\ell$ in $S^3$ are given by the formulas: $\sigma(\ell) = \sigma(H(L))$, $n(\ell) = n(H(L)) + 1$, where $\sigma(H(L))$ and $n(H(L))$ are the signature and the nullity of the matrix $H(L)$, respectively [13]. The absolute value of the determinant, $\text{det}(H(L))$, of the modified Goeritz matrix $H(L)$ associated to a diagram $L$ of a link $\ell$ is clearly an invariant of the link type $\ell$. Let $\Delta_k(t)$ denote the Alexander polynomial of a knot $k$. Then it is well known that $|\Delta_k(-1)| = |\text{det}(G(K))| = |\text{det}(H(K))|$ for any diagram $K$ of the knot $k$.

Now two symmetric rational matrices $A_1$ and $A_2$ are said to be $R$-equivalent if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

$Q_1 : A \to RAR^t$, where $R$ is a nonsingular rational matrix,

$Q_2 : A \to A \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Any $n \times n$ nonzero symmetric rational matrix $A$ can be transformed by $Q_1$ into a matrix of the form:

$\begin{pmatrix} B & 0 \\ 0 & O \end{pmatrix}$,

where $B$ is a nonsingular matrix. In particular, if $A$ is a symmetric integral matrix, then $A$ may be transformed by $T_1$ into the same form. The matrix $B$ is called a nonsingular matrix associated to $A$.

Let $A$ be an $n \times n$ symmetric integral matrix of rank $r$ and $B$ a nonsingular integral matrix associated to $A$. Then there is a sequence
$B_1, B_2, \cdots, B_r$, called the $\sigma$-series of $A$, of principal minors of $B$ such

that

(1) $B_i$ is of order $i$ and is a principal minor of $B_{i+1}$,

(2) For $i = 1, 2, \cdots, r - 1$, no consecutive matrices $B_i$ and $B_{i+1}$ are

both singular.

Denote $D_i = \det(B_i)$. Then for any prime integer $p$, we define

$$c_p(B) = (-1, -D_r)_p \prod_{i=1}^{r-1} (D_i, -D_{i+1})_p,$$

where $(a, b)_p$ denotes the Hilbert symbol. If $D_{i+1} = 0$, then $(D_i, -D_{i+1})_p

(D_{i+1}, -D_{i+2})_p$ is interpreted to be $(D_i, -h)_p (h, -D_{i+1})_p$, where $h$ is an

arbitrary nonzero integer. Note that $c_p(B)$ is independent of the choice

of $\sigma$-series of $B$ [5, 11].

**Definition 2.1.** Let $B$ be a nonsingular integral matrix of order $r$. Then the Minkowski units $C_p(B)$ of $B$ is defined as follows:

(1) For $p = 2$, $C_2(B) = c_2(B)(-1)^{\beta}$, where

$$\beta = \left[\frac{r}{4}\right] + \left\{1 + \left[\frac{r}{2}\right]\right\} \frac{(d + 1)}{2} + \frac{(d^2 - 1)m}{8},$$

and [ ] denotes the Gaussian symbol, $m$ the power of 2 occurring

in $\det(B)$, and $d = 2^{-m}\det(B)$.

(2) For any odd prime integer $p$,

$$C_p(B) = c_p(B)(\det(B), p)_{p}^{\alpha},$$

where $\alpha$ denotes the exponent of $p$ occurring in $\det(B)$.

(3) For $p = \infty$, $C_{\infty}(B) = \prod C_p(B)$, where the product extends over

all prime integer $p$'s.

Let $A$ be an $n \times n$ symmetric integral matrix of rank $r$ and let $B$ and

$B'$ be any two nonsingular integral matrices of order $r$ associated to $A$. Then $C_p(B) = C_p(B')$ for any prime integer $p$, including $p = \infty$. The Minkowski unit $C_p(A)$ of $A$ is defined to be the Minkowski unit $C_p(B)$

of $B$.

**Theorem 2.2.** Let $\ell$ be an oriented link in $S^3$ and let $H(L)$ be

the modified Goeritz matrix associated to a diagram $L$ of $\ell$. Then the Minkowski unit $C_p(H(L))$ of $H(L)$ is an invariant of the link type $\ell$, denoted by $C_p(\ell)$, for any prime integer $p$, including $p = \infty$. 
Proof. Let $L_1$ and $L_2$ be two diagrams of the link $\ell$ and let $H(L_1)$ and $H(L_2)$ be the modified Goeritz matrices associated to $L_1$ and $L_2$, respectively. By [11, Lemma 2.4], it suffices to show that $H(L_1)$ and $H(L_2)$ are $R$-equivalent matrices.

$T_1$: Suppose that $H(L_2) = UH(L_1)U^t$ with unimodular integral matrix $U$. Then it is obvious from $Q_1$ that $H(L_1)$ and $H(L_2)$ are $R$-equivalent.

$T_2$: Suppose that $H(L_2) = \begin{pmatrix} H(L_1) & O & O \\ O & 1 & 0 \\ O & 0 & -1 \end{pmatrix}$. Observe that

\[
\begin{pmatrix}
I & O & O \\
O & 1 & -1 \\
O & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
H(L_1) & O & O \\
O & 1 & 0 \\
O & 0 & -1
\end{pmatrix}
\begin{pmatrix}
I & O & O \\
O & 1 & \frac{1}{2} \\
O & -1 & \frac{1}{2}
\end{pmatrix}
= \begin{pmatrix} H(L_1) & O & O \\ O & 0 & 1 \\ O & 1 & 0 \end{pmatrix},
\]

where $I$ denotes the identity matrix with the same order as $H(L_1)$.

By $Q_2$, \( \begin{pmatrix} H(L_1) & O & O \\ O & 0 & 1 \\ O & 1 & 0 \end{pmatrix} \) is $R$-equivalent to $H(L_1)$. Since $H(L_1)$ and $H(L_2)$ are transformed into each other by a finite sequence of $T_1$, $T_2$, or their inverses, they are $R$-equivalent matrices from the above observations. This completes the proof.

\[\square\]

Remark 2.3. (1) The set of modified Goeritz matrices $H$ obtained from the various diagrams of a link $\ell$ contains $M + M^t$ for some Seifert matrix $M$ of $\ell$. This implies that $C_p(\ell) = C_p(H)$ is equal to the Minkowski unit $C_p(\ell)$ defined by Murasugi [11].

(2) Let $A$ be a symmetric integral matrix and let $B$ be a nonsingular matrix associated to $A$. Let $\nu$ denote the number of odd primes of the form $4s + 3$ occurring with odd powers in the prime factor decomposition of $\det(B)$. It follows that $C_{\infty}(A) = (-1)^\gamma$, where $\gamma = \left\lceil \frac{\sigma(A) - 2\nu}{2} \right\rceil + \left\lceil \frac{\sigma(A) - 2\nu}{4} \right\rceil$ [4].

3. The Minkowski units of 2-periodic knots

Let $\ell = k_* \cup f_*$ be a 2-component oriented link in $S^3$ such that the component $f_*$ is unknotted and the linking number $\lambda$ of $k_*$ and $f_*$, denoted by $\lambda = \text{link}(k_*, f_*)$, is an odd integer. Then the inverse image $k = p_2^{-1}(k_*)$ of $k_*$ in the 2-fold cyclic branched covering $p_2 : \Sigma^3 \to S^3$ branched over $f_*$ is a 2-periodic knot in $\Sigma^3 \cong S^3$ whose factor knot is
the knot $k_*$. Conversely, every 2-periodic knots in $S^3$ arises in the this manner.

Now let $L = K_* \cup F_*$ be a regular diagram of $\ell = k_* \cup f_*$ in $\mathbb{R}^2$ which has the form as shown in Figure 2, where the points $a_1, a_2, \ldots, a_m$ are identified with the points $b_1, b_2, \ldots, b_m$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Let $w$ denote the number of white regions in the coloured diagram which does not intersect with the trivial component $F_*$ and let $a$ and $b$ denote the number of the crossings of type II in $K_* = L - F_*$ with incidence number $+1$ and $-1$, respectively.

In [6, Section 3], the authors discovered a relationship among the modified Goeritz matrices of the 2-periodic knot (or link) $k_*$, its factor $k_*$, and the link $\ell = k_* \cup f_*$ which can be summarized as the following Theorem 3.1.

**Theorem 3.1.** Let $\ell = k_* \cup f_*$ be an oriented 2-component link in $S^3$ such that $f_*$ is unknotted and $\lambda = \text{link}(k_*, f_*)$ is an odd integer. Let $L$ be a link diagram of $\ell$ as shown in Figure 2. Then

1. The modified Goeritz matrix $H(L)$ of $\ell$ associated to $L$ equivalent to the symmetric integral matrix of the form:

$$H(L) = \begin{pmatrix}
M & P & Q & O \\
P^t & N_1 & R & J \\
Q^t & R^t & N_2 & J \\
O & J^t & J^t & S
\end{pmatrix} \oplus (-I_a \oplus I_b) \oplus E_r,$$
where \( M \) (a \( w \times w \) matrix), \( P, Q, R, N_1, N_2 \) are some integral matrices, \( S = \begin{pmatrix} O & O \\ O & 2 \end{pmatrix} \), \( r \) is the positive integer with \( \lambda = 2r - m \), \( E_r = -I_r \oplus I_{m-r-1} \) if \( r \) is even, \( E_r = -I_{r+1} \oplus I_{m-r} \) if \( r \) is odd, and \( J \) is the \( \frac{m-1}{2} \times \left( \frac{m-1}{2} + 1 \right) \)
matrix of the form: for \( m = 1 \), \( J = \emptyset \) and for \( m > 1 \),

\[
J = \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 & 0 \\
0 & 1 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -1
\end{pmatrix}.
\]

(2) The modified Goeritz matrix \( H(K_*) \), \( K_* = L - F_* \), of the component \( k_* \) of \( \ell \) is given by

\[
H(K_*) = \begin{pmatrix} M & P + Q \\ P^t + Q^t & N_1 + N_2 + R + R^t \end{pmatrix} \oplus (-I_a \oplus I_b).
\]

(3) Let \( p_2 : \Sigma^3 \to S^3 \) be the 2-fold cyclic branched covering space branched over \( f_* \). Then the modified Goeritz matrix \( H(K) \) of the 2-periodic knot \( k = p_2^{-1}(k_*) \) in \( \Sigma^3 \cong S^3 \) is given by the symmetric matrix of the form:

\[
H(K) = \begin{pmatrix} M & P & O & Q \\ P^t & N_1 + N_2 & Q^t & R + R^t \\ O & Q & M & P \\ Q^t & R + R^t & P^t & N_1 + N_2 \end{pmatrix} \oplus (-I_a \oplus I_b) \oplus (-I_a \oplus I_b).
\]

Lemma 3.2. Let \( H(L) \), \( H(K_*) \), and \( H(K) \) be the modified Goeritz matrices in Theorem 3.1. Then

(1) There exists a nonsingular rational matrix \( R \) such that

\[
R(H(K) \oplus T(r))R^t = 2\{H(K_*) \oplus H(L)\} \oplus -I_{2a} \oplus I_{2b},
\]

where \( T(r) \) denotes the diagonal matrix given by

\[
T(r) = \begin{cases}
4(-I_{\frac{m-1}{2}} \oplus I_{\frac{m-1}{2}+1}) \oplus 2(-I_{2a+r} \oplus I_{2b+m-r-1}) & \text{if } r \text{ even,} \\
4(-I_{\frac{m-1}{2}} \oplus I_{\frac{m-1}{2}+1}) \oplus 2(-I_{2a+r+1} \oplus I_{2b+m-r}) & \text{if } r \text{ odd.}
\end{cases}
\]

(2) \( \det(H(K)) = \frac{1}{2} \det(H(K_*)) \det(H(L)) (-1)^{\frac{m-1}{2}} \).
Proof. (1) Let
\[ U = \begin{pmatrix}
I_w & O & O & -QD & O \\
O & I_s & -I_s & (N_2 - R_1)D & O \\
O & 0 & I_s & O & D \\
O & O & O & O & Z \\
O & O & O & 1 & 1
\end{pmatrix} \oplus I_{a+b} \oplus I_{t(r)}, \]
where \( s = \frac{m-1}{2}, Z = (1 \, 1 \, \cdots \, 1), D = (d_{ij})_{1 \leq i, j \leq s} \) such that \( d_{ij} = 1 \) for \( i \geq j \), otherwise all zero, and \( t(r) = m + 1 \) or \( m - 1 \) according as \( r \) is odd or even.

Now define \( V = I_{w+s} \oplus U \oplus I_{a+b} \oplus I_{a+b} \) and
\[ W = \begin{pmatrix}
I_{w+s} & I_{w+s} & O \\
I_{w+s} & -I_{w+s} & O \\
O & O & I_{2(a+b)}
\end{pmatrix} \oplus \begin{pmatrix}
I_s & -\frac{1}{2}N_2 + I_s & O \\
I_s & -\frac{1}{2}N_2 - I_s & O \\
O & O & 1
\end{pmatrix}^{-1} \oplus I_{2(a+b)+t(r)}. \]

Then \( V \) and \( W \) are nonsingular rational matrices and we obtain that
\[
W \{ H(K) \oplus 4(I_s \oplus -I_s \oplus (1)) \oplus 2(-I_a \oplus I_b \oplus -I_a \oplus I_b \oplus E_r) \} W^t
= (XV)\{2(H(K_s) \oplus H(L)) \oplus (-I_a \oplus I_b \oplus -I_a \oplus I_b) \}(XV)^t
\]
for an appropriate permutation matrix \( X \). This implies the result.

(2) By (1), \( \det(H(K)) \det(R)^2 = 2^{2w-1} \det(H(K_s)) \det(H(L))(\det(R))^{\frac{m-1}{2}} \).

Since \( 2|\det(H(K))| = |\det(H(K_s))||\det(H(L))| \) [6] and \( \det(R)^2 = 2^{2w} \).
This implies the result. \( \square \)

**Lemma 3.3.** Let \( A \) be an \( n \times n \) nonsingular integral matrix and let \( m \) denote the power of 2 occurring in \( \det(A) \).

(1) Let \( d = 2^{-m} \det(A) \). Then
\[
C_2(2A) = \begin{cases}
C_2(A)(-1)^{(\frac{d^2-1}{8})n} & \text{if } n \text{ is odd}, \\
C_2(A)(2, d)_2 & \text{if } n \text{ is even}.
\end{cases}
\]

(2) Let \( p \) be any odd prime integer and let \( \alpha \) be the power of the odd prime \( p \) occurring in \( \det(A) \). Then
\[
C_p(2A) = C_p(A)(-1)^{(\frac{p^2-1}{8})\alpha}.
\]

(3) \( C_\infty(2A) = C_\infty(A) \).

**Proof.** Let \( B_1, B_2, \cdots, B_n \) be a \( \sigma \)-series of \( A \), where \( B_n = A \), and let \( D_i = \det(B_i)(i = 1, 2, \cdots, n) \). Then \( 2B_1, 2B_2, \cdots, 2B_n \) is a \( \sigma \)-series of \( 2A \). Let \( \tilde{D}_i = \det(2B_i) \). Then \( \tilde{D}_i = 2^i D_i \) and so, for any prime integer \( p \),
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\[
c_p(2A) = (-1, -\bar{D}_n)_p \prod_{i=1}^{n-1} (\bar{D}_i, -\bar{D}_{i+1})_p
\]
\[
= \{(1, -D_n)_p \prod_{i=1}^{n-1} (D_i, -D_{i+1})_p \} \epsilon(p)
\]
\[
= c_p(A) \epsilon(p),
\]

where
\[
\epsilon(p) = (-1, 2^n)_p \prod_{i=1}^{n-1} (2^i, -2^{i+1} D_{i+1})_p (2^{i+1}, D_i)_p
\]
\[
= \begin{cases} 
1 & \text{if } n \text{ is odd}, \\
(2, \det(A))_p & \text{if } n \text{ is even}.
\end{cases}
\]

In order to show (1), let \( m \) denote the power of 2 occurring in \( \det(A) \) and let \( d = 2^{-m} \det(A) \). Let \( \bar{m} \) be the power of 2 occurring in \( \det(2A) \) and let \( \bar{d} = 2^{-\bar{m}} \det(2A) \). Then \( \bar{m} = m + n \) and \( d = \bar{d} \). By Definition 2.1,

\[
C_2(2A) = c_2(2A)(-1)^\bar{\beta}
\]
\[
= \begin{cases} 
c_2(A)(-1)^\bar{\beta} & \text{if } n \text{ is odd}, \\
c_2(A)(2, \det(A))_2(-1)^\bar{\beta} & \text{if } n \text{ is even},
\end{cases}
\]

where \( \bar{\beta} = \left[ \frac{n}{4} \right] + \left\{ 1 + \left[ \frac{n}{2} \right] \right\} \left( \frac{d+1}{2} \right) + \left( \frac{d^2-1}{2} \right) \bar{m} = \left( \left[ \frac{n}{4} \right] + \left\{ 1 + \left[ \frac{n}{2} \right] \right\} \left( \frac{d+1}{2} \right) + \left( \frac{d^2-1}{2} \right) \right) + \left( \frac{d^2-1}{2} \right) \bar{m} \). Since \( \det(A) = 2^m d \) and \( (2,2)_2 = 1 \), we obtain that

\[
C_2(2A) = \begin{cases} 
c_2(A)(-1)^{\left( \frac{d^2-1}{2} \right) m} & \text{if } n \text{ is odd}, \\
c_2(A)(2, d)_2 & \text{if } n \text{ is even}.
\end{cases}
\]

(2) Let \( \alpha \) denote the power of \( p \) occurring in \( \det(A) \). By Definition 2.1, for any odd prime integer \( p \),

\[
C_p(2A) = c_p(2A)(\det(2A), p)_p^\alpha = c_p(2A)(\det(A), p)_p^\alpha (2^n, p)_p^\alpha
\]
\[
= \begin{cases} 
c_p(A)(2, p)_p^\alpha & \text{if } n \text{ is odd}, \\
c_p(A)(2, \det(A))_p & \text{if } n \text{ is even}.
\end{cases}
\]

Note that \( (2, \det(A))_p = (2, p)_p^\alpha \) and \( (2, p)_p = (-1)^{\frac{p^2-1}{8}} \). Hence \( C_p(2A) = c_p(A)(-1)^{\frac{(p^2-1)\alpha}{8}} \).
(3) Since \( \sigma(2A) = \sigma(A) \) and the number \( \nu \) of odd primes of the form \( 4s + 3 \) occurring with odd powers in the prime factor decomposition of \( \det(2A) \) is equal to that of \( \det(A) \), it follows Remark 2.3(2) that \( C_\infty(2A) = C_\infty(A) \).

From Lemma 3.2, Lemma 3.3, [11, (2.5)], and the properties of Hilbert symbol [5], we obtain the following

**Lemma 3.4.** For any odd prime integer \( p \),

1. \( C_p(T(r)) = 1 \).
2. \( C_p(H(K) \oplus T(r)) = C_p(H(K)) \).
3. \( C_p(2\{H(K_* \oplus H(L))\}) = C_p(H(K_* ) \oplus H(L))(-1)^{(p^2-1)\alpha \over 8} \), where \( \alpha \) denotes the power of \( p \) occurring in \( \det(H(K_* ) \oplus H(L)) \).

Let \( \Delta_{k_*}(t) \) and \( \Delta_{k_* \cup f_*}(t_1, t_2) \) denote the Alexander polynomials of \( k_* \) and \( \ell = k_* \cup f_* \), respectively. Then

**Theorem 3.5.** Let \( k \) be a 2-periodic knot in \( S^3 \) with the fixed point set \( f \) and let \( k_* \) be its factor knot and \( f_* \) be the orbit of \( f \). Then

1. For any odd prime integer \( p \),
   \[
   C_p(k)(-1)^{(p^2-1)\alpha \over 8} = C_p(k_*)_C_p(k_* \cup f_*)(p, p)^{\alpha_1 \alpha_2},
   \]
   where \( \alpha, \alpha_1, \) and \( \alpha_2 \) denote the powers of \( p \) occurring in \( |\Delta_k(-1)| \), \( |\Delta_{k_* \cup f_*}(-1, -1)| \), and \( |\Delta_{k_*}(-1)| \), respectively.

2. \[
   C_\infty(k)(-1)^{n_k(2\nu_2 + 2\nu_1 + 2) \over 4} = C_\infty(k_*)_C_\infty(k_* \cup f_*)(-1)^{\sigma(k_*) - 2\nu_1 \over 4} + \sigma(k_* \cup f_*) - 2\nu_2 \over 4},
   \]
   where \( \nu, \nu_1, \) and \( \nu_2 \) be the number of odd primes of the form \( 4s + 3 \) occurring with odd powers in the prime factor decomposition of \( |\Delta_k(-1)| \), \( |\Delta_{k_*}(-1)| \), and \( |\Delta_{k_* \cup f_*}(-1, -1)| \), respectively, and \( [ \ ] \) denotes the Gaussian symbol.

**Proof.** From [11, Lemma 2.4] and Lemma 3.2(1), for any prime integer \( p \), it follows that

\[
C_p(H(K) \oplus T(r)) = C_p(2\{H(K_* \oplus H(L)) \oplus -I_{2a} \oplus I_{2b}) \).
\]

1. By [11, (2.5)], Lemma 3.4, and the fact that \( C_p(-I_{2a} \oplus I_{2b}) = 1 \), we obtain that for any odd prime \( p \),
   \[
   C_p(H(K)) = C_p(H(K_* \oplus H(L))(-1)^{(p^2-1)\alpha \over 8},
   \]
where $\alpha$ denotes the powers of $p$ occurring in $\det(H(K_*) \oplus H(L))$ and

$$C_p(H(K)) = C_p(H(K_*))C_p(H(L))(\det(H(K_*), p^{-\alpha_1}\det(H(L), p^{-\alpha_2})$$

$$(\det(H(K_*)), \det(H(L)))_p(-1)^{\frac{(p^2-1)(\alpha)}{8}},$$

where $\alpha_1$ and $\alpha_2$ denote the powers of $p$ occurring in $\det(H(L))$ and $\det(H(K_*))$, respectively. Let $d(k_*) = p^{-\alpha_2}\det(H(K_*)), d(\ell) = p^{-\alpha_1}\det(H(L)).$ Then $\alpha_1 + \alpha_2 = \alpha$ and

$$\det(H(K_*), p^{-\alpha_1}\det(H(L), p^{-\alpha_2}\det(H(K_*)), \det(H(L)))_p$$

$$= (d(k_*), d(\ell))_p(p, p^{-\alpha_1\alpha_2} = (p, p)^{\alpha_1\alpha_2}.$$ 

It follows from Lemma 3.2(2) that $\alpha$ is equal to the power of $p$ occurring in $\det(H(K)).$ This implies the result.

(2) Let $\nu, \nu_1, \nu_2$ be the number of odd primes of the form $4s + 3$ occurring with odd powers in the prime factor decomposition of $\det(H(K))$, $\det(H(K_*)), \det(H(L))$, respectively, and let $\gamma = \left[\frac{\sigma(k) - 2\nu}{2}\right] + \left[\frac{\sigma(k) - 2\nu}{4}\right].$ Then $C_\infty(k) = C_\infty(H(K)) = (-1)^\gamma.$ Since $2\det(H(K)) = \det(H(K_*))$ and $\det(H(L))$ and $\sigma(k) = \sigma(k_*) + \sigma(\ell) + \lambda$ [6], $\nu = \nu_1 + \nu_2$ and $\gamma = \left[\frac{\sigma(k) - 2\nu_1}{2}\right] + \left[\frac{\sigma(k) - 2\nu_2}{2}\right] + \left[\frac{\sigma(k) - 2\nu + 2\lambda + 2}{4}\right].$ This implies the result and we complete the proof of Theorem 3.5.

References


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